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Singularities in auxetic elastic bimaterials

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ABSTRACT

In this paper we study the effects of negative Poisson's ratios on elastic problems containing singularities. Materials with a negative Poisson's ratio are termed auxetic. We present a brief review of such materials. The elasticity problem of a bimateral wedge is presented, then two particular cases of this problem are investigated: the free-edge problem and the interface crack problem. We study the effect on the stress singularity due to one portion of the bimaterial becoming auxetic. We find that the auxetic material has a significant effect on the singularity order, even causing the singularity to vanish for certain values of the elastic constants.

1. Introduction

Poisson observed that a material stretched under axial tensile forces not only elongates longitudinally, but it also contracts laterally. Most common materials undergo a transverse contraction when stretched in one direction and a transverse expansion when compressed; the magnitude of this transverse deformation is governed by Poisson's ratio. Poisson's ratio has two theoretical limits for a linear elastic, isotropic material: $-1 < \nu < 0.5$. The upper limit, $\nu \to 0.5$, represents the incompressible limit for the material, and the lower limit, $\nu \to -1$, is required for the strain energy to be a positive definite function (Fung et al., 1965).

Lakes (1987) presented a new foam structure which exhibited a negative Poisson's ratio. This was achieved by converting a conventional foam using heating and compression techniques to create a reentrant structure. This type of material is called *auxetic*: the material expands laterally upon longitudinal tensile loading and contracts laterally under longitudinal compression. Auxetic materials are not completely new: Love (1944) in his treatise book on elasticity presents an example of "single crystal pyrite" with a Poisson's ratio of $-1/7$. However, this material is cubic and Poisson's ratios do not follow the isotropic limits. For anisotropic solids, no such limits exist as shown by Ting and Chen (2005).

Auxetic materials are very rare in nature as discussed by Stavroulakis (2005). Some examples are silicon polymorphs (Kimizuka et al., 2000; Yeganeh-Haeri et al., 1992;Alderson and Evans, 2002; Keskar and Chelikowsky, 1992), zeolite (Grima et al., 2005), and silicates (Grima et al., 2005). Additionally, auxetic behavior can be created in foams and similar materials by varying the geometric structure (Grima and Evans, 2000, 2006; Smith et al., 2000; Grima et al., 2012). The auxetic behavior in these materials can be explained in terms of their geometry and deformation mechanisms. Auxetic behavior has also been observed to occur locally on the nanoscale in certain materials (Alderson et al., 2005; Franke and Magerle, 2011), and auxetic behavior has been noted in certain plate problems (Bhullar et al., 2010). A review article by Yang et al. (2004) summarizes much of the state of the art as to current understanding of both natural and man-made auxetic materials.

The effects of auxetic behavior on material properties have been investigated (see, for example, Lakes and Drugan, 2002) to study if material properties such as hardness, toughness, indentation resistance, and acoustic response can be enhanced. For example, in Lakes (1987) and Grima and Evans (2000) the shear modulus, μ, is predicted to increase as the Poisson's ratio approaches the lower theoretical limit ($\nu \to -1.0$) if Young's modulus (E) is constant. Indentation resistance has been investigated and enhancements in hardness have been found. Auxetic foams have been found to be up to three times more difficult to indent than conventionally processed polymers (Alderson et al., 2000). Foams with negative Poisson's ratios were also found to have higher resilience than conventional foams (Lakes, 1987). Finally, auxetic materials are predicted to become very tough according to classical elasticity theory when the Poisson's ratio approaches the lower limit of -1.0. For example, the fracture toughness of negative Poisson's ratio open cell copper foams are enhanced by 80%, 130%, and 160% for permanent volumetric compression ratio values of 2.0, 2.5 and 3.0, respectively, compared to a conventional foam with a positive Poisson ratio (Choi and Lakes, 1996).

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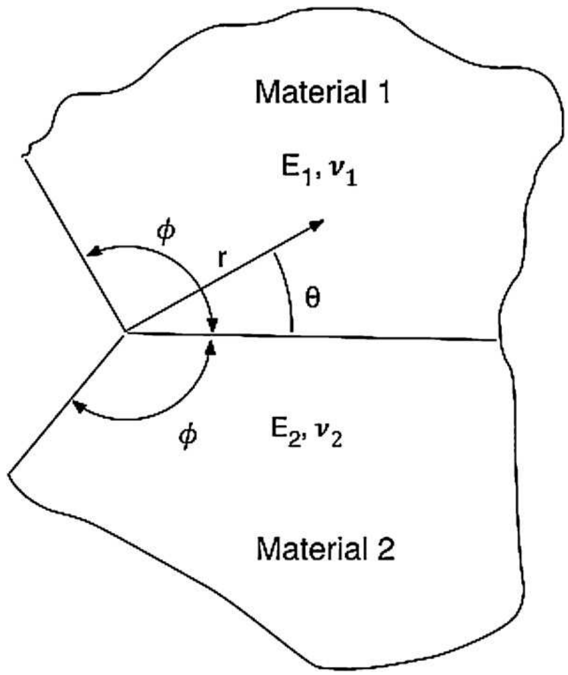

Fig. 1. General elastic bimaterial problem.

The motivation for the present paper is to investigate problems in elastic bimaterials when one portion of the bimaterial is auxetic. Typical problems of this nature are the free-edge singularity (which can be useful in investigating delaminatation failures in bimaterials) and the interface crack problem. To generalize the problem, we present results for both the free-edge and interface crack problems over the entire range of permissible Poisson ratios where the effect of a material becoming auxetic on the stress singularity order may be readily seen.

2. Singularities in bimaterials

We consider the general bimaterial problem shown in Fig. 1. Both Material 1 and Material 2 are taken to be linear, elastic, isotropic, and homogeneous. Material j has Poisson's ratio ν_j, shear modulus μ_j, and Young's modulus E_j, $j = 1, 2$. We will assume the two materials are perfectly bonded at the planar interface so displacements and tractions are continuous across the interface. The local polar coordinate system r, θ is positioned at the root of the notch, and the notch-face angle ϕ is assumed to be symmetric with respect to the interface. We focus on two specific problems: the *free-edge problem* ($\phi = \pi/2$ in Fig. 1) and the *interface crack problem* ($\phi = \pi$). Of particular interest in both problems is the nature of the increase in stress (if any) as $r \to 0$.

The singularity order δ has been studied extensively in isotropic bimaterials. For interface cracks, solutions of the type $\sigma \sim r^\delta$ have been obtained where δ is complex valued. The free-edge problem has been extensively studied by Bogy (1968, 1970) and others, and the stress field solution was of the type r^δ where δ was real-valued depending on the elastic properties of the two materials. Dundurs (1969) reduced the number of parameters that the stress field depends on from three to two and used these parameters to express the solution in a more compact way.

The plane problem of bonded dissimilar wedges of arbitrary angle subjected to general forms of loading was studied by Hein and Erdogan (1971). The solution was obtained by solving two simpler problems where the solution is given by the sum of solutions of the two separated problems. The dependence of singularity order in the stress field on the wedge angles and the elastic constants of the materials was also investigated by Hein and Erdogan (1971). Numerical results of several angle geometries for all relevant material constants combinations were produced. Finally, we note that Green's function methods have also been used successfully on the free-edge problem. The anisotropic problem was considered by Tewary (1991), where the Green's function was represented by an exact integral. Both the displacement and the stress Green's function were obtained. A Green's function method was also employed by Martin (2003) where a reduced equation for calculating the singularity order was found. An anisotropic bimaterial problem was also considered by Berger et al. (1998) where it was noted that the free edge singularity vanishes for certain free-surface angles and elastic constants combinations.

3. Formulation

In this section, we review the analytical solution for the bimaterial wedge as introduced in previous papers, see for example Bogy (1968), Ding and Kumosa (1994). To be definite, we consider plane-strain problems; similar results are available for plane-stress problems. Typical solutions use an Airy stress function in polar coordinates, $\psi(r, \theta)$ (Williams, 1952), written in each portion of the bimaterial,

$$\psi_j(r,\theta) = r^\delta F_j(\theta), \quad j = 1, 2 \tag{1}$$

where $F_j(\theta)$ can be determined from the governing differential equation for ψ, $\nabla^4\psi = 0$. The stress components in each portion of the bimaterial are then

$$\begin{aligned}\sigma_{rr}^{(j)} &= (\delta+1)r^\delta\left\{\delta[A_j\sin\delta\theta - B_j\cos\delta\theta]\right.\\ &\quad \left.-(\delta+2)[C_j\sin(\delta+2)\theta + D_j\cos(\delta+2)\theta]\right\},\end{aligned} \tag{2}$$

$$\begin{aligned}\sigma_{r\theta}^{(j)} &= (\delta+1)r^\delta\left\{\delta[A_j\cos\delta\theta + B_j\sin\delta\theta]\right.\\ &\quad \left.+(\delta+2)[C_j\cos(\delta+2)\theta - D_j\sin(\delta+2)\theta]\right\},\end{aligned} \tag{3}$$

$$\begin{aligned}\sigma_{\theta\theta}^{(j)} &= (\delta+1)(\delta+2)r^\delta\left\{A_j\sin\delta\theta + B_j\cos\delta\theta\right.\\ &\quad \left.+C_j\sin(\delta+2)\theta + D_j\cos(\delta+2)\theta\right\},\end{aligned} \tag{4}$$

where $j = 1, 2$ indicates if the stress component is in Material 1 or 2.

Using similar notation, the displacements in each portion of the bimaterial are

$$\begin{aligned}u_r^{(j)} &= \frac{r^{\delta+1}}{E_j}\left\{\left(\delta+\nu_j(\delta+2)\right)[A_j\sin\delta\theta - B_j\cos\delta\theta]\right.\\ &\quad \left.-(\delta+2)(1+\nu_j)[C_j\sin(\delta+2)\theta + D_j\cos(\delta+2)\theta]\right\},\end{aligned} \tag{5}$$

$$\begin{aligned}u_\theta^{(j)} &= \frac{r^{\delta+1}}{E_j}\{-(\nu_j(\delta+2)+(\delta+6))[A_j\cos\delta\theta + B_j\sin\delta\theta]\\ &\quad +(\delta+2)(1+\nu_j)[C_j\cos(\delta+2)\theta + D_j\sin(\delta+2)\theta]\}.\end{aligned} \tag{6}$$

For continuity of displacement and traction at the interface, $\theta = 0$,

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0), \quad u_\theta^{(1)}(r,0) = u_\theta^{(2)}(r,0), \tag{7}$$

$$\sigma_{\theta\theta}^{(1)}(r,0) = \sigma_{\theta\theta}^{(2)}(r,0), \quad \sigma_{r\theta}^{(1)}(r,0) = \sigma_{r\theta}^{(2)}(r,0). \tag{8}$$

Finally, the traction-free surface conditions are

$$\sigma_{\theta\theta}^{(1)}(r,\phi) = \sigma_{\theta\theta}^{(2)}(r,-\phi) = \sigma_{r\theta}^{(1)}(r,\phi) = \sigma_{r\theta}^{(2)}(r,-\phi) = 0. \tag{9}$$

Substituting the stress and displacement equations in the boundary and continuity conditions, we obtain eight linear equations in the eight unknown values of A_j, B_j, C_j, and D_j which can be written in matrix form as $\mathcal{A}x = 0$. For a non-trivial solution we then have $\det\mathcal{A} = 0$, which yields the singularity order δ.

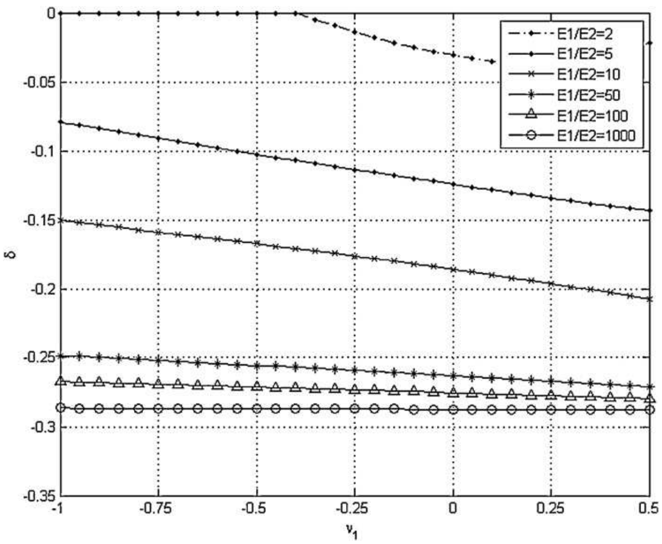

Fig. 2. Singularity order δ for the free-edge problem as a function of ν_1 (with $\nu_2 = 0.30$) for $2 \le \mathcal{E} = E_1/E_2 \le 1000$.

4. The free-edge problem

For this problem, we consider the effects of an auxetic material on the singularity at the intersection between an interface and the free surface of the bimaterial. Following Martin (2003), the singularity order for $\phi = \pi/2$ can be computed from

$$\det \mathcal{A} = -4^4(\delta+1)^2[(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1]^2 \Delta(\delta) = 0 \qquad (10)$$

where

$$\begin{aligned}\Delta(\delta) &= (\beta^2-1)S^4 + \left[1 + 2(\delta+1)^2(\alpha-\beta)\beta\right] S^2 \\ &\quad + (\delta+1)^2\left[(\delta+1)^2(\alpha-\beta)^2 - \alpha^2\right],\end{aligned} \qquad (11)$$

$S = -\sin\frac{1}{2}(\delta+1)\pi$, and α, β are Dundurs constants (Dundurs, 1969); for plane-strain problems,

$$\alpha = \frac{\mathcal{E}-1}{\mathcal{E}+1}, \quad \beta = \frac{(1-2\nu_2)\mu_1 - (1-2\nu_1)\mu_2}{2(1-\nu_2)\mu_1 + 2(1-\nu_1)\mu_2}, \quad \mathcal{E} = \frac{E_1}{E_2}. \qquad (12)$$

In the majority of the literature (see, for example, Hutchinson and Suo, 1992), physically admissible values for the Dundurs constants are restricted to lie within the parallelogram enclosed by $\alpha = \pm 1$ and $\alpha - 4\beta = \pm 1$ in the (α, β) plane. However, this result assumes that Poisson's ratio is positive. If we permit auxetic materials, the parallelogram is 50% larger; it is enclosed by $\alpha = \pm 1$ and $\alpha - \frac{8}{3}\beta = \pm 1$.

To examine the influence of one portion of the bimaterial becoming auxetic, we calculate the singularity order, δ, from Eq. (11) as we vary ν_1. For this calculation we fix $\nu_2 = 0.30$ and allow the elastic moduli ratio, $\mathcal{E} = E_1/E_2$, to vary. The first set of results is shown in Fig. 2, where we plot δ as a function of ν_1. We take values of ν_1 across the entire range of permissible values of the Poisson ratio, $-1 < \nu_1 < 0.5$. Note in the figure that the singularity order, δ, becomes increasingly negative as $\mathcal{E}$ is increased for fixed ν_1. Also, once Material 1 becomes auxetic ($\nu_1 < 0$), the singularity order is significantly affected by the Poisson ratio. In certain cases the singularity order decreases over 18% due to Material 1 becoming auxetic. Finally, in Fig. 2 we note that for $\mathcal{E} = 2$, the singularity disappears when ν_1 decreases below -0.4. This is explored in more detail in Fig. 3 for moduli ratios of $1.5 \le \mathcal{E} \le 4.5$. In Fig. 3, note that if $\mathcal{E} \ge 3$, the singularity cannot be eliminated.

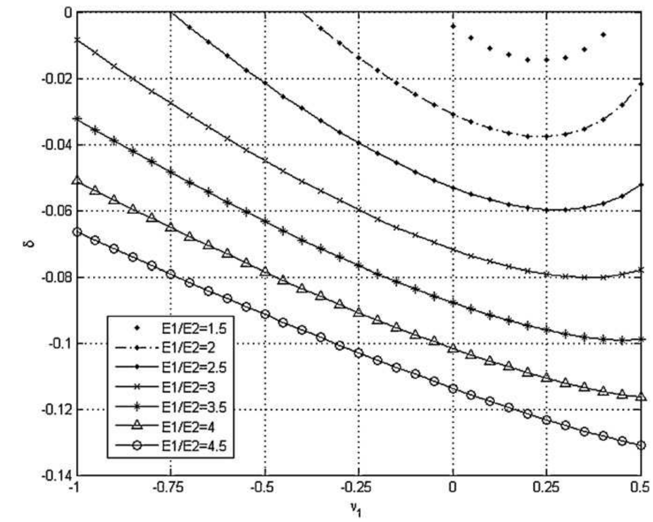

Fig. 3. Singularity order δ for the free-edge problem as a function of ν_1 (with $\nu_2 = 0.30$) for $1.5 \le \mathcal{E} \le 4.5$.

5. The interface crack problem

For this problem, we investigate the effect of an auxetic material on the singularity at the tip of an interface crack, that is, when $\phi = \pi$. The stress field near the tip of the crack varies, for example, as (Atkinson, 1979)

$$\sigma \sim r^{-1/2}\cos(\mathrm{Im}(\delta)\log r) \qquad (13)$$

where the second term arises from the imaginary part of the singularity order and is usually referred to in the literature as the oscillatory part of the singularity. This can be computed in plane-strain as (Gu and Belytschko, 1994) as

$$\mathrm{Im}(\delta) = \frac{1}{2\pi}\log\left[\frac{(3-4\nu_1)\mu_2 + \mu_1}{(3-4\nu_2)\mu_1 + \mu_2}\right]. \qquad (14)$$

When the two portions of the bimaterial are closely matched elastically, the imaginary part of the singularity order is very small as expected. In Fig. 4 we have plotted $\mathrm{Im}(\delta)$ for a variety of ratios μ_1/μ_2 as a function of ν_1. As with the free-edge problem, we have kept $\nu_2 = 0.30$ for these calculations. We note in the figure that, in

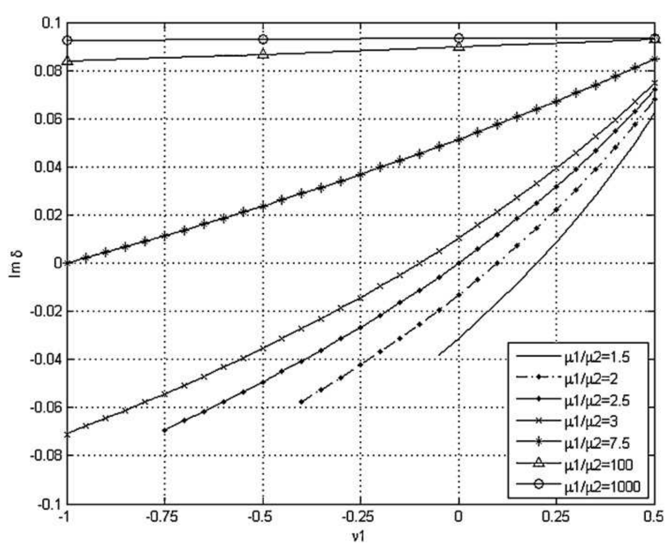

Fig. 4. Imaginary part of the singularity order δ for the interface crack problem as a function of ν_1 (with $\nu_2 = 0.30$) for $1.5 \le E_1/E_2 \le 1000$.

Table 1
Values of ν_1 where $\mathrm{Im}(\delta) = 0$ ($\nu_2 = 0.3$).

μ_1/μ_2	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
ν_1	0.2	0.1	0.0	−0.1	−0.2	−0.3	−0.4	−0.5	−0.6	−0.7	−0.8	−0.9

general, a negative Poisson's ratio causes the oscillatory singularity to decrease when compared to cases where the Poisson's ratio is positive. Of particular interest are cases where the decrease due to ν_1 becoming negative can actually drive $\mathrm{Im}(\delta)$ to zero. In Table 1 we present values for ν_1 where $\mathrm{Im}(\delta) = 0$ for various ratios μ_1/μ_2. Note in the table that if $\mu_1/\mu_2 > 2.5$, the material must be auxetic in order for the crack-tip singularity to be purely real.

6. Summary

In this paper, we have investigated stress singularities in elastic bimaterials where we allow the Poisson's ratio of one portion of the bimaterial to vary completely over $-1 < \nu < 0.5$. Our motivation for this study comes from recent discoveries of auxetic materials, and how this might affect stress singularities in such problems. We found that when one portion of the bimaterial becomes auxetic, the effect on either the free-edge or interface crack singularity can be profound, even causing it to vanish given appropriate values of the remaining elastic constants. As more auxetic materials are developed, this fact could lead to strategies helping to suppress delamination or fracture failures in these bimaterials.

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