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On the long-time limit of the Garvin–Alterman–Loewenthal solution for a buried blast load

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ABSTRACT

The Garvin–Alterman–Loewenthal solution refers to the problem of a line blast load suddenly applied in the interior of an elastic half-space. It is expected that the long-time asymptotic limit of this solution should be equal to the solution of a related static problem. This expectation is justified here. First, the solution of the static problem is constructed. Then, the asymptotic limit of the transient problem is found, correcting previously published results.

1. Introduction

At time $t = 0$, there is an underground explosion, generating elastic waves. Determining the subsequent wave motion can be modeled as a variant of Lamb's problem. Garvin (1956) considered a plane-strain version in an elastic half-plane with a discontinuous change in pressure at a point S inside the half-plane. This problem can be solved by the Cagniard–de Hoop technique. Garvin (1956) gave the resulting displacement components on the flat traction-free boundary of the half-plane; see Kausel (2006, Section 5.5) for an exposition. There have been numerous studies of related problems; see, for example, Borejko (1987), Tsai and Ma (1991), Ma and Huang (1996), Wang and Achenbach (1996), Georgiadis et al. (1999) and Sánchez-Sesma and Iturrarán-Viveros (2006).

Thirteen years after Garvin's paper was published, Alterman and Loewenthal (1969) gave formulas for the displacement components at any arbitrary point inside the half-plane. Their solution has been reviewed and clarified recently by Sánchez-Sesma et al. (2013). It is valuable because it is exact and so it can be used for benchmarking purposes.

The Garvin problem is an initial value problem: how does the solution behave for long times? Physically, we expect the solution to approach that of a related static problem. That problem is an elastic half-plane containing a singularity at the point S. If r and α are polar coordinates at S (see Fig. 1), the displacement vector should be directed away from S and it should be singular as r^{-1}. We construct this solution in Section 2.

Next, we determine the long-time asymptotic limit of the Garvin–Alterman–Loewenthal solution, starting with the formulas given by Sánchez-Sesma et al. (2013). It turns out that this is not straightforward: indeed, the long-time results given by Sánchez-Sesma et al. (2013, Section 3.4) are incorrect (but not the dynamic solution itself). In Section 3, we confirm that the long-time limit is the static solution described above. The fact that these two solutions agree perfectly implies not only that they corroborate each other, but provides also a strong indication that the dynamic solution may be free from errors, because the static and dynamic solutions were obtained independently.

2. Static solution using integral transforms

Consider a two-dimensional elastic half-plane subjected in its interior to a dilatative line source. We start the derivation of the static solution to the problem at hand by considering a full space containing both the actual source and an image source placed symmetrically with respect to the position in the plane that will ultimately form the free surface. Then from the known analytical solution to this problem, we can infer the stresses that act at the interface between the upper and lower half-planes forming the full space. If we then separate the upper and lower half-planes and apply external tractions equal to the known internal stresses at the now free surface, equilibrium will be preserved so that the lower half-plane with the actual source and the tractions at the free surface will elicit exactly the same displacement field as the full space with the two sources. Applying next tractions at the surface which are equal in magnitude but opposite in sign to those inferred in the previous step, we cause that surface to be stress free. Hence, it

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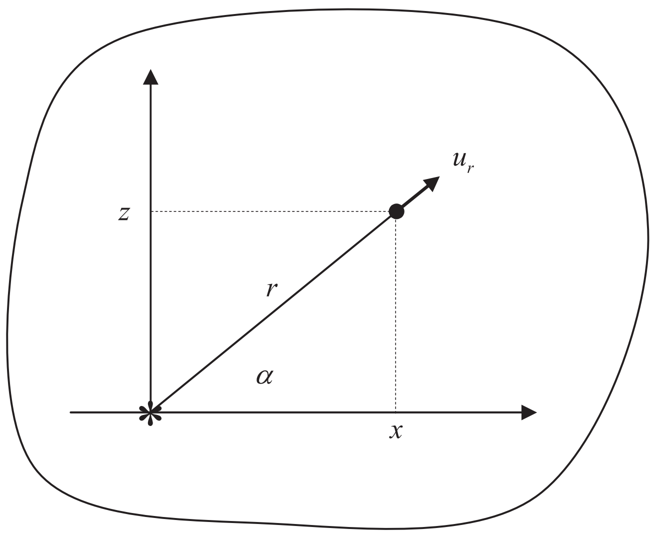

Fig. 1. Full space subjected to one source at the origin.

suffices to find the displacement field elicited by those surface tractions and subtract these from the full space solution. The latter are obtained by means of integral transform techniques.

2.1. Full space containing two sources

With reference to Kausel (2006, p. 44, Eq. (3.51)), the displacement field elicited by a line of pressure (dilatative source) acting at the origin of coordinates in a full space is

$$\mathbf{u} = u_r \hat{\mathbf{r}} = \frac{1}{2\pi\mu r}\hat{\mathbf{r}}, \tag{1}$$

where μ is the shear modulus and $\hat{\mathbf{r}}$ is a unit vector along the direction with angle of inclination α with respect to the horizontal direction x (see Fig. 1).

Next, consider a full space subjected to two sources which are vertically aligned and are separated by a vertical distance $2z_0$. For convenience, we change the positive direction z to point down into the lower half-plane (Fig. 2). The mid-plane between the two sources will ultimately represent the free surface of a half-space, and $z_0 > 0$ will be the depth of the source. Placing the origin of coordinates at the intersection of the mid-plane with the line connecting the sources, then from Eq. (1), the response at some arbitrary point is

$$\mathbf{u} = \frac{1}{2\pi\mu}\left(\frac{\hat{\mathbf{r}}_1}{r_1} + \frac{\hat{\mathbf{r}}_2}{r_2}\right), \tag{2}$$

where $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ are unit vectors pointing away from the source and its image, respectively,

$r_1 = \sqrt{x^2 + (z - z_0)^2}$ is the source-receiver distance and

$r_2 = \sqrt{x^2 + (z + z_0)^2}$ is the image source-receiver distance.

We also introduce polar coordinates, writing

$$x = r_1 \sin\theta_1 = r_2 \sin\theta_2, \quad z - z_0 = r_1 \cos\theta_1, \quad z + z_0 = r_2 \cos\theta_2, \tag{3}$$

so that Eq. (2) becomes

$$u_x = \frac{1}{2\pi\mu}\left(\frac{\sin\theta_1}{r_1} + \frac{\sin\theta_2}{r_2}\right), \quad u_z = \frac{1}{2\pi\mu}\left(\frac{\cos\theta_1}{r_1} + \frac{\cos\theta_2}{r_2}\right). \tag{4}$$

We use Hooke's law and calculate the stresses, τ_{xz} and σ_z, at the mid-plane ("free surface", $z = 0$), where $r_1 = r_2 = r = \sqrt{x^2 + z_0^2}$. We find that $\tau_{xz} = 0$ (as expected, by symmetry) and

$$\sigma_z(x, 0) = \frac{2(x^2 - z_0^2)}{\pi(x^2 + z_0^2)^2} = p_z(x), \tag{5}$$

say, where σ_z is positive when tensile. Clearly, we can now remove the upper half-space containing the image source and preserve equilibrium in the newly formed free surface by application of an external traction equal in magnitude to and with the same spatial distribution as σ_z. This traction is *upwards* when positive (i.e., tensile).

2.2. Fourier transform solution

To solve the problem of the source acting on a lower half-plane with a free surface condition, it suffices to start from the full space solution for the two sources already described and add the displacement field caused by a downward (i.e., compressive) external traction p_z applied on the lower half-plane which is equal and opposite to the stress defined by Eq. (5). Doing this cancels exactly the internal stresses at the interface between the lower and upper half-planes.

The Fourier transform of $p_z(x)$ is

$$P_z(k) = \int_{-\infty}^{\infty} p_z(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x = -2|k|\, \mathrm{e}^{-|k|z_0}, \quad z_0 > 0. \tag{6}$$

From Kausel (2006), modified to account for a z-axis pointing down, the static stress-displacement relationship in the transform domain for a lower half-space and a downward traction applied at the free surface is

$$\begin{pmatrix} \Delta U_x \\ -\mathrm{i}\Delta U_z \end{pmatrix}_{z=0} = \frac{1-\nu}{k\mu}\begin{pmatrix} \operatorname{sgn} k & a^2 \\ a^2 & \operatorname{sgn} k \end{pmatrix}\begin{pmatrix} 0 \\ -\mathrm{i}P_z \end{pmatrix}, \quad a^2 = \frac{1-2\nu}{2(1-\nu)}, \tag{7}$$

where ν is Poisson's ratio, ΔU_x and ΔU_z are the Fourier transforms of Δu_x and Δu_z, respectively, and Δu_x and Δu_z are the displacements which need to be added to the full space solution so as to model a half-space; Δu_z points down when positive. On the other hand, for a harmonically distributed source acting on the surface, the

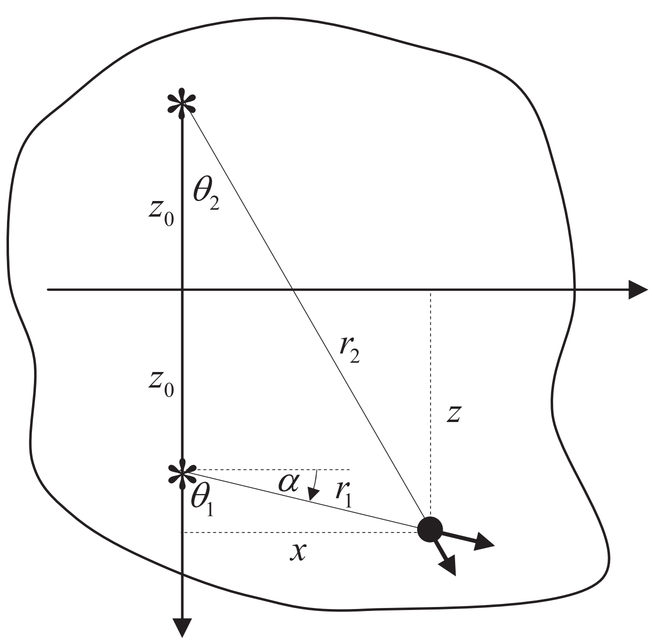

Fig. 2. Full space subjected to two sources.

transfer matrix needed to extend in the transform domain the displacements from the surface to some arbitrary depth $z > 0$ can be shown to be given by

$$\mathbf{T} = \begin{pmatrix} 1-|k|zb & -kzb \\ kzb & 1-|k|zb \end{pmatrix} \mathrm{e}^{-|k|z}, \quad b = \frac{1-a^2}{1+a^2}, \quad z > 0,$$

so

$$\begin{pmatrix} \Delta U_x \\ -\mathrm{i}\Delta U_z \end{pmatrix} = \frac{1-\nu}{k\mu} \begin{pmatrix} 1-|k|zb & -kzb \\ kzb & 1-|k|zb \end{pmatrix} \begin{pmatrix} \mathrm{sgn}\,k & a^2 \\ a^2 & \mathrm{sgn}\,k \end{pmatrix} \begin{pmatrix} 0 \\ -\mathrm{i}P_z \end{pmatrix} \mathrm{e}^{-|k|z}$$
$$= \frac{1-\nu}{k\mu} \begin{pmatrix} \mathrm{sgn}\,k-(1-a^2)kz & a^2-(1-a^2)|k|z \\ a^2+(1-a^2)|k|z & \mathrm{sgn}\,k+(1-a^2)kz \end{pmatrix} \begin{pmatrix} 0 \\ -\mathrm{i}P_z \end{pmatrix} \mathrm{e}^{-|k|z}.$$

Hence, using Eq. (6),

$$\begin{pmatrix} \Delta U_x \\ \Delta U_z \end{pmatrix} = \frac{2(1-\nu)}{\mu} \begin{pmatrix} \mathrm{i}[a^2\mathrm{sgn}\,k-(1-a^2)kz] \\ -[1+(1-a^2)|k|z] \end{pmatrix} \mathrm{e}^{-|k|(z+z_0)}. \tag{8}$$

The incremental displacements, Δu_x and Δu_z, are then obtained by carrying out an inverse Fourier transform of Eq. (8).

2.3. Vertical displacement

We begin by inverting the exponential term,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-|k|(z+z_0)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{-k(z+z_0)}\cos kx\,\mathrm{d}k$$
$$= \frac{z+z_0}{\pi[x^2+(z+z_0)^2]} = \frac{z+z_0}{\pi r_2^2} = \frac{\cos\theta_2}{\pi r_2}.$$

Differentiating this formula with respect to z gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (-|k|)\mathrm{e}^{-|k|(z+z_0)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{1}{\pi}\frac{\partial}{\partial z}\left(\frac{z+z_0}{r_2^2}\right)$$
$$= \frac{1}{\pi}\left(\frac{1}{r_2^2} - \frac{2(z+z_0)^2}{r_2^4}\right) = -\frac{\cos 2\theta_2}{\pi r_2^2}.$$

It follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |k|z(1-a^2)\mathrm{e}^{-|k|(z+z_0)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{z}{\pi r_2^2}(1-a^2)\cos 2\theta_2.$$

Hence, inverting Eq. (8) gives

$$\Delta u_z = -\frac{2(1-\nu)}{\mu}\frac{1}{2\pi}\int_{-\infty}^{\infty}[1+|k|z(1-a^2)]\mathrm{e}^{-|k|(z+z_0)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$
$$= -\frac{2(1-\nu)}{\mu}\left[\frac{\cos\theta_2}{\pi r_2} + \frac{z}{\pi r_2^2}(1-a^2)\cos 2\theta_2\right]$$
$$= \frac{-1}{\pi\mu r_2}\left[2(1-\nu)\cos\theta_2 + \frac{z}{r_2}\cos 2\theta_2\right].$$

The total vertical displacement is then obtained by adding u_z from Eq. (4):

$$u_z = \frac{1}{2\pi\mu}\left\{\frac{\cos\theta_1}{r_1} - \frac{1}{r_2}\left[(3-4\nu)\cos\theta_2 + \frac{2z}{r_2}\cos 2\theta_2\right]\right\}. \tag{9}$$

This displacement is positive when pointing down.

2.4. Horizontal displacement

From Eq. (8),

$$\Delta u_x = \frac{2(1-\nu)}{\mu}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}[a^2\mathrm{sgn}\,k - kz(1-a^2)]\mathrm{e}^{-|k|(z+z_0)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$
$$= \frac{2(1-\nu)}{\pi\mu}\int_0^{\infty}[a^2 - kz(1-a^2)]\mathrm{e}^{-k(z+z_0)}\sin kx\,\mathrm{d}k.$$

But

$$\int_0^{\infty}\mathrm{e}^{-k(z+z_0)}\sin kx\,\mathrm{d}k = \frac{x}{r_2^2} = \frac{\sin\theta_2}{r_2},$$

and the derivative of this formula with respect to z gives

$$\int_0^{\infty} k\mathrm{e}^{-k(z+z_0)}\sin kx\,\mathrm{d}k = \frac{\sin 2\theta_2}{r_2^2}.$$

Hence

$$\Delta u_x = \frac{2(1-\nu)}{\pi\mu}\left\{a^2\frac{\sin\theta_2}{r_2} - (1-a^2)z\frac{\sin 2\theta_2}{r_2^2}\right\}.$$

Adding u_x from Eq. (4) gives the total horizontal displacement,

$$u_x = \frac{1}{2\pi\mu}\left\{\frac{\sin\theta_1}{r_1} + \frac{1}{r_2}\left[(3-4\nu)\sin\theta_2 - \frac{2z}{r_2}\sin 2\theta_2\right]\right\}. \tag{10}$$

This displacement is positive from left to right.

3. Long-time asymptotics

3.1. Preliminaries

We recall some formulas for the Garvin–Alterman–Loewenthal solution for an impulsive blast line load, as given by Sánchez-Sesma et al. (2013). They are given in terms of a dimensionless time, $\tau = t\beta/r_2$, where t is time and β is the shear wave speed. From Sánchez-Sesma et al. (2013, Eq. (19)): for sufficiently large τ (so that all the Heaviside functions therein take the value 1),

$$\pi\mu u_x = \frac{\tau\sin\theta_1}{2r_1\sqrt{\tau^2-\tau_P^2}} - \frac{\tau\sin\theta_2}{2r_2\sqrt{\tau^2-\tau_{PP}^2}}$$
$$-\frac{4}{r_2}\,\mathrm{Im}\left\{\frac{q_{\alpha\alpha}^3}{R_{\alpha\alpha}}\sqrt{q_{\alpha\alpha}^2+1}\,\frac{\partial q_{\alpha\alpha}}{\partial\tau}\right\}$$
$$+\frac{2}{r_2}\,\mathrm{Im}\left\{\frac{q_{\alpha\beta}}{R_{\alpha\beta}}(1+2q_{\alpha\beta}^2)\sqrt{q_{\alpha\beta}^2+1}\,\frac{\partial q_{\alpha\beta}}{\partial\tau}\right\}, \tag{11}$$

$$\pi\mu u_z = \frac{\tau\cos\theta_1}{2r_1\sqrt{\tau^2-\tau_P^2}} + \frac{\tau\cos\theta_2}{2r_2\sqrt{\tau^2-\tau_{PP}^2}}$$
$$-\frac{1}{r_2}\,\mathrm{Re}\left\{\frac{(1+2q_{\alpha\alpha}^2)^2}{R_{\alpha\alpha}}\frac{\partial q_{\alpha\alpha}}{\partial\tau}\right\} + \frac{2}{r_2}\,\mathrm{Re}\left\{\frac{q_{\alpha\beta}^2}{R_{\alpha\beta}}(1+2q_{\alpha\beta}^2)\frac{\partial q_{\alpha\beta}}{\partial\tau}\right\} \tag{12}$$

where $\tau_{PP} = \beta/\alpha$, $\tau_P = \tau_{PP}r_1/r_2$ and α is the compressional wave speed. The quantity $q_{\alpha\alpha}$ solves (Sánchez-Sesma et al., 2013, Eq. (11a)); this equation is Eq. (16) below. Similarly, $q_{\alpha\beta}$ solves solves (Sánchez-Sesma et al., 2013, Eq. (11b)); this is Eq. (17) below. The Rayleigh functions, $R_{\alpha\alpha}$ and $R_{\alpha\beta}$, are defined by Sánchez-Sesma et al. (2013, Eq. (18)); thus, $R_{\alpha\alpha} = R(q_{\alpha\alpha})$ and $R_{\alpha\beta} = R(q_{\alpha\beta})$ with

$$R(Q) = (2Q^2+1)^2 - 4Q^2\sqrt{Q^2+1}\sqrt{Q^2+a^2}; \tag{13}$$

as before, $a = \beta/\alpha$ is given by Eq. (7) in terms of Poisson's ratio. Henceforth, we write Q for $q_{\alpha\alpha}$ or $q_{\alpha\beta}$, as they satisfy similar equations.

3.2. Analysis

We are interested in large (dimensionless) time τ. It is convenient to introduce $T = \tau\mathrm{e}^{\mathrm{i}\theta_2}$ so that we are interested in large $|T|$. Leading-order estimates show that $Q \sim T$. (To see this, replace the square-roots in Eqs. (16) and (17) by Q.) As we require a more accurate estimate, we start with

$$Q = T\left(1 + \frac{A}{T^2} + \frac{B}{T^4} + \cdots\right), \tag{14}$$

where the coefficients A and B are to be found. (We could include terms such as C/T and D/T^3 inside the parentheses in Eq. (14), but subsequent calculation would show that such terms must be absent.) A quick inspection of Eqs. (11) and (12) suggests that we will need B in order to estimate the Rayleigh functions correctly: however, it turns out that the value of B will not be needed.

3.3. Rayleigh functions

From Eq. (14), we obtain

$$Q^2 \sim T^2\left(1+\frac{2A}{T^2}+\frac{A^2+2B}{T^4}\right),$$
$$Q^2+c^2 \sim T^2\left(1+\frac{2A+c^2}{T^2}+\frac{A^2+2B}{T^4}\right),$$
$$\sqrt{Q^2+c^2} \sim T\left(1+\frac{2A+c^2}{2T^2}+\frac{8B-4Ac^2-c^4}{8T^4}\right),$$

using $\sqrt{1+x}\sim 1+\frac{1}{2}x-\frac{1}{8}x^2$; the constant c^2 will be selected later.

We estimate the terms in Eq. (13). Thus,

$$2Q^2+1 \sim 2T^2\left(1+\frac{4A+1}{2T^2}+\frac{A^2+2B}{T^4}\right),$$
$$(2Q^2+1)^2 \sim 4T^4\left(1+\frac{4A+1}{T^2}+\frac{(4A+1)^2+8(A^2+2B)}{4T^4}\right),$$
$$\sqrt{Q^2+1}\sqrt{Q^2+a^2} \sim T^2\left(1+\frac{4A+a^2+1}{2T^2}+\frac{8A^2+16B-(a^2-1)^2}{8T^4}\right),$$
$$4Q^2\sqrt{Q^2+1}\sqrt{Q^2+a^2} \sim 4T^4 \times\left(1+\frac{8A+a^2+1}{2T^2}+\frac{8A(a^2+1)+16(3A^2+2B)-(a^2-1)^2}{8T^4}\right).$$

Hence some calculation gives

$$R \sim 2T^2\left(1-a^2+\frac{8A(1-a^2)+2+(a^2-1)^2}{4T^2}\right). \tag{15}$$

Surprisingly, the terms in B cancel. For later calculations, we require

$$\frac{1}{R} \sim \frac{1}{2T^2(1-a^2)}\left(1-\frac{8A(1-a^2)+2+(a^2-1)^2}{4T^2(1-a^2)}\right).$$

3.4. Calculation of A

We find A by substituting into the governing equation for Q. (This is simpler than substituting into an explicit but complicated formula for Q. The same method could be used to find B.)

With $Q = q_{\alpha\alpha}$, (Sánchez-Sesma et al., 2013, Eq. (11a)) gives

$$\begin{aligned}\tau = T\mathrm{e}^{-\mathrm{i}\theta_2} &= \cos\theta_2\sqrt{Q^2+a^2}-\mathrm{i}Q\sin\theta_2\\ &\sim T\cos\theta_2\left(1+\frac{2A+a^2}{2T^2}\right)-\mathrm{i}T\sin\theta_2\left(1+\frac{A}{T^2}\right)\\ &= T\mathrm{e}^{-\mathrm{i}\theta_2}+\frac{1}{2T}\{(2A+a^2)\cos\theta_2-2\mathrm{i}A\sin\theta_2\}.\end{aligned} \tag{16}$$

As the coefficient of T^{-1} must vanish, $(2A+a^2)\cos\theta_2 - 2\,\mathrm{i}A\sin\theta_2 = 0$, we obtain

$$A = -\frac{1}{2}a^2\mathrm{e}^{\mathrm{i}\theta_2}\cos\theta_2 = A_{\alpha\alpha}$$

say. Thus

$$q_{\alpha\alpha} \sim T\left(1-\frac{a^2\mathrm{e}^{\mathrm{i}\theta_2}}{2T^2}\cos\theta_2\right) = \tau\mathrm{e}^{\mathrm{i}\theta_2}-\frac{a^2}{2\tau}\cos\theta_2.$$

This agrees with the exact formula for $q_{\alpha\alpha}$, Sánchez-Sesma et al., 2013, Eq. (15).

Similarly, with $Q = q_{\alpha\beta}$, Sánchez-Sesma et al., 2013, Eq. (11b) gives

$$\begin{aligned}T\mathrm{e}^{-\mathrm{i}\theta_2} &= H\sqrt{Q^2+a^2}+Z\sqrt{Q^2+1}-\mathrm{i}XQ\\ &\sim TH\left(1+\frac{2A+a^2}{2T^2}\right)+TZ\left(1+\frac{2A+1}{2T^2}\right)\\ &\quad -\mathrm{i}TX\left(1+\frac{A}{T^2}\right),\end{aligned} \tag{17}$$

where $H = z_0/r_2, X = x/r_2$ and $Z = z/r_2$. As $H+Z-\mathrm{i}X = \mathrm{e}^{-\mathrm{i}\theta_2}$ (see Eq. (3)), the terms in T^1 balance. The terms in T^{-1} give

$$H(2A+a^2)+Z(2A+1)-2\mathrm{i}XA = 0$$

whence

$$A = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta_2}(a^2H+Z) = A_{\alpha\beta},$$

say. An estimate for $q_{\alpha\beta}$ follows readily.

3.5. Vertical displacement

The vertical displacement is given by Eq. (12). From Eq. (14), we have

$$\frac{\partial Q}{\partial\tau} \sim \mathrm{e}^{\mathrm{i}\theta_2}\left(1-\frac{A}{T^2}\right). \tag{18}$$

Then, we find

$$\frac{(1+2Q^2)^2}{R}\frac{\partial Q}{\partial\tau} \sim \frac{2T^2\mathrm{e}^{\mathrm{i}\theta_2}}{1-a^2}\left(1+\frac{4A(1-a^2)+1-2a^2-a^4}{4T^2(1-a^2)}\right),$$
$$\frac{Q^2}{R}(1+2Q^2)\frac{\partial Q}{\partial\tau} \sim \frac{T^2\mathrm{e}^{\mathrm{i}\theta_2}}{1-a^2}\left(1+\frac{4A(1-a^2)-a^4-1}{4T^2(1-a^2)}\right).$$

Hence

$$\begin{aligned}&-\frac{(1+2q_{\alpha\alpha}^2)^2}{R_{\alpha\alpha}}\frac{\partial q_{\alpha\alpha}}{\partial\tau}+2\frac{q_{\alpha\beta}^2}{R_{\alpha\beta}}(1+2q_{\alpha\beta}^2)\frac{\partial q_{\alpha\beta}}{\partial\tau}\\ &\sim\frac{2T^2\mathrm{e}^{\mathrm{i}\theta_2}}{1-a^2}\left[\left(1+\frac{4A_{\alpha\beta}(1-a^2)-a^4-1}{4T^2(1-a^2)}\right)\right.\\ &\quad\left.-\left(1+\frac{4A_{\alpha\alpha}(1-a^2)+1-2a^2-a^4}{4T^2(1-a^2)}\right)\right]\\ &\sim\frac{\mathrm{e}^{\mathrm{i}\theta_2}}{1-a^2}(2A_{\alpha\beta}-2A_{\alpha\alpha}-1).\end{aligned} \tag{19}$$

Now

$$2A_{\alpha\beta}-2A_{\alpha\alpha} = a^2\mathrm{e}^{\mathrm{i}\theta_2}\cos\theta_2-\mathrm{e}^{\mathrm{i}\theta_2}(a^2H+Z). \tag{20}$$

Then, from Eq. (12),

$$\begin{aligned}\pi\mu u_z &\sim \frac{\cos\theta_1}{2r_1}+\frac{\cos\theta_2}{2r_2}\\ &\quad+\frac{1}{r_2(1-a^2)}\left[-\cos\theta_2+(a^2\cos\theta_2-a^2H-Z)\cos 2\theta_2\right]\\ &= \frac{\cos\theta_1}{2r_1}-\frac{(1+a^2)\cos\theta_2}{2(1-a^2)r_2}-\frac{Z\cos 2\theta_2}{r_2}\end{aligned}$$

using $H+Z=\cos\theta_2$. Finally, using

$$a^2 = \frac{1-2\nu}{2(1-\nu)},\quad 1-a^2 = \frac{1}{2(1-\nu)},\quad 1+a^2 = \frac{3-4\nu}{2(1-\nu)}, \tag{21}$$

we find precise agreement with the known static result, Eq. (9), derived in Section 2.2.

3.6. Horizontal displacement

The horizontal displacement is given by Eq. (11). We find

$$\frac{Q^3}{R}\sqrt{Q^2+1}\frac{\partial Q}{\partial \tau}\sim\frac{T^2 e^{i\theta_2}}{2(1-a^2)}\left(1+\frac{4A(1-a^2)-a^4-1}{4T^2(1-a^2)}\right),$$

$$\frac{Q(2Q^2+1)}{R}\sqrt{Q^2+1}\frac{\partial Q}{\partial \tau}\sim\frac{T^2 e^{i\theta_2}}{1-a^2}\left(1+\frac{4A(1-a^2)+1-2a^2-a^4}{4T^2(1-a^2)}\right).$$

Hence

$$\begin{aligned}&-4\frac{q_{\alpha\alpha}^3}{R_{\alpha\alpha}}\sqrt{q_{\alpha\alpha}^2+1}\frac{\partial q_{\alpha\alpha}}{\partial\tau}+2\frac{q_{\alpha\beta}(2q_{\alpha\beta}^2+1)}{R_{\alpha\beta}}\sqrt{q_{\alpha\beta}^2+1}\frac{\partial q_{\alpha\beta}}{\partial\tau}\\ &\sim\frac{2T^2e^{i\theta_2}}{1-a^2}\left[\left(1+\frac{4A_{\alpha\beta}(1-a^2)+1-2a^2-a^4}{4T^2(1-a^2)}\right)\right.\\ &\left.-\left(1+\frac{4A_{\alpha\alpha}(1-a^2)-a^4-1}{4T^2(1-a^2)}\right)\right]\quad\sim\frac{e^{i\theta_2}}{1-a^2}(2A_{\alpha\beta}-2A_{\alpha\alpha}+1),\end{aligned}$$

which is almost the same as Eq. (19). Using Eq. (20), Eq. (11) gives

$$\begin{aligned}\pi\mu u_x&\sim\frac{\sin\theta_1}{2r_1}-\frac{\sin\theta_2}{2r_2}+\frac{1}{r_2(1-a^2)}[\sin\theta_2+\{a^2\cos\theta_2-a^2H-Z\}\sin 2\theta_2]\\ &=\frac{\sin\theta_1}{2r_1}+\frac{(1+a^2)\sin\theta_2}{2(1-a^2)r_2}-\frac{Z\sin 2\theta_2}{r_2}.\end{aligned}$$

Using Eq. (21), we find agreement with the static result, Eq. (10), found in Section 2.2.

4. Conclusion

This article has presented two independent solutions for the long-time asymptotic limit of the dynamic problem of a line blast load suddenly applied within an elastic half-plane, the so-called generalized Garvin problem. The need for these solutions arose after the writers detected an error in the limits given in an earlier article by Sánchez-Sesma et al. (2013), an error which resulted from a naïve asymptotic approximation. However, as demonstrated herein, obtaining the correct limit is not entirely trivial. Thus, for verification purposes, it was necessary to arrive at the same limits by two independent methods: perfect agreement was found. The correct long-time limits are given by Eqs. (9) and (10).

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