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N masses on an infinite string and related one-dimensional scattering problems

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HIGHLIGHTS

- Exact solution for a finite periodic row.
- Exact solution for a finite periodic row apart from one scatterer.
- Approximate solution for disordered almost-periodic row.

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ABSTRACT

One-dimensional time-harmonic waves interact with a finite number of scatterers: they could be beads on a long string, for example. If the scatterers are identical and equally spaced, such periodic problems can be solved exactly. One problem solved here arises when one scatterer in a periodic row is forced to oscillate, giving the Green function for the row. Our main interest is with disordered problems, where a periodic configuration is disturbed. Two problems are studied. First, just one scatterer in a finite periodic row is displaced: an exact solution is obtained for the transmission coefficient and its average over all allowable displacements. Second, a similar problem is treated where each scatterer is displaced by a small distance from its position in the periodic row. The main tools used are perturbation theory and transfer matrices.

1. Introduction

Wave propagation in one-dimensional periodic media is a classical topic: one thinks of passbands, stopbands, the Kronig–Penney problem and the little book by Léon Brillouin [1]. Reflection and transmission by a finite periodic row of N scatterers have also been studied extensively: explicit formulas are available for the complex reflection and transmission coefficients, R_N^{per} and T_N^{per}, respectively (see Section 4). The limit $N \to \infty$, giving a semi-infinite periodic row, is discussed briefly in Section 5.

A related problem arises when one scatterer in the finite periodic row is forced to oscillate. The solution of this problem is given in Section 6. It can be viewed as a Green function for the structure.

We may think of periodic media as being at one end of a spectrum of one-dimensional problems. At the other end are random media. Here, the paradigm is *localization*; see, for example, [2, Chapter 7]. We are motivated by disordered periodic media, where the problem is almost periodic. For some recent work on this problem, see, for example, [3–6]. (Further references will be mentioned later.) In particular, Poddubny et al. [6] have shown that localization can be suppressed in certain situations.

With these applications in mind, we describe some calculations in which a finite periodic row is perturbed. Thus, in Section 7, we consider reflection and transmission by a row in which one of the scatterers is displaced by a distance ε. The

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transmission coefficient is calculated exactly, as is the average transmission coefficient. These results do not assume that ε is small (compared to the spacing or wavelength) but they are complicated. We then approximate these results for small $k\varepsilon$, and we show that they can be obtained more easily by assuming that $k\varepsilon$ is small from the outset. This latter approach is then developed for the more difficult problem where each scatterer is displaced by a small distance, independently of all the others (Section 8). Again, explicit results are obtained for the average reflection and transmission coefficients, correct to second order in $k\varepsilon$.

The purpose of this work is to obtain some benchmark solutions with minimal assumptions. The methods used are rather elementary. The main tool used is the transfer matrix for each scatterer; their properties are reviewed in Section 2. Each scatterer can be quite general: we do not assume point scatterers.

2. Transfer matrices

Consider one scattering region or "cell", $|x| < a$. Outside the cell, the governing differential equation is $u''(x)+k^2u(x) = 0$. Thus, we can write

$$u(x) = \begin{cases} A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx}, & x < -a, \\ C\mathrm{e}^{\mathrm{i}kx} + D\mathrm{e}^{-\mathrm{i}kx}, & x > a, \end{cases}$$

where A, B, C and D are constants. (At this stage, we do not have to say anything about what is in the cell, except we shall assume that there are no losses.) The suppressed time-dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$, so that the $\mathrm{e}^{\mathrm{i}kx}$ terms give waves going to the right (x increasing) whereas the $\mathrm{e}^{-\mathrm{i}kx}$ terms give waves going to the left. The amplitudes on the right are related to those on the left using a *transfer matrix* $\mathbb{T}$:

$$\begin{pmatrix} C \\ D \end{pmatrix} = \mathbb{T} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Considerations of energy conservation and time-reversal invariance show that $\mathbb{T}$ must have the structure (see [7] or [8, Chapter 1])

$$\mathbb{T} = \begin{pmatrix} w^* & z \\ z^* & w \end{pmatrix} \quad \text{with } \det \mathbb{T} = |w|^2 - |z|^2 = 1, \tag{1}$$

where the asterisk denotes complex conjugation.

If we want to step to the left, we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathbb{T}^{-1} \begin{pmatrix} C \\ D \end{pmatrix} \quad \text{with } \mathbb{T}^{-1} = \begin{pmatrix} w & -z \\ -z^* & w^* \end{pmatrix}. \tag{2}$$

In terms of reflection and transmission coefficients, we have

$$u(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + r_+\mathrm{e}^{-\mathrm{i}kx}, & x < -a, \\ t_+\mathrm{e}^{\mathrm{i}kx}, & x > a, \end{cases}$$

$$u(x) = \begin{cases} t_-\mathrm{e}^{-\mathrm{i}kx}, & x < -a, \\ r_-\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx}, & x > a. \end{cases}$$

These give

$$\begin{pmatrix} 1 \\ r_+ \end{pmatrix} = \mathbb{T}^{-1} \begin{pmatrix} t_+ \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} r_- \\ 1 \end{pmatrix} = \mathbb{T} \begin{pmatrix} 0 \\ t_- \end{pmatrix}.$$

Comparison with Eqs. (1) and (2) shows that

$$t_+ = t_- \equiv t, \qquad 1 - |t|^2 = |r_\pm|^2 \equiv |r|^2, \qquad r_+^* t + r_- t^* = 0, \qquad w = t^{-1}, \qquad z = r_-/t = -r_+^*/t^*.$$

If the scatterer is moved from $x = 0$ to $x = b$, the new reflection coefficients are $r_+\mathrm{e}^{2\mathrm{i}kb}$ and $r_-\mathrm{e}^{-2\mathrm{i}kb}$, whereas the transmission coefficient remains unchanged. Hence, moving the scatterer within the cell changes z to $z\mathrm{e}^{-2\mathrm{i}kb}$ but leaves w unchanged.

For a point scatterer at $x = 0$, we have $a = 0$,

$$u(0^+) = u(0^-) \quad \text{and} \quad u'(0^+) - u'(0^-) = Mu(0), \tag{3}$$

where M is a real constant. We find $r_+ = r_- = r$, say, $t = 1 + r$,

$$r = \frac{M}{2\mathrm{i}k - M} \quad \text{and} \quad t = \frac{2\mathrm{i}k}{2\mathrm{i}k - M}. \tag{4}$$

(If M is not real, $r^*t + rt^* \neq 0$.)

3. Multiple cells

Let us consider a periodic row of identical cells, each of width $2a$. The cells are centred at $x = nd$ with $d \geq 2a$. To the left of the cell at $x = nd$, we can write

$$u(x) = A_n e^{ik(x-nd)} + B_n e^{-ik(x-nd)} \quad \text{for } (n-1)d + a < x < nd - a. \tag{5}$$

To the right of the cell at $x = nd$,

$$\begin{aligned} u(x) &= A_{n+1} e^{ik(x-[n+1]d)} + B_{n+1} e^{-ik(x-[n+1]d)} \\ &= A_{n+1} e^{-ikd} e^{ik(x-nd)} + B_{n+1} e^{ikd} e^{-ik(x-nd)} \quad \text{for } nd + a < x < (n+1)d - a. \end{aligned} \tag{6}$$

Using Eq. (2),

$$\begin{pmatrix} A_{n+1} e^{-ikd} \\ B_{n+1} e^{ikd} \end{pmatrix} = \mathbb{T} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

whence

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = P \begin{pmatrix} A_n \\ B_n \end{pmatrix} \quad \text{with } P = \begin{pmatrix} w^* e^{ikd} & z e^{ikd} \\ z^* e^{-ikd} & w e^{-ikd} \end{pmatrix}. \tag{7}$$

Therefore, for multiple cells, we shall need an expression for powers of P. Indeed, there is a closed-form expression for P^n. To state it, let

$$W = w e^{-ikd} \quad \text{and} \quad Z = z e^{ikd}, \text{ with } |W|^2 - |Z|^2 = 1. \tag{8}$$

Then we have

$$P = \begin{pmatrix} W^* & Z \\ Z^* & W \end{pmatrix} \quad \text{and} \quad P^n = \begin{pmatrix} X_n^* & Z U_{n-1} \\ Z^* U_{n-1} & X_n \end{pmatrix} \tag{9}$$

for $n \geq 1$, where

$$X_n(\xi) = W U_{n-1}(\xi) - U_{n-2}(\xi), \tag{10}$$

$$2\xi = W + W^* = \text{trace } P = \text{Re}\left\{ w e^{-ikd} \right\} \tag{11}$$

and U_n is a Chebyshev polynomial of the second kind, defined by

$$U_{m-1}(\cos\theta) = \frac{\sin m\theta}{\sin\theta}, \quad m = 0, 1, 2, \ldots. \tag{12}$$

From Eq. (12), $U_0 = 1$, $U_{-1} = 0$ and $U_{-2} = -1$. These give $X_0 = 1$ so that Eq. (9) gives $P^0 = I$, as expected.

There are many proofs of the formula for P^n, Eq. (9), and it has been rediscovered on many occasions. It was stated by Abelès in 1948 [9, Eq. (6)]; see also [10, Eq. (A8)]. For a review and a neat proof, see [7]. For textbook treatments, see [11, Section 1.6.5] and [8, Section 1.4.4].

We note a few useful properties. As $\det P = 1$, we have

$$\det P^n = |X_n|^2 - |Z|^2 U_{n-1}^2 = 1. \tag{13}$$

Also, as $P^m P^n = P^{m+n}$, we obtain

$$X_m X_n + |Z|^2 U_{m-1} U_{n-1} = X_{m+n}, \tag{14}$$

$$X_m U_{n-1} + X_n^* U_{m-1} = U_{m+n-1}. \tag{15}$$

Finally, using $w = t^{-1}$, Eqs. (8) and (11), we obtain

$$2\xi |t|^2 = t e^{ikd} + t^* e^{-ikd}. \tag{16}$$

The eigenvalues of P satisfy $\lambda^2 - 2\xi\lambda + 1 = 0$. When $|\xi| \leq 1$, the eigenvalues can be written as $e^{\pm iqd}$ where $\xi = \cos qd$ and q is real. In this case, we are in a passband for the periodic structure. When $\xi > 1$ (the case $\xi < -1$ is similar), we can write $\xi = \cosh\eta$ and then the eigenvalues are $e^{\pm\eta}$: this exponential behaviour implies that we are in a stopband for the periodic structure.

4. A finite periodic row

Suppose there are N identical cells, located at $x = nd$, $n = 0, 1, 2, \dots, N-1$. We call this a "slab". To the left of the slab, we have

$$u(x) = A_0 e^{ikx} + B_0 e^{-ikx} \quad \text{for } x < -a.$$

To the right of the slab, we have

$$u(x) = A_N e^{ik(x-Nd)} + B_N e^{-ik(x-Nd)} \quad \text{for } x > (N-1)d + a.$$

Thus (for $N \ge 1$)

$$\begin{pmatrix} A_N \\ B_N \end{pmatrix} = P^N \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{17}$$

Explicitly, using Eq. (9), we have

$$A_N = X_N^* A_0 + Z U_{N-1} B_0, \qquad B_N = Z^* U_{N-1} A_0 + X_N B_0. \tag{18}$$

For a wave incident from the left, we write

$$A_0 = 1, \qquad B_0 = R_N^{\text{per}}, \qquad A_N e^{-iNkd} = T_N^{\text{per}}, \qquad B_N = 0, \tag{19}$$

where R_N^{per} is the reflection coefficient and T_N^{per} is the transmission coefficient. Then Eq. (18) gives

$$R_N^{\text{per}} = -\frac{Z^* U_{N-1}(\xi)}{X_N(\xi)}, \qquad T_N^{\text{per}} = \frac{e^{-iNkd}}{X_N(\xi)}. \tag{20}$$

These satisfy $|R_N^{\text{per}}|^2 + |T_N^{\text{per}}|^2 = 1$ (use Eq. (13)). As $z^* = -r_+/t$ and $w = 1/t$, we obtain (using Eqs. (8) and (10))

$$R_N^{\text{per}} = \frac{r_+ U_{N-1}(\xi)}{U_{N-1}(\xi) - t e^{ikd} U_{N-2}(\xi)}, \qquad T_N^{\text{per}} = \frac{t e^{-i(N-1)kd}}{U_{N-1}(\xi) - t e^{ikd} U_{N-2}(\xi)}. \tag{21}$$

These expressions for R_N^{per} and T_N^{per} agree with those found by Mauguin in 1936 [12, p. 234]; see also [13, p. 109] and [14, p. 314, problem 3]. In particular, from Eq. (13),

$$\left|T_N^{\text{per}}\right|^{-2} = |X_N|^2 = 1 + |Z|^2 U_{N-1}^2(\xi), \quad |Z| = |r|/|t|. \tag{22}$$

This is [8, Eq. (1.105)].

It is interesting to note that similar problems arise in surface science. The difference is that $u'' + k^2 u = 0$ is replaced by $u'' - k_0^2 u = 0$ outside the slab, where k_0 is real, and there is no incident field. See, for example, [15, Section 3.3].

5. A semi-infinite periodic row

What happens if we let $N \to \infty$ so as to obtain a semi-infinite row? The answer depends on the magnitude of ξ (defined by Eq. (11)).

Suppose first that $\xi > 1$. (The case $\xi < -1$ is similar.) Put $\xi = \cosh \eta$ with $\eta > 0$ giving

$$U_{N-1}(\xi) = \frac{\sinh N\eta}{\sinh \eta} \sim \frac{e^{N\eta}}{2 \sinh \eta} \quad \text{as } N \to \infty.$$

It follows that $T_N^{\text{per}} \to 0$ and

$$R_N^{\text{per}} \sim \frac{r_+}{1 - t e^{ikd} e^{-\eta}} \equiv R_\infty^{\text{per}} \quad \text{as } N \to \infty.$$

It can be verified (using Eq. (16)) that $|R_\infty^{\text{per}}| = 1$: no energy passes through the row. This is as expected: we are in a stopband ($|\xi| > 1$). In detail, from Eq. (22),

$$\left|T_N^{\text{per}}\right| \sim 2|t/r| \sinh \eta \, e^{-N\eta} \quad \text{as } N \to \infty.$$

Alternatively, in a passband ($|\xi| < 1$), put $\xi = \cos\theta$, whence

$$R_N^{\text{per}} = \frac{r_+ \sin N\theta}{\sin N\theta - t e^{ikd} \sin(N-1)\theta}, \qquad T_N^{\text{per}} = \frac{t e^{-i(N-1)kd} \sin\theta}{\sin N\theta - t e^{ikd} \sin(N-1)\theta}. \tag{23}$$

These formulas do not have limits as $N \to \infty$. This fact is known [13,16]; see also [8, Section 4.6]. In detail, from Eq. (22), we obtain (see [8, Eq. (1.106)])

$$\frac{1}{\left|T_N^{\mathrm{per}}\right|^2} = 1 + \left(\frac{|r|\sin N\theta}{|t|\sin\theta}\right)^2 .$$

For some additional papers where the limit $N \to \infty$ is considered, see [17–20].

We note that the problem of reflection by a semi-infinite periodic row can be solved directly [13] and [14, Section 62].

6. A finite periodic row with internal forcing

Consider a finite periodic row of N identical scatterers (as in Section 4) except that one scatterer is forced and there is no incident wave; we can assume that the forced scatterer is in the cell at $x = nd$. There are n scatterers to the left and $N-n-1$ to the right.

This problem can be solved using transfer matrices. We give its solution for two reasons. First, the solution itself is of interest: it gives the Green function for the finite row. Second, the methods used will be adapted to problems in which there are random perturbations to the periodic row.

To the left of the cell at $x = nd$, we can write u as Eq. (5), where

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = P^n \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} . \tag{24}$$

To the right of the cell at $x = nd$, we can write u as Eq. (6), where

$$\begin{pmatrix} A_N \\ B_N \end{pmatrix} = P^{N-n-1} \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} . \tag{25}$$

To connect these two expansions, we use

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = P \begin{pmatrix} A_n \\ B_n \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} , \tag{26}$$

where the second term on the right is the prescribed forcing. Hence

$$\begin{pmatrix} A_N \\ B_N \end{pmatrix} = P^N \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} + P^{N-n-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

We are interested in finding the field in the cell containing the forced scatterer. As there is no incident wave, $A_0 = B_N = 0$, and then Eqs. (9), (24) and (25) give

$$X_n A_n - Z U_{n-1} B_n = 0, \qquad Z^* U_{N-n-2} A_{n+1} + X_{N-n-1} B_{n+1} = 0. \tag{27}$$

These are combined with Eq. (26) and solved for A_n, B_n, A_{n+1} and B_{n+1}. Thus, using Eq. $(27)_2$,

$$\begin{aligned} 0 &= Z^* U_{N-n-2}\{W^* A_n + Z B_n + f_1\} + X_{N-n-1}\{Z^* A_n + W B_n + f_2\} \\ &= Z^*\{W^* U_{N-n-2} + X_{N-n-1}\} A_n + \{W X_{N-n-1} + |Z|^2 U_{N-n-2}\} B_n + f_3, \end{aligned}$$

with $f_3 = Z^* U_{N-n-2} f_1 + X_{N-n-1} f_2$. Combining this equation with Eq. $(27)_1$ gives a 2×2 system for A_n and B_n. The determinant of the system simplifies (for a very similar calculation, see Section 7.1). Hence,

$$A_n = -Z U_{n-1} f_3 / X_N, \qquad B_n = -X_n f_3 / X_N . \tag{28}$$

For a simple example, consider point scatterers. The scatterer at $x = nd$ is forced. The conditions Eq. (3) are amended there to

$$u(nd^+) = u(nd^-) \quad \text{and} \quad u'(nd^+) - u'(nd^-) = M u(nd) + 1,$$

implying that $f_1 = \mathrm{e}^{\mathrm{i}kd}/(2\mathrm{i}k)$ and $f_2 = f_1^*$. Also, Eqs. (4) and (11), and $w = 1/t$ give $2\xi = 2\cos kd + (M/k)\sin kd$. Of particular interest is $u(nd)$, the response at the forcing location. From Eqs. (5) and (28), this quantity is

$$A_n + B_n = -X_N^{-1}(Z U_{n-1} + X_n)(Z^* U_{N-n-2} f_1 + X_{N-n-1} f_2). \tag{29}$$

As $X_n = W U_{n-1} - U_{n-2}$, $W = w\mathrm{e}^{-\mathrm{i}kd}$ and $w = 1 + \frac{1}{2}\mathrm{i}(M/k)$,

$$\begin{aligned} 2X_n \mathrm{e}^{\mathrm{i}kd}\sin kd &= U_{n-1}\,(2 + \mathrm{i}(M/k))\sin kd - 2U_{n-2}\mathrm{e}^{\mathrm{i}kd}\sin kd \\ &= 2U_{n-1}\sin kd + 2\mathrm{i}U_{n-1}(\xi - \cos kd) + \mathrm{i}U_{n-2}(\mathrm{e}^{2\mathrm{i}kd} - 1) \\ &= -2\mathrm{i}\mathrm{e}^{\mathrm{i}kd}U_{n-1} + \mathrm{i}U_n + \mathrm{i}U_{n-2}\mathrm{e}^{2\mathrm{i}kd} = -\mathrm{i}(\mathrm{e}^{\mathrm{i}kd}V_{n-1} - V_n), \end{aligned}$$

where $V_m = U_m(\xi) - \mathrm{e}^{\mathrm{i}kd}U_{m-1}(\xi)$. Next, we find that

$$ZU_{n-1} + X_n = (Z + W)U_{n-1} - U_{n-2} = 2\xi U_{n-1} - U_{n-2} - \mathrm{e}^{\mathrm{i}kd}U_{n-1} = V_n,$$

using $Z = z\mathrm{e}^{\mathrm{i}kd}$, $z = r/t = -\frac{1}{2}\mathrm{i}(M/k)$ and $Z + W = \mathrm{e}^{-\mathrm{i}kd} + (M/k)\sin kd = 2\xi - \mathrm{e}^{\mathrm{i}kd}$. Similarly

$$\begin{aligned}2\mathrm{i}k(Z^*U_{m-1}f_1 + X_mf_2) &= Z^*\mathrm{e}^{\mathrm{i}kd}U_{m-1} - \mathrm{e}^{-\mathrm{i}kd}X_m\\ &= (Z^*\mathrm{e}^{\mathrm{i}kd} - W\mathrm{e}^{-\mathrm{i}kd})U_{m-1} + \mathrm{e}^{-\mathrm{i}kd}U_{m-2} = -\mathrm{e}^{-\mathrm{i}kd}V_m,\end{aligned}$$

using $Z^*\mathrm{e}^{\mathrm{i}kd} - W\mathrm{e}^{-\mathrm{i}kd} = -\mathrm{e}^{-2\mathrm{i}kd} - \mathrm{e}^{-\mathrm{i}kd}(M/k)\sin kd = 1 - 2\xi\mathrm{e}^{-\mathrm{i}kd}$. Hence, Eq. (29) gives

$$u(nd) = A_n + B_n = \frac{\sin kd}{k}\,\frac{V_nV_{N-n-1}}{\mathrm{e}^{\mathrm{i}kd}V_{N-1} - V_N}. \tag{30}$$

We note that $u(nd) = u([N - n - 1]d)$, as expected by symmetry.

7. A finite periodic row apart from one scatterer

Consider a finite periodic row of N identical cells (as in Section 4) except that one scatterer (the "impurity" or "defect") is changed; we can assume that it is in the cell at $x = nd$. There are n scatterers to the left and $N - n - 1$ to the right. As in Section 4, there is a wave incident from the left and the problem is to calculate the reflection and transmission coefficients, R_N^n and T_N^n. (A related problem, discussed recently [21], concerns the calculation of scattering resonances in the presence of one defect.) We begin with an exact treatment: the impurity is characterized by its transfer matrix, P_n. Then, we suppose that the impurity is the same as all the other scatterers except that it is displaced by an amount ε from its periodic location. This is a form of "positional disorder".

We proceed as in Section 6. To the left of the cell at $x = nd$, we have Eqs. (5) and (24). To the right of the cell at $x = nd$, we have Eqs. (6) and (25). Let P_n be the transfer matrix for the impurity in the cell at $x = nd$, so that (cf. Eq. (7))

$$\begin{pmatrix} A_{n+1}\\ B_{n+1}\end{pmatrix} = P_n\begin{pmatrix} A_n\\ B_n\end{pmatrix} \quad \text{with } P_n = \begin{pmatrix} p_n^* & q_n\\ q_n^* & p_n\end{pmatrix}, \qquad \det P_n = 1. \tag{31}$$

Combining Eqs. (24), (25) and (31) gives

$$\begin{pmatrix} A_N\\ B_N\end{pmatrix} = Q_n\begin{pmatrix} A_0\\ B_0\end{pmatrix} \quad \text{with } Q_n = P^{N-n-1}P_nP^n = \begin{pmatrix} Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}. \tag{32}$$

Now, consider a wave incident from the left and write

$$A_0 = 1, \qquad B_0 = R_N^n, \qquad A_N\mathrm{e}^{-\mathrm{i}Nkd} = T_N^n, \qquad B_N = 0. \tag{33}$$

Solving Eq. (32), noting that $\det Q_n = 1$, we obtain

$$R_N^n = -Q_{21}/Q_{22}, \qquad T_N^n = \mathrm{e}^{-\mathrm{i}Nkd}/Q_{22}. \tag{34}$$

The entries in the matrix Q_n can be calculated, using Eqs. (9) and (32):

$$Q_n = \begin{pmatrix} X_{N-n-1}^* & ZU_{N-n-2}\\ Z^*U_{N-n-2} & X_{N-n-1}\end{pmatrix}\begin{pmatrix} p_n^* & q_n\\ q_n^* & p_n\end{pmatrix}\begin{pmatrix} X_n^* & ZU_{n-1}\\ Z^*U_{n-1} & X_n\end{pmatrix}.$$

For example, we obtain

$$\begin{aligned}Q_{21} &= Z^*U_{N-n-2}[p_n^*X_n^* + q_nZ^*U_{n-1}] + X_{N-n-1}[q_n^*X_n^* + p_nZ^*U_{n-1}]\\ &= p_nZ^*X_{N-n-1}U_{n-1} + p_n^*Z^*X_n^*U_{N-n-2} + q_nZ^{*2}U_{N-n-2}U_{n-1} + q_n^*X_{N-n-1}X_n^*,\end{aligned} \tag{35}$$

$$\begin{aligned}Q_{22} &= Z^*U_{N-n-2}[q_nX_n + p_n^*ZU_{n-1}] + X_{N-n-1}[p_nX_n + q_n^*ZU_{n-1}]\\ &= p_nX_{N-1} + (p_n^* - p_n)|Z|^2U_{N-n-2}U_{n-1} + q_nZ^*X_nU_{N-n-2} + q_n^*ZX_{N-n-1}U_{n-1},\end{aligned} \tag{36}$$

using Eq. (14), $X_{N-n-1}X_n = X_{N-1} - |Z|^2U_{N-n-2}U_{n-1}$.

7.1. A check on the calculation

Let us check the calculations in the periodic case. Then, $P_n = P$, $p_n = W$ and $q_n = Z$. As $WX_{N-1} = X_1X_{N-1} = X_N - |Z|^2U_{N-2}$, we obtain $Q_{22} = X_N + |Z|^2\Omega_0$, where

$$\begin{aligned}\Omega_0 &= -U_{N-2} + (W^* - W)U_{N-n-2}U_{n-1} + X_nU_{N-n-2} + X_{N-n-1}U_{n-1}\\ &= -U_{N-2} + (W^* + W)U_{N-n-2}U_{n-1} - U_{N-n-2}U_{n-2} - U_{N-n-3}U_{n-1}\\ &= -U_{N-2} + (2\xi U_{n-1} - U_{n-2})U_{N-n-2} - U_{N-n-3}U_{n-1}\\ &= -U_{N-2} + U_nU_{N-n-2} - U_{n-1}U_{N-n-3},\end{aligned}$$

using the recurrence relation for $U_n(\xi)$. From Eq. (12), we have

$$2U_{p-1}U_{m-1}\sin^2\theta = \cos(p-m)\theta - \cos(p+m)\theta, \tag{37}$$

so that

$$\begin{aligned}&2U_nU_{N-n-2}\sin^2\theta = \cos(2n+2-N)\theta - \cos N\theta,\\ &2U_{n-1}U_{N-n-3}\sin^2\theta = \cos(2n+2-N)\theta - \cos(N-2)\theta.\end{aligned}$$

Also $2U_{N-2}\sin^2\theta = \cos(N-2)\theta - \cos N\theta$ whence $\Omega_0 = 0$, as expected. Similarly, one can check that $Q_{21} = Z^*U_{N-1}$ when $P_n = P$.

7.2. An application: one displaced scatterer

Suppose that the scatterer at $x = nd$ is displaced to $x = nd + \varepsilon$ (with $|\varepsilon| < d$). Then $p_n = W$ and $q_n = Z\mathrm{e}^{-2\mathrm{i}k\varepsilon}$ (exactly). Hence, from Eq. (36), we have

$$Q_{22}(\varepsilon) = \mathcal{A}\mathrm{e}^{2\mathrm{i}k\varepsilon} + \mathcal{B} + \mathcal{C}\mathrm{e}^{-2\mathrm{i}k\varepsilon} = X_N - \mathcal{A}E - \mathcal{C}E^*, \tag{38}$$

where $E(\varepsilon) = 1 - \mathrm{e}^{2\mathrm{i}k\varepsilon}$, $\mathcal{A} = |Z|^2X_{N-n-1}U_{n-1}$, $\mathcal{C} = |Z|^2X_nU_{N-n-2}$, $\mathcal{B} = X_N - \mathcal{A} - \mathcal{C}$ and we have noted that $Q_{22}(0) = \mathcal{A} + \mathcal{B} + \mathcal{C} = X_N$. With $Y = \mathrm{e}^{2\mathrm{i}k\varepsilon}$, we have

$$YQ_{22}(\varepsilon) = \mathcal{A}Y^2 + \mathcal{B}Y + \mathcal{C} = \mathcal{A}(Y - Y_1)(Y - Y_2), \tag{39}$$

say. Then, by partial fractions,

$$\frac{1}{Q_{22}(\varepsilon)} = \frac{1}{\mathcal{A}(Y_1 - Y_2)}\left(\frac{Y_1}{Y - Y_1} - \frac{Y_2}{Y - Y_2}\right) = \frac{1}{2\mathrm{i}k\mathcal{A}(Y_1 - Y_2)}\left(\frac{2\mathrm{i}kY_1\mathrm{e}^{-2\mathrm{i}k\varepsilon}}{1 - Y_1\mathrm{e}^{-2\mathrm{i}k\varepsilon}} - \frac{2\mathrm{i}kY_2\mathrm{e}^{-2\mathrm{i}k\varepsilon}}{1 - Y_2\mathrm{e}^{-2\mathrm{i}k\varepsilon}}\right).$$

This gives the transmission coefficient, T_N^n, using Eq. (34). A similar but more complicated calculation could be given for the reflection coefficient.

It is of interest to calculate the average transmission coefficient, $\langle T_N^n\rangle$, using

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2}\frac{\mathrm{d}\varepsilon'}{Q_{22}(\varepsilon')} &= \frac{1}{2\mathrm{i}k\varepsilon\mathcal{A}(Y_1 - Y_2)}\left[\log\frac{1 - Y_1\mathrm{e}^{-2\mathrm{i}k\varepsilon'}}{1 - Y_2\mathrm{e}^{-2\mathrm{i}k\varepsilon'}}\right]_{-\varepsilon/2}^{\varepsilon/2}\\ &= \frac{1}{2\mathrm{i}k\varepsilon\mathcal{A}(Y_1 - Y_2)}\log\left(\frac{(1 - Y_1\mathrm{e}^{-\mathrm{i}k\varepsilon})(1 - Y_2\mathrm{e}^{\mathrm{i}k\varepsilon})}{(1 - Y_1\mathrm{e}^{\mathrm{i}k\varepsilon})(1 - Y_2\mathrm{e}^{-\mathrm{i}k\varepsilon})}\right).\end{aligned} \tag{40}$$

When this formula is multiplied by $\mathrm{e}^{-\mathrm{i}Nkd}$, it gives $\langle T_N^n\rangle$ exactly. Again, the average reflection coefficient, $\langle R_N^n\rangle$, could be determined, but we defer this calculation until Section 7.4.

7.3. Approximation of an exact solution for small $k\varepsilon$

Let us approximate the exact formula, Eq. (40), for $0 < k\varepsilon \ll 1$. For a non-trivial result, we must approximate the logarithmic term with an error that is $O((k\varepsilon)^4)$ as $k\varepsilon \to 0$. Write

$$\log\left(\frac{1 - Y\mathrm{e}^{-\mathrm{i}k\varepsilon}}{1 - Y\mathrm{e}^{\mathrm{i}k\varepsilon}}\right) = \log\left(\frac{1 - \Upsilon(\mathrm{e}^{-\phi} - 1)}{1 - \Upsilon(\mathrm{e}^{\phi} - 1)}\right) \quad \text{with } \Upsilon = \frac{Y}{1 - Y}, \phi = \mathrm{i}k\varepsilon.$$

Then, as $\log(1-x) \sim -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$ as $x \to 0$, and using

$$\mathrm{e}^{\pm\phi} - 1 \sim \pm\phi + \frac{1}{2}\phi^2 \pm \frac{1}{6}\phi^3, \qquad (\mathrm{e}^{\pm\phi} - 1)^2 \sim \phi^2 \pm \phi^3, \qquad (\mathrm{e}^{\pm\phi} - 1)^3 \sim \pm\phi^3,$$

we obtain

$$\begin{aligned}\log\left(1 - \Upsilon(\mathrm{e}^{\pm\phi} - 1)\right) &\sim -\Upsilon(\mathrm{e}^{\pm\phi} - 1) - \frac{1}{2}\Upsilon^2(\mathrm{e}^{\pm\phi} - 1)^2 - \frac{1}{3}\Upsilon^3(\mathrm{e}^{\pm\phi} - 1)^3 \\ &\sim \mp\Upsilon\phi - \frac{1}{2}\Upsilon(1+\Upsilon)\phi^2 \mp \Upsilon\left(\frac{1}{6} + \frac{1}{2}\Upsilon + \frac{1}{3}\Upsilon^2\right)\phi^3.\end{aligned}$$

Hence

$$\log\left(\frac{1 - Y\mathrm{e}^{-\mathrm{i}k\varepsilon}}{1 - Y\mathrm{e}^{\mathrm{i}k\varepsilon}}\right) \sim 2\Upsilon\phi + 2\Upsilon\left(\frac{1}{6} + \frac{1}{2}\Upsilon + \frac{1}{3}\Upsilon^2\right)\phi^3 = \frac{2\mathrm{i}k\varepsilon Y}{1 - Y}\left(1 - \frac{(1+Y)}{6(1-Y)^2}(k\varepsilon)^2\right).$$

Then, using Eq. (40),

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \frac{\mathrm{d}\varepsilon'}{Q_{22}(\varepsilon')} &\simeq \frac{1}{\mathcal{A}(Y_1 - Y_2)}\left(\frac{Y_1}{1 - Y_1} - \frac{Y_2}{1 - Y_2}\right) - \frac{(k\varepsilon)^2}{6\mathcal{A}(Y_1 - Y_2)}\left(\frac{Y_1(1+Y_1)}{(1-Y_1)^3} - \frac{Y_2(1+Y_2)}{(1-Y_2)^3}\right) \\ &= \frac{1}{\mathcal{A}(1-Y_1)(1-Y_2)} - \frac{(k\varepsilon)^2\Lambda}{6[\mathcal{A}(1-Y_1)(1-Y_2)]^3} = \frac{1}{X_N} - (k\varepsilon)^2\frac{\Lambda}{6X_N^3},\end{aligned} \tag{41}$$

where we have used Eq. (39) with $\varepsilon = 0$ ($Y = 1$) and

$$\Lambda = \mathcal{A}^2\left\{1 + Y_1 + Y_2 - 6Y_1Y_2 + Y_1Y_2(Y_1 + Y_2) + Y_1^2Y_2^2\right\}.$$

This expression simplifies. From Eq. (39), we have $\mathcal{A}Y_1Y_2 = \mathcal{C}$ and $\mathcal{A}(Y_1 + Y_2) = -\mathcal{B}$ so that

$$\Lambda = \mathcal{A}^2 - \mathcal{A}\mathcal{B} - 6\mathcal{A}\mathcal{C} - \mathcal{B}\mathcal{C} + \mathcal{C}^2 = -X_N(\mathcal{A} + \mathcal{C}) + 2(\mathcal{A} - \mathcal{C})^2. \tag{42}$$

Note that Λ, $\mathcal{A}$ and $\mathcal{C}$ depend on N and n. Thus, we obtain

$$\langle T_N^n\rangle \simeq T_N^{\mathrm{per}}\left(1 - (k\varepsilon)^2\frac{\Lambda}{6X_N^2}\right) \tag{43}$$

for small ε, where T_N^{per} is the transmission coefficient for a finite periodic row (see Eq. (20)).

7.4. *Approximation assuming $k\varepsilon$ is small from the outset*

As an alternative to approximating the exact solution (as done in Section 7.3), we could assume that $k\varepsilon$ is small at an earlier stage in the calculation. This has the advantage that more complicated problems may be handled later.

We have $E = 1 - \mathrm{e}^{2\mathrm{i}k\varepsilon} \simeq -2\mathrm{i}k\varepsilon + 2(k\varepsilon)^2$ and $E^* \simeq 2\mathrm{i}k\varepsilon + 2(k\varepsilon)^2$. Then Eq. (38) gives

$$Q_{22}(\varepsilon) \simeq X_N + 2\mathrm{i}k\varepsilon(\mathcal{A} - \mathcal{C}) - 2(k\varepsilon)^2(\mathcal{A} + \mathcal{C}),$$

whence

$$\begin{aligned}\frac{1}{Q_{22}(\varepsilon)} &\simeq \frac{1}{X_N}\left\{1 + \frac{2\mathrm{i}k\varepsilon}{X_N}(\mathcal{A} - \mathcal{C}) - \frac{2(k\varepsilon)^2}{X_N}(\mathcal{A} + \mathcal{C})\right\}^{-1} \\ &\simeq \frac{1}{X_N}\left\{1 - \frac{2\mathrm{i}k\varepsilon}{X_N}(\mathcal{A} - \mathcal{C}) + \left[\frac{2\mathrm{i}k\varepsilon}{X_N}(\mathcal{A} - \mathcal{C})\right]^2 + \frac{2(k\varepsilon)^2}{X_N}(\mathcal{A} + \mathcal{C})\right\} \\ &\simeq \frac{1}{X_N} - \frac{2\mathrm{i}k\varepsilon}{X_N^2}(\mathcal{A} - \mathcal{C}) + \frac{2(k\varepsilon)^2}{X_N^3}\left\{X_N(\mathcal{A} + \mathcal{C}) - 2(\mathcal{A} - \mathcal{C})^2\right\}.\end{aligned} \tag{44}$$

Hence, integrating with respect to ε, we recover Eq. (41) with Eq. (42).

To calculate the average reflection coefficient, we need $Q_{21}(\varepsilon)$, defined by Eq. (35). We have

$$\begin{aligned}Q_{21}(\varepsilon) &= Z^*\{U_{N-1} - \mathcal{D}E - \mathcal{F}E^*\} \\ &\simeq Z^*\{U_{N-1} + 2\mathrm{i}k\varepsilon(\mathcal{D} - \mathcal{F}) - 2(k\varepsilon)^2(\mathcal{D} + \mathcal{F})\},\end{aligned} \tag{45}$$

where $\mathcal{D} = X_{N-n-1}X_n^*$, $\mathcal{F} = |Z|^2 U_{N-n-2}U_{n-1}$ and we have used $Q_{21}(0) = Z^* U_{N-1}$. Hence, combining Eqs. (34), (44) and (45), we obtain

$$\frac{X_N Q_{21}}{Z^* Q_{22}} \simeq U_{N-1} + \frac{2\mathrm{i}k\varepsilon}{X_N}\{X_N(\mathcal{D}-\mathcal{F}) - U_{N-1}(\mathcal{A}-\mathcal{C})\} - \frac{2(k\varepsilon)^2}{X_N^2}\Omega,$$

where $\Omega = U_{N-1}\Lambda - 2X_N(\mathcal{A}-\mathcal{C})(\mathcal{D}-\mathcal{F}) + X_N^2(\mathcal{D}+\mathcal{F})$ and Λ is given by Eq. (42). Finally, Eqs. (20) and (34) give

$$\langle R_N^n \rangle \simeq R_N^{\mathrm{per}} + (k\varepsilon)^2 \frac{Z^*\Omega}{6X_N^3}. \tag{46}$$

8. A perturbed finite row

Suppose that every scatterer in a finite row is perturbed independently, so that the scatterer at $x = nd$ is displaced to $x = nd + \varepsilon_n$, $n = 0, 1, 2, \ldots, N-1$. Then $p_n = W$ and $q_n = Z\mathrm{e}^{-2\mathrm{i}k\varepsilon_n}$ (exactly). Hence, from Eq. (31),

$$P_n(\varepsilon_n) = \begin{pmatrix} W^* & Z\mathrm{e}^{-2\mathrm{i}k\varepsilon_n} \\ Z^*\mathrm{e}^{2\mathrm{i}k\varepsilon_n} & W \end{pmatrix},$$

with $P_n(0) = P$. For an irregular row of N scatterers, Eq. (17) is replaced by

$$\begin{pmatrix} A_N \\ B_N \end{pmatrix} = \mathcal{P}_N \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \tag{47}$$

where

$$\mathcal{P}_N(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{N-1}) = P_{N-1}P_{N-2}\cdots P_1 P_0 = \begin{pmatrix} \mathcal{P}_{11}^N & \mathcal{P}_{12}^N \\ \mathcal{P}_{21}^N & \mathcal{P}_{22}^N \end{pmatrix}, \tag{48}$$

say. Then, for the usual scattering problem, with a wave incident from the left,

$$A_0 = 1, \qquad B_0 = R_N, \qquad A_N\mathrm{e}^{-\mathrm{i}Nkd} = T_N, \qquad B_N = 0$$

and the problem is to calculate R_N and T_N. As $\det \mathcal{P}_N = 1$, solving Eq. (47) gives

$$R_N = -\mathcal{P}_{21}^N/\mathcal{P}_{22}^N, \qquad T_N = \mathrm{e}^{-\mathrm{i}Nkd}/\mathcal{P}_{22}^N. \tag{49}$$

8.1. Small perturbations

Many authors have started from Eqs. (47) and (48), with $\mathcal{P}_N$ written as the product of N random 2×2 matrices; see, for example, [22, Chapter 8] and [23–26]. We shall proceed differently. We begin by approximating $\mathcal{P}_N$ for small perturbations. Thus, for small $k\varepsilon_n$, we have

$$P_n(\varepsilon_n) \simeq P + \delta_n S_1 + \delta_n^2 S_2,$$

where $\delta_n = k\varepsilon_n$,

$$S_1 = 2\mathrm{i}\begin{pmatrix} 0 & -Z \\ Z^* & 0 \end{pmatrix}, \qquad S_2 = -2\begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix}. \tag{50}$$

Note that, with this approximation, $\det P_n = 1 + O(\delta_n^4)$ as $\delta_n \to 0$.

Then, correct to second order, we find that

$$\mathcal{P}_N = P^N + \sum_{j=0}^{N-1}\left\{\delta_j L_j(S_1) + \delta_j^2 L_j(S_2)\right\} + \sum_{j=0}^{N-2}\sum_{k=j+1}^{N-1}\delta_j\delta_k M_{jk},$$

where $L_j(S) = P^{N-j-1}SP^j$, $M_{jk} = P^{N-1-k}S_1P^{k-j-1}S_1P^j$ and the second sum is absent when $N = 1$. Note that L_j and M_{jk} depend on N.

We shall estimate R_N and T_N, given by Eq. (49). We start with $\mathcal{P}_{22}^N$. As $[P^N]_{22} = X_N$, we have

$$\begin{aligned}\frac{X_N}{\mathcal{P}_{22}^N} &= \left\{1 + \frac{1}{X_N}\sum_{j=0}^{N-1}\delta_j\left[L_j(S_1)\right]_{22} + \frac{1}{X_N}\sum_{j=0}^{N-1}\delta_j^2\left[L_j(S_2)\right]_{22} + \frac{1}{X_N}\sum_{j=0}^{N-2}\sum_{k=j+1}^{N-1}\delta_j\delta_k\left[M_{jk}\right]_{22}\right\}^{-1} \\ &\simeq 1 - \frac{1}{X_N}\sum_{j=0}^{N-1}\delta_j\left[L_j(S_1)\right]_{22} - \frac{1}{X_N}\sum_{j=0}^{N-1}\delta_j^2\left[L_j(S_2)\right]_{22} - \frac{1}{X_N}\sum_{j=0}^{N-2}\sum_{k=j+1}^{N-1}\delta_j\delta_k\left[M_{jk}\right]_{22} \\ &\quad + \frac{1}{X_N^2}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\delta_j\delta_k\left[L_j(S_1)\right]_{22}\left[L_k(S_1)\right]_{22}.\end{aligned} \tag{51}$$

8.2. Average reflection and transmission

Next, we calculate an average, defining $\langle f\rangle = \varepsilon^{-N}\int_{-\varepsilon/2}^{\varepsilon/2}\cdots\int_{-\varepsilon/2}^{\varepsilon/2} f\,\mathrm{d}\varepsilon_0\cdots\mathrm{d}\varepsilon_{N-1}$. As $\langle\varepsilon_n\rangle = 0$ and $\langle\varepsilon_n^2\rangle = \frac{1}{12}\varepsilon^2$, we obtain, using Eqs. (20) and (49),

$$\langle T_N\rangle = T_N^{\mathrm{per}} - (k\varepsilon)^2\frac{\mathrm{e}^{-\mathrm{i}Nkd}}{12X_N^3}\,\mathcal{S}_N^T = T_N^{\mathrm{per}}\left\{1-\frac{(k\varepsilon)^2\mathcal{S}_N^T}{12X_N^2}\right\}, \tag{52}$$

where

$$\mathcal{S}_N^T = X_N\sum_{j=0}^{N-1}\left[L_j(S_2)\right]_{22} - \sum_{j=0}^{N-1}\left[L_j(S_1)\right]_{22}^2. \tag{53}$$

We note that $\mathcal{S}_1^T = 0$, as expected, because displacing a single scatterer ($N = 1$) does not change the transmission coefficient.

For $\langle R_N\rangle$, we multiply Eq. (51) by $\mathcal{P}_{21}^N$ and gather terms to give

$$\begin{aligned}\frac{X_N}{\mathcal{P}_{22}^N}\mathcal{P}_{21}^N = [P^N]_{21} &+ \sum_{j=0}^{N-1}\delta_j^2[L_j(S_2)]_{21} - \frac{1}{X_N}\sum_{j=0}^{N-1}\delta_j^2[L_j(S_1)]_{21}[L_j(S_1)]_{22}\\ &- \frac{[P^N]_{21}}{X_N}\sum_{j=0}^{N-1}\delta_j^2\left[L_j(S_2)\right]_{22} + \frac{[P^N]_{21}}{X_N^2}\sum_{j=0}^{N-1}\delta_j^2\left[L_j(S_1)\right]_{22}^2,\end{aligned}$$

omitting linear terms and those containing $\delta_j\delta_k$ with $j \neq k$ (as all such terms have zero mean). For the average of this quantity, replace δ_j^2 by $\frac{1}{12}(k\varepsilon)^2$. Then, Eqs. (9), (20) and (49) give

$$\langle R_N\rangle = R_N^{\mathrm{per}} - \frac{(k\varepsilon)^2}{12X_N^3}\left\{X_N\mathcal{S}_N^R - Z^*U_{N-1}\mathcal{S}_N^T\right\} = R_N^{\mathrm{per}}\left\{1-\frac{(k\varepsilon)^2\mathcal{S}_N^T}{12X_N^2}\right\} - \frac{(k\varepsilon)^2\mathcal{S}_N^R}{12X_N^2}, \tag{54}$$

where

$$\mathcal{S}_N^R = X_N\sum_{j=0}^{N-1}[L_j(S_2)]_{21} - \sum_{j=0}^{N-1}[L_j(S_1)]_{21}[L_j(S_1)]_{22}. \tag{55}$$

8.3. Calculation of $\mathcal{S}_N^T$ and $\mathcal{S}_N^R$

To evaluate $\mathcal{S}_N^T$ and $\mathcal{S}_N^R$, we begin by noting that the matrices S_1 and S_2 have the structure (see Eq. (50))

$$S = \begin{pmatrix}0 & \sigma\\ \tau & 0\end{pmatrix}.$$

Then, from $L_n(S) = P^{N-n-1}SP^n$ and (9), we obtain

$$[L_n(S)]_{21} = \tau X_{N-n-1}X_n^* + \sigma(Z^*)^2U_{N-n-2}U_{n-1},$$
$$[L_n(S)]_{22} = \tau ZX_{N-n-1}U_{n-1} + \sigma Z^*X_nU_{N-n-2}.$$

For S_1, $\sigma = -2\mathrm{i}Z$ and $\tau = 2\mathrm{i}Z^*$, giving

$$\begin{aligned}[L_n(S_1)]_{22} &= 2\mathrm{i}|Z|^2[X_{N-n-1}U_{n-1} - X_nU_{N-n-2}]\\ &= 2\mathrm{i}|Z|^2[U_{N-n-2}U_{n-2} - U_{N-n-3}U_{n-1}] = 2\mathrm{i}|Z|^2U_{2n-N},\end{aligned}$$
$$[L_n(S_1)]_{21} = 2\mathrm{i}Z^*[X_{N-n-1}X_n^* - |Z|^2U_{N-n-2}U_{n-1}].$$

Inspection of Eq. (53) shows that we require

$$[L_n(S_1)]_{22}^2 = -4|Z|^4U_{2n-N}^2 = \frac{2|Z|^4}{\sin^2\theta}\{\cos(4n-2N+2)\theta - 1\},$$

where $\cos\theta = \xi = (W+W^*)/2 = \mathrm{Re}\,W$. For Eq. (55), we also require

$$[L_n(S_1)]_{21}[L_n(S_1)]_{22} = 4Z^*|Z|^2U_{2n-N}[|Z|^2U_{n-1}U_{N-n-2} - X_{N-n-1}X_n^*].$$

We simplify this expression using Eq. (15) for $U_{2n-N}X_n^*$, then Eq. (14) for $X_{2n-N+1}X_{N-n-1}$. The result is

$$[L_n(S_1)]_{21}[L_n(S_1)]_{22} = 4Z^*|Z|^2[X_nU_{n-1} - X_{N-n-1}U_{3n-N}].$$

Then we use Eqs. (10) and (37), giving

$$[L_n(S_1)]_{21}[L_n(S_1)]_{22} = \frac{2Z^*|Z|^2}{\sin^2\theta}\left\{W - \cos\theta - W\cos(4n-2N+2)\theta + \cos(4n-2N+3)\theta\right\}.$$

Note that $W - \cos\theta = (W - W^*)/2 = \mathrm{i}\,\mathrm{Im}\,W$.

To sum the series in Eqs. (53) and (55) involving $L_j(S_1)$, we use [27, Eq. 1.341.3]

$$\sum_{j=0}^{N-1}\cos(2jx+\alpha) = \frac{\sin Nx}{\sin x}\cos[(N-1)x+\alpha]. \tag{56}$$

This gives

$$\sum_{j=0}^{N-1}\left[L_j(S_1)\right]_{22}^2 = \frac{2|Z|^4}{\sin^2\theta}\left(\frac{\sin 2N\theta}{\sin 2\theta} - N\right)$$

and

$$\sum_{j=0}^{N-1}[L_j(S_1)]_{21}[L_j(S_1)]_{22} = \frac{Z^*|Z|^2}{\sin^2\theta}(W-W^*)\left(N - \frac{\sin 2N\theta}{\sin 2\theta}\right).$$

For S_2, $\sigma = -2Z$ and $\tau = -2Z^*$, giving

$$\begin{aligned}[L_n(S_2)]_{22} &= -2|Z|^2[X_{N-n-1}U_{n-1} + X_nU_{N-n-2}]\\ &= -2|Z|^2[2WU_{N-n-2}U_{n-1} - U_{N-n-3}U_{n-1} - U_{N-n-2}U_{n-2}]\\ &= -2|Z|^2\sin^{-2}\theta\,[(W-\cos\theta)\cos(2n-N+1)\theta + \ell_N],\end{aligned}$$

where $\ell_N = \cos(N-2)\theta - W\cos(N-1)\theta$. Similarly,

$$\begin{aligned}[L_n(S_2)]_{21} &= -2Z^*[X_{N-n-1}X_n^* + |Z|^2U_{N-n-2}U_{n-1}]\\ &= -2Z^*[X_{N-1} + (W^*-W)X_{N-n-1}U_{n-1}],\end{aligned}$$

using $X_n^* = X_n + (W^*-W)U_{n-1}$ and Eq. (14). Also,

$$2\sin^2\theta X_{N-n-1}U_{n-1} = W\cos(2n-N+1)\theta - \cos(2n-N+2)\theta + \ell_N.$$

Summing, using Eq. (56), gives

$$\sum_{j=0}^{N-1}[L_j(S_2)]_{22} = -\frac{2|Z|^2}{\sin^2\theta}\left\{\frac{\sin N\theta}{2\sin\theta}(W-W^*) + N\ell_N\right\},$$

$$\sum_{j=0}^{N-1}[L_j(S_2)]_{21} = \frac{Z^*}{\sin^2\theta}\left\{\frac{\sin N\theta}{2\sin\theta}(W-W^*)^2 + N\ell_N^{(21)}\right\},$$

where

$$\ell_N^{(21)} = (W-W^*)\ell_N - 2\sin^2\theta X_{N-1} = 2|Z|^2\cos(N-1)\theta;$$

the last simplification makes use of $W^2 = W(W+W^*-W^*) = 2W\cos\theta - |W|^2$.

Finally, Eqs. (53) and (55) give

$$\begin{aligned}\mathcal{S}_N^T &= -\frac{2X_N|Z|^2}{\sin^2\theta}\left(\frac{\sin N\theta}{2\sin\theta}(W-W^*) + N\ell_N\right) - \frac{2|Z|^4}{\sin^2\theta}\left(\frac{\sin 2N\theta}{\sin 2\theta} - N\right)\\ &= -2|Z|^2(\mathcal{H}_N + N\mathcal{G}_N),\end{aligned} \tag{57}$$

$$\begin{aligned}\mathcal{S}_N^R &= \frac{X_NZ^*}{\sin^2\theta}\left(\frac{\sin N\theta}{2\sin\theta}(W-W^*)^2 + N\ell_N^{(21)}\right) + \frac{Z^*|Z|^2}{\sin^2\theta}(W-W^*)\left(\frac{\sin 2N\theta}{\sin 2\theta} - N\right)\\ &= Z^*(W-W^*)(\mathcal{H}_N + N\mathcal{G}_N) - 2NZ^*X_NX_{N-1},\end{aligned} \tag{58}$$

where

$$\mathcal{H}_N = \frac{X_N\sin N\theta}{2\sin^3\theta}(W-W^*) + \frac{|Z|^2\sin 2N\theta}{\sin^2\theta\sin 2\theta}, \qquad \mathcal{G}_N = \frac{1}{\sin^2\theta}\left(X_N\ell_N - |Z|^2\right).$$

This completes the determination of $\langle T_N\rangle$ and $\langle R_N\rangle$, as given by Eqs. (52) and (54), respectively.

8.4. Discussion

We have given estimates for $\langle T_N \rangle$ and $\langle R_N \rangle$, correct to second order in $k\varepsilon$. One natural next step would be to try and use these estimates to determine an effective wavenumber for the slab of scatterers. We could also estimate other averaged quantities, such as $\langle |T_N| \rangle$ or $\langle \log |T_N| \rangle$. Such quantities often arise in studies of localization and delocalization. Note that our estimates involve terms proportional to $N(k\varepsilon)^2$ (such as $N\mathcal{G}_N$ in Eq. (57)), so they are not uniform in N. This is expected. Localization predicts decay of the transmission coefficient as $e^{-\gamma Nd}$ when $N \to \infty$, where γ^{-1} is the localization length. When $\gamma d \ll 1$ and N is fixed, the approximation $e^{-\gamma Nd} \simeq 1 - \gamma Nd$ leads to exactly the kind of terms that we have found. Note also that our estimates fail whenever $\sin\theta = 0$, that is, at band edges. This is also expected: "under pretty reasonable hypotheses, Anderson localization occurs in a vicinity of the edges of the gap" [28].

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