Wave Motion 51 (2014) 1161–1169

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Shear-wave resonances in a fluid–solid–solid layered structure

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HIGHLIGHTS

- Properties of inhomogeneous layer modeled with exponentials.
- Exact solutions found using hypergeometric functions.
- Accurate asymptotic approximations developed.

ARTICLE INFO

Article history:
Received 17 December 2013
Accepted 4 July 2014
Available online 21 July 2014

Keywords:
Layered media
Hypergeometric function
Fluid–solid interaction

ABSTRACT

An inhomogeneous solid layer is bounded on one side by a fluid half-space and on the other by a homogeneous solid half-space. An acoustic wave in the fluid is incident on the layer. Experiments suggest that some kind of shear-wave resonance of the layer exists. Here, the layer is modeled with exponential variations of the material properties (Epstein model). Solutions in terms of hypergeometric functions are found. Genuine resonances are found but only when the layer is not bonded to the solid half-space; these are analogous to Jones frequencies in fluid–solid interaction problems. When the solid half-space is present, the resonances become complex: they are scattering frequencies. Simple but accurate asymptotic approximations are found using known estimates for hypergeometric functions with large parameters.

1. Introduction

Shear waves in marine sediments is the title of a 600-page edited book, published in 1991 [1]. Its subject has long been of interest to underwater acousticians. The basic model considered is a fluid (the ocean) on top of an inhomogeneous solid layer (the sediment) on top of a homogeneous solid (the basement). Such configurations (usually without the fluid) have also been studied in the context of seismology and soil dynamics.

Our motivation comes from studies by Godin and Chapman [2,3], and others, which show some kind of resonance behavior, attributed to shear waves in the inhomogeneous layer; see, especially, [2, Fig. 1]. In fact, these are not genuine resonance frequencies; they are complex scattering frequencies close to the real-frequency axis. We shall investigate these scattering frequencies, using mainly analytical methods.

Various analytical formulas have been used to represent the variations of the material properties through the inhomogeneous layer. For an isotropic elastic solid, lying between planes $z = 0$ and $z = h$, the relevant quantities are the Lamé moduli, $\lambda(z)$ and $\mu(z)$, and the density, $\rho(z)$. We shall assume exponential variations, giving models of Epstein type: in 1930, Epstein [4] considered acoustic waves in a continuously varying medium (not a layer), and he gave solutions in terms of hypergeometric functions; we shall encounter such functions later. For elastodynamic problems with Epstein models,

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http://dx.doi.org/10.1016/j.wavemoti.2014.07.002
0165-2125/

see, for example, Rao [5], Vrettos [6], Muravskii [7], Rao and Li [8] and Manolis et al. [9]; all these authors assume that ρ and λ/μ are constants. The same assumptions are made by Godin and Chapman [2,3], but they use power-law variations of $\mu(z)$. Robins [10] allows ρ to vary but not the Lamé moduli. For two other approaches, see [11,12]. For more details on acoustic models, see [13, Chapter 3].

We begin by recalling the governing equations for the fluid–solid–solid problem. We consider two-dimensional motions, with plane-strain conditions in the solid regions. There is a plane time-harmonic acoustic wave in the fluid, incident upon the fluid–solid interface. Our focus is on normal incidence because then the whole problem decouples into two subproblems, one involving the acoustic pressure and the z-component of the elastic displacement (we call this the "compressional problem"), and one involving the other component of the displacement ("shear problem"). If the shear problem has any non-trivial solutions, such solutions do not couple to the fluid, and so the potential for resonance would arise. Indeed, such real resonance frequencies do exist but only when the layer is not bonded to the homogeneous elastic half-space. (This is a simple consequence of Sturm–Liouville theory.) When the layer is bonded to the half-space, we find complex scattering frequencies. In both cases, the frequencies are found by setting an appropriate 2×2 determinant to zero. We also give a brief discussion of a semi-infinite smoothly inhomogeneous half-space (so that there is no interface at $z = h$).

The numerical results are compared with simple asymptotic approximations. These are obtained by approximating the relevant determinants, which consist of products of hypergeometric functions. Unusually, we have to estimate such functions when their argument is fixed but their parameters are large; for example, $F(1+\delta, 1+\delta; 1+2\delta; \zeta)$ when ζ is fixed but $\delta \to \infty$. Fortunately, an appropriate (but complicated) asymptotic approximation was given by G.N. Watson almost 100 years ago. (For a recent review of this topic, see [14].) It turns out that the asymptotic approximations give excellent agreement with the numerical results.

2. Formulation of the problem

We consider a three-part layered medium with two flat interfaces, at $z = 0$ and $z = h > 0$.

In the semi-infinite region $z < 0$ (the "water"), there is a homogeneous compressible inviscid fluid with density ρ_f and sound speed c_f. The pressure P satisfies the wave equation for $z < 0$.

In the semi-infinite region $z > h$ (the "substrate"), there is a homogeneous isotropic elastic solid with density ρ_s and Lamé moduli μ_s and λ_s.

In the region $0 < z < h$ (the "layer"), there is an inhomogeneous isotropic elastic solid with density $\rho(z)$ and Lamé moduli $\mu(z)$ and $\lambda(z)$.

For plane-strain motions in the solid regions, the elastodynamic displacement vector has components $u_1(x, z, t)$ and $u_3(x, z, t)$ in the x and z directions, respectively. The governing equations are

$$\rho\frac{\partial^2 u_1}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u_1}{\partial x^2} + \mu\frac{\partial^2 u_1}{\partial z^2} + (\lambda+\mu)\frac{\partial^2 u_3}{\partial x\,\partial z} + \mu'(z)\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right), \tag{1}$$

$$\rho\frac{\partial^2 u_3}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u_3}{\partial z^2} + \mu\frac{\partial^2 u_3}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u_1}{\partial x\,\partial z} + \lambda'(z)\frac{\partial u_1}{\partial x} + (\lambda'+2\mu')\frac{\partial u_3}{\partial z}. \tag{2}$$

The relevant stresses are

$$\sigma_{33} = \lambda\frac{\partial u_1}{\partial x} + (\lambda+2\mu)\frac{\partial u_3}{\partial z}, \qquad \sigma_{13} = \mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right). \tag{3}$$

At the water-layer interface, the boundary conditions are

$$\frac{\partial P}{\partial z} + \rho_f\frac{\partial^2 u_3}{\partial t^2} = 0, \qquad P + \sigma_{33} = 0 \quad \text{and} \quad \sigma_{13} = 0 \quad \text{at } z = 0. \tag{4}$$

At the layer-substrate interface, the boundary conditions are

$$u_1, u_3, \sigma_{13} \text{ and } \sigma_{33} \text{ are continuous across } z = h. \tag{5}$$

2.1. Time-harmonic motions

Suppose now that

$$P(x, z, t) = p(z)\,e^{i(\xi x - \omega t)}, \qquad u_1(x, z, t) = i\,u(z)\,e^{i(\xi x - \omega t)}, \qquad u_3(x, z, t) = w(z)\,e^{i(\xi x - \omega t)}.$$

(The factor i in u_1 is inserted for algebraic convenience.) In the water, we have

$$p''(z) + \{(\omega/c_f)^2 - \xi^2\}p(z) = 0.$$

In the solid regions, Eqs. (1) and (2) become

$$\mu u'' + \mu' u' + [\rho\omega^2 - (\lambda+2\mu)\xi^2]u + (\lambda+\mu)\xi w' + \mu'\xi w = 0, \tag{6}$$

$$(\lambda+2\mu)w'' + (\lambda'+2\mu')w' + (\rho\omega^2 - \mu\xi^2)w - (\lambda+\mu)\xi u' - \lambda'\xi u = 0. \tag{7}$$

In particular, in the substrate ($z > h$), where $u \equiv u_s$ and $w \equiv w_s$, we have

$$\mu_s u_s'' + [\rho_s\omega^2 - (\lambda_s + 2\mu_s)\xi^2]u_s + (\lambda_s + \mu_s)\xi w_s' = 0, \tag{8}$$

$$(\lambda_s + 2\mu_s)w_s'' + (\rho_s\omega^2 - \mu_s\xi^2)w_s - (\lambda_s + \mu_s)\xi u_s' = 0. \tag{9}$$

At the water-layer interface, the boundary conditions, Eq. (4), give

$$p'(0) - \rho_f\omega^2 w(0) = 0, \tag{10}$$

$$p(0) - \lambda(0)\xi u(0) + [\lambda(0) + 2\mu(0)]w'(0) = 0, \tag{11}$$

$$\mu(0)[u'(0) + \xi w(0)] = 0. \tag{12}$$

At the layer-substrate interface, the boundary conditions, Eq. (5), give

$$u(h) = u_s(h), \qquad \mu(h)[u'(h) + \xi w(h)] = \mu_s[u_s'(h) + \xi w_s(h)], \tag{13}$$

$$w(h) = w_s(h), \qquad \lambda(h)\xi u(h) - [\lambda(h) + 2\mu(h)]w'(h) = \lambda_s\xi u_s(h) - (\lambda_s + 2\mu_s)w_s'(h). \tag{14}$$

2.2. Normal incidence: $\xi = 0$

When $\xi = 0$, the problem simplifies and decouples. In the water, $p'' + (\omega/c_f)^2 p = 0$. In the layer, Eqs. (6) and (7) reduce to

$$\mu u'' + \mu' u' + \rho\omega^2 u = 0, \tag{15}$$

$$(\lambda + 2\mu)w'' + (\lambda' + 2\mu')w' + \rho\omega^2 w = 0. \tag{16}$$

In the substrate, Eqs. (8) and (9) reduce to

$$\mu_s u_s'' + \rho_s\omega^2 u_s = 0, \tag{17}$$

$$(\lambda_s + 2\mu_s)w_s'' + \rho_s\omega^2 w_s = 0. \tag{18}$$

The water-layer interface conditions, Eqs. (10)–(12), become

$$p'(0) - \rho_f\omega^2 w(0) = 0, \qquad p(0) + [\lambda(0) + 2\mu(0)]w'(0) = 0, \qquad \mu(0)u'(0) = 0,$$

and the layer-substrate interface conditions, Eqs. (13) and (14), become

$$u(h) = u_s(h), \qquad \mu(h)u'(h) = \mu_s u_s'(h), \tag{19}$$

$$w(h) = w_s(h), \qquad [\lambda(h) + 2\mu(h)]w'(h) = (\lambda_s + 2\mu_s)w_s'(h). \tag{20}$$

Thus the problem of calculating p, w and w_s (the "compressional problem") decouples from the problem of calculating u and u_s (the "shear problem"). This means that if the forcing comes from a pressure wave in the water, at normal incidence, then the horizontal components of the displacement in the solid, u and u_s, may not be determined uniquely (they could be zero). For a more general incident field, a Fourier analysis suggests that some components with $\xi = 0$ may be present.

3. The shear problem

Let us restate the problem. From Eqs. (15) and (17), we have

$$[\mu(z)u'(z)]' + \rho(z)\omega^2 u(z) = 0, \quad 0 < z < h, \tag{21}$$

$$\mu_s u_s''(z) + \rho_s\omega^2 u_s(z) = 0, \quad z > h, \tag{22}$$

with

$$\mu(0)u'(0) = 0, \tag{23}$$

$$u(h) = u_s(h), \qquad \mu(h)u'(h) = \mu_s u_s'(h). \tag{24}$$

In the substrate ($z > h$), we assume that waves propagate away from the layer-substrate interface whence

$$u_s(z) = A\exp(\mathrm{i}z\omega/c_s) \quad \text{where } c_s = \sqrt{\mu_s/\rho_s}$$

is the speed of shear waves in the substrate and A is an arbitrary constant. Then substituting in the two conditions, Eq. (24), and eliminating A gives

$$u(h) + \frac{\mathrm{i}T}{\omega}c(h)u'(h) = 0 \quad \text{with } T = \frac{\rho(h)c(h)}{\rho_s c_s}, \tag{25}$$

where $c(h) = \sqrt{\mu(h)/\rho(h)}$ is the speed of shear waves at the bottom of the layer. The boundary condition Eq. (25) (including the dimensionless parameter T) was given by Godin and Chapman [2, Eq. (35)]; in their application, $T \simeq 0.1$.

3.1. A Sturm–Liouville problem

When $T = 0$, the shear problem simplifies to a boundary value problem for $u(z)$, with a differential equation in $0 < z < h$, namely Eq. (21), together with boundary conditions, Eq. (23) at $z = 0$ and (from Eq. (25) with $T = 0$)

$$u(h) = 0.$$

If $\mu(z) > 0$ and $\rho(z) > 0$ for $0 \le z \le h$, this problem is a regular Sturm–Liouville problem for u. The general theory of such problems asserts that there are infinitely many eigenvalues, all real, giving resonance frequencies for the shear problem. At these frequencies, the layer can oscillate without coupling to the water. In the context of acoustic scattering by a bounded elastic object, these frequencies are known as *Jones frequencies* [15,16].

Mature software packages exist for solving Sturm–Liouville problems numerically; for analytical and computational aspects, see, for example, Zettl's book [17].

For a homogeneous layer, with $\mu = \mu_{\text{lay}}$ and $\rho = \rho_{\text{lay}}$, both constants, we obtain

$$u(z) = \cos k_n z, \qquad k_n = \frac{\omega_n}{c_{\text{lay}}} = \frac{(2n-1)\pi}{2h}, \qquad c_{\text{lay}} = \sqrt{\frac{\mu_{\text{lay}}}{\rho_{\text{lay}}}}, \quad n = 1, 2, \ldots. \tag{26}$$

Let us compare with the experimental results in [2]. From Eq. (26), the first two frequencies are

$$f_1 = \frac{\omega_1}{2\pi} = \frac{c_{\text{lay}}}{4h} \quad \text{and} \quad f_2 = 3f_1.$$

If we take $h = 25.5$ m and $f_1 = 1.06$ Hz from [2], we obtain $c_{\text{lay}} = 108$ m/s, which is plausible. But the experiments give $f_2 = 2.16$ Hz, which is not close to $3f_1 = 3.18$ Hz. Thus, we confirm that assuming a homogeneous layer does not yield good agreement with the marine experiments.

3.2. Approximations when $T \ll 1$

When $T \ne 0$, we no longer have a Sturm–Liouville problem. For example, reconsider the homogeneous layer but now bonded to the substrate. Then we find that $u(z) = \cos kz$ where $k = \omega/c_{\text{lay}}$ solves

$$\cos kh - \mathrm{i}T \sin kh = 0 \quad \text{with } T = (\rho_{\text{lay}} c_{\text{lay}})/(\rho_s c_s).$$

Thus $kh = (n - 1/2)\pi - \mathrm{i}\xi$ where $\tanh \xi = T$. This means that we do not have genuine resonance frequencies, we have complex "scattering frequencies". We notice that ξ is small when T is small: all the scattering frequencies are close to the real-ω axis. However, the real parts of the scattering frequencies are unchanged from those given in Eq. (26) for $T = 0$. We will see similar results when the layer is not homogeneous (Section 4.4).

4. Inhomogeneous layer: Epstein model

To model an inhomogeneous solid, we assume that the shear modulus varies exponentially with distance from the interface. Such an assumption has been investigated by many authors (see Section 1 for citations), but mainly within infinite or semi-infinite media: we use it within a layer of finite thickness, h. Godin and Chapman [2,3] have used power-law forms for $\mu(z)$. Unlike previous workers, we also allow $\rho(z)$ to vary with depth.

4.1. The model: exponential μ and ρ

Following Vrettos [6] and others, we consider solids in which the shear modulus varies as

$$\mu(z) = \mu_0 + \mu_1(1 - \mathrm{e}^{-\alpha z}), \quad \alpha > 0, \tag{27}$$

where μ_0, μ_1 and α are constants. We have $\mu(0) = \mu_0$ and $\mu'(0) = \alpha\mu_1$. We assume that $\mu_1 > 0$ because we want $\mu(z)$ to increase with z.

Vrettos [6, Eq. (16)] makes a change of independent variable, writing

$$\zeta = \zeta_0 \, \mathrm{e}^{-\alpha z} \quad \text{with } \zeta_0 = \mu_1/(\mu_0 + \mu_1) \tag{28}$$

so that Eq. (27) becomes $\mu = (\mu_0 + \mu_1)(1 - \zeta)$. The range $0 < z < h$ is mapped to $0 < \zeta_h < \zeta < \zeta_0 < 1$. In particular, the water-layer interface at $z = 0$ is mapped to $\zeta = \zeta_0$ whereas $\zeta = \zeta_h = \zeta_0 \, \mathrm{e}^{-\alpha h}$ corresponds to the layer-substrate interface at $z = h$.

For the density, we take

$$\rho(z) = \rho_0 + \rho_1(1 - \mathrm{e}^{-\alpha z}) = \rho_0 + \rho_1 - (\rho_1/\zeta_0)\zeta$$

so that $\rho = \rho_0$ at $z = 0$. Setting $\rho_1 = 0$ will give a constant density, ρ_0, in the layer.

Let us write $U(\zeta) = u(z)$. Then we have

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\mathrm{d}u}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z} = -\alpha\zeta \frac{\mathrm{d}U}{\mathrm{d}\zeta}, \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \alpha^2\zeta^2 \frac{\mathrm{d}^2 U}{\mathrm{d}\zeta^2} + \alpha^2\zeta \frac{\mathrm{d}U}{\mathrm{d}\zeta}.$$

Also

$$\frac{\mathrm{d}\mu}{\mathrm{d}z} = -(\mu_0 + \mu_1)\frac{\mathrm{d}\zeta}{\mathrm{d}z} = (\mu_0 + \mu_1)\alpha\zeta.$$

Using a prime to denote $\mathrm{d}/\mathrm{d}\zeta$, Eq. (21) becomes

$$\zeta(1-\zeta)U'' + (1-2\zeta)U' + \zeta^{-1}\left(\theta_0^2 - \theta_1^2\zeta\right)U = 0, \tag{29}$$

where

$$\theta_0^2 = \frac{(\rho_0+\rho_1)\omega^2}{(\mu_0+\mu_1)\alpha^2} \quad \text{and} \quad \theta_1^2 = \frac{\rho_1\omega^2}{\mu_1\alpha^2}. \tag{30}$$

Eq. (29) is to be solved for $U(\zeta)$, $\zeta_h < \zeta < \zeta_0$, subject to

$$U'(\zeta_0) = 0 \quad \text{and} \quad U(\zeta_h) - \frac{\mathrm{i}T}{\omega}c(h)\alpha\zeta_h U'(\zeta_h) = 0, \tag{31}$$

where

$$[c(h)]^2 = \left.\frac{\mu}{\rho}\right|_{z=h} = \frac{\mu_0 + \mu_1(1-\mathrm{e}^{-\alpha h})}{\rho_0 + \rho_1(1-\mathrm{e}^{-\alpha h})} = \frac{(\mu_0+\mu_1)(1-\zeta_h)}{\rho_0 + \rho_1 - (\rho_1/\zeta_0)\zeta_h}.$$

4.2. Solution of Eq. (29)

Eq. (29) is similar to the hypergeometric differential equation [18, 15.10.1],

$$z(1-z)f''(z) + (c - (a+b+1)z)f'(z) - abf(z) = 0. \tag{32}$$

To change one into the other, put $U(\zeta) = \zeta^\delta q(\zeta)$ in Eq. (29), where δ is to be chosen. We obtain

$$\zeta(1-\zeta)q'' + [2\delta + 1 - 2(\delta+1)\zeta]q' + \zeta^{-1}[\delta^2 + \theta_0^2 - (\delta^2 + \delta + \theta_1^2)\zeta]q = 0.$$

Comparing with Eq. (32), we choose $\delta^2 + \theta_0^2 = 0$ and then q solves the hypergeometric equation with $c = 2\delta + 1$, $a + b + 1 = 2\delta + 2$ and $ab = \delta^2 + \delta + \theta_1^2$, giving

$$a = \delta + \frac{1-\phi_1}{2}, \qquad b = \delta + \frac{1+\phi_1}{2}, \qquad c = a + b, \qquad \phi_1 = \sqrt{1 - 4\theta_1^2}.$$

Hence

$$\begin{aligned} U(\zeta) = {} & A\zeta^{-\delta}F\left(-\delta + \frac{1}{2}(1-\phi_1),\ -\delta + \frac{1}{2}(1+\phi_1);\ 1 - 2\delta;\ \zeta\right) \\ & + B\zeta^{\delta}F\left(\delta + \frac{1}{2}(1-\phi_1),\ \delta + \frac{1}{2}(1+\phi_1);\ 1 + 2\delta;\ \zeta\right), \end{aligned} \tag{33}$$

where $\delta = \mathrm{i}\theta_0$, A and B are arbitrary constants, and F is the Gauss hypergeometric function. Application of the boundary conditions, Eq. (31), gives a homogeneous pair of equations for A and B. Setting the determinant to zero gives an equation from which the scattering frequencies can be determined.

In what follows, we simplify slightly by assuming that the density is constant, $\rho(z) = \rho_0$. Thus $\rho_1 = 0$, $\theta_1 = 0$, $\phi_1 = 1$ and Eq. (33) reduces to

$$U(\zeta) = A\zeta^{-\delta}F(-\delta, 1-\delta;\ 1-2\delta;\ \zeta) + B\zeta^{\delta}F(\delta, 1+\delta;\ 1+2\delta;\ \zeta). \tag{34}$$

Also, using $(\mathrm{d}/\mathrm{d}z)\{z^aF(a,b;c;z)\} = az^{a-1}F(a+1,b;c;z)$ [18, 15.5.3],

$$U'(\zeta) = -A\delta\zeta^{-\delta-1}F(1-\delta, 1-\delta;\ 1-2\delta;\ \zeta) + B\delta\zeta^{\delta-1}F(1+\delta, 1+\delta;\ 1+2\delta;\ \zeta). \tag{35}$$

(The expression for U' would be more complicated if we had differentiated Eq. (33) with $\phi_1 \neq 1$.)

4.3. An inhomogeneous layer but no substrate

For an inhomogeneous layer without a substrate, we obtain a Sturm–Liouville problem. Combining Eqs. (34) and (35) with the boundary conditions, $U'(\zeta_0) = 0$ and $U(\zeta_h) = 0$, gives

$$AF(1-\delta, 1-\delta;\ 1-2\delta;\ \zeta_0) - B\zeta_0^{2\delta}F(1+\delta, 1+\delta;\ 1+2\delta;\ \zeta_0) = 0, \tag{36}$$

$$AF(-\delta, 1-\delta;\ 1-2\delta;\ \zeta_h) + B\zeta_h^{2\delta}F(\delta, 1+\delta;\ 1+2\delta;\ \zeta_h) = 0. \tag{37}$$

Denote the determinant of this 2×2 system by $\Delta_0(\delta)$. Recall that $\delta = \mathrm{i}\theta_0$ and θ_0 is proportional to the frequency, ω (see Eq. (30)). Thus Sturm–Liouville theory implies that there are infinitely many real values of θ_0 for which $\Delta_0(\mathrm{i}\theta_0) = 0$.

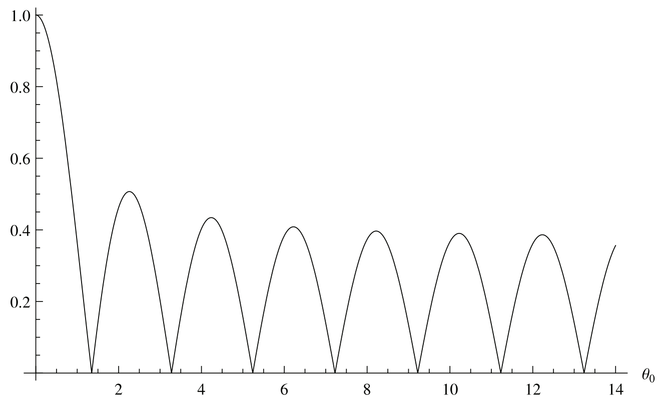

Fig. 1. A graph of $|\Delta_0(\mathrm{i}\theta_0)|/\Delta_0(0)$ as a function of θ_0, for $\zeta_0 = 0.99$ and $\zeta_h = 0.5$. The first seven zeros are $\theta_0 \simeq 1.35$, 3.28, 5.24, 7.23, 9.22, 11.23 and 13.24.

As an example, suppose that $\mu_0 = 0.02\mu(h)$ and $\mu_1 = 2\mu(h)$, whence $\zeta_h/\zeta_0 = \mathrm{e}^{-\alpha h} = 0.51$ and $\zeta_0 = 0.99$. (These numerical values are comparable to the data in [2].) A plot of $|\Delta_0(\mathrm{i}\theta_0)|$ as a function of θ_0 (obtained using Mathematica) is given in Fig. 1, where the first seven zeros are seen.

For large θ_0, an asymptotic analysis (see Appendix) shows that the zeros occur (approximately) when

$$\cos(\theta_0[\xi_h - \xi_0]) = 0, \tag{38}$$

where $\xi_h = \xi(\zeta_h)$, $\xi_0 = \xi(\zeta_0)$ and $\xi(\zeta)$ is defined by

$$\mathrm{e}^{-\xi(\zeta)} = \zeta^{-1}\left(2 - \zeta - 2\sqrt{1-\zeta}\right).$$

For $\zeta_0 = 0.99$ and $\zeta_h = 0.5$, we obtain $\xi_0 = 0.201$ and $\xi_h = 1.763$. Then Eq. (38) predicts zeros at $\theta_0 \simeq 1.01 + 2.01m$, $m = 0, 1, 2, \ldots$, which is in good agreement with the numerical results.

Before proceeding to the shear problem, it is convenient to introduce some shorthand notation. Thus let

$$\mathcal{F}(\delta; \zeta) = F(1+\delta, 1+\delta; 1+2\delta; \zeta), \tag{39}$$

$$\mathcal{G}(\delta; \zeta) = F(\delta, 1+\delta; 1+2\delta; \zeta), \tag{40}$$

so that Eqs. (36) and (37) become

$$A\mathcal{F}(-\delta; \zeta_0) - B\zeta_0^{2\delta}\mathcal{F}(\delta; \zeta_0) = 0, \tag{41}$$

$$A\mathcal{G}(-\delta; \zeta_h) + B\zeta_h^{2\delta}\mathcal{G}(\delta; \zeta_h) = 0, \tag{42}$$

with

$$\Delta_0(\delta) = \zeta_h^{2\delta}\mathcal{F}(-\delta; \zeta_0)\mathcal{G}(\delta; \zeta_h) + \zeta_0^{2\delta}\mathcal{F}(\delta; \zeta_0)\mathcal{G}(-\delta; \zeta_h). \tag{43}$$

We observe that $\Delta_0(\delta) = (\zeta_0\zeta_h)^{2\delta}\Delta_0(-\delta)$: as expected, when δ is a zero of Δ_0, so is $-\delta$.

4.4. *The shear problem: layer and substrate*

For the full shear problem, the boundary conditions are Eq. (31). The first of these yields Eq. (41). The boundary condition at $\zeta = \zeta_h$ is more complicated; it gives

$$A\mathcal{G}(-\delta; \zeta_h) + B\zeta_h^{2\delta}\mathcal{G}(\delta; \zeta_h) - T\sqrt{1-\zeta_h}\left\{A\mathcal{F}(-\delta; \zeta_h) - B\zeta_h^{2\delta}\mathcal{F}(\delta; \zeta_h)\right\} = 0,$$

where we have used $c(h)\alpha\delta/\omega = \mathrm{i}\sqrt{1-\zeta_h}$. Setting the determinant to zero gives

$$\Delta(\delta) \equiv \Delta_0(\delta) + T\sqrt{1-\zeta_h}\,\Delta_1(\delta) = 0, \tag{44}$$

where Δ_0 is given by Eq. (43) and

$$\Delta_1(\delta) = \zeta_h^{2\delta}\mathcal{F}(-\delta; \zeta_0)\mathcal{F}(\delta; \zeta_h) - \zeta_0^{2\delta}\mathcal{F}(\delta; \zeta_0)\mathcal{F}(-\delta; \zeta_h). \tag{45}$$

For small T, we can write $\delta = \delta_0 + T\delta_1$, where $\Delta_0(\delta_0) = 0$. Then, a simple perturbation argument shows that the correction $\delta_1 \simeq -\sqrt{1-\zeta_h}\,\Delta_1(\delta_0)/\Delta_0'(\delta_0)$. At first sight, this is not very convenient because it requires differentiation of hypergeometric functions with respect to their parameters. However, for large θ_0, we can use the asymptotic approximation Eq. (A.5), whence

$$\delta_1 \simeq 1/(\xi_h - \xi_0),$$

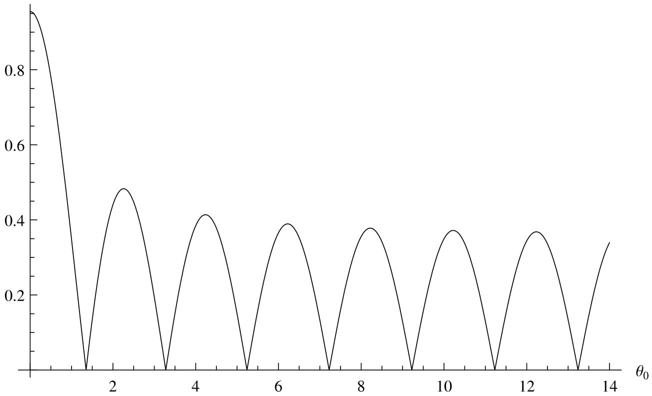

Fig. 2. A graph of $|\Delta(\mathrm{i}\theta)|/\Delta(0)$ as a function of θ_0, where $\theta = \theta_0 - 0.064\mathrm{i}$ and θ_0 is real. Also $\zeta_0 = 0.99$, $\zeta_h = 0.5$ and $T = 0.1$.

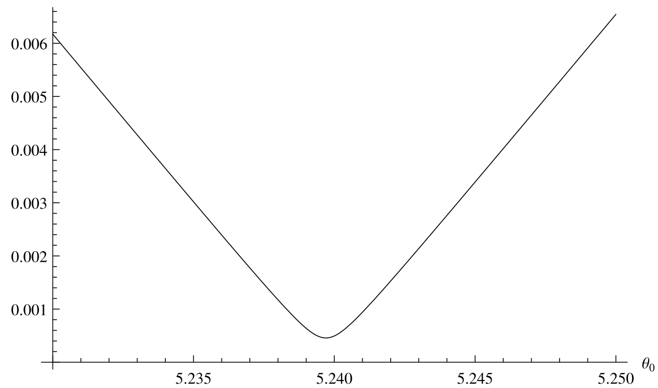

Fig. 3. A graph of $|\Delta(\mathrm{i}\theta_0)|/\Delta(0)$ as a function of θ_0, where $\theta = \theta_0 - 0.064\mathrm{i}$ and θ_0 is real. Also $\zeta_0 = 0.99$, $\zeta_h = 0.5$ and $T = 0.1$. This is a magnification of Fig. 2.

where ξ_h and ξ_0 are defined below Eq. (38). Then, with $\delta = \mathrm{i}\theta$ and $\delta_0 = \mathrm{i}\theta_0$, we find that $\Delta(\mathrm{i}\theta)$ is approximately zero when

$$\theta \simeq \theta_0 - \mathrm{i}T/(\xi_h - \xi_0). \tag{46}$$

Thus, using this approximation, the complex scattering frequencies are of the form

$$\omega = \omega_0 - \mathrm{i}\gamma,$$

where ω_0 is any solution of the Sturm–Liouville layer problem (Section 4.3) and γ is real, positive and independent of ω_0: all the complex scattering frequencies are given by displacing the (real) resonance frequencies by the same fixed amount, γ, away from the real axis. (The fact that γ_0 is positive is a consequence of causality.)

With the values of ξ_0 and ξ_h used previously, namely, $\xi_0 = 0.201$ and $\xi_h = 1.763$, together with $T = 0.1$, Eq. (46) gives $\theta \simeq \theta_0 - 0.064\mathrm{i}$, with θ_0 determined approximately by Eq. (38). Plotting $\Delta(\mathrm{i}(\theta_0 - 0.064\mathrm{i}))$ as a function of θ_0 (real) gives a graph (Fig. 2) that is almost indistinguishable from the graph of $\Delta_0(\mathrm{i}\theta_0)$ given in Fig. 1. If we magnify by a factor of more than 100, and look near the third zero, we obtain the result shown in Fig. 3: the asymptotic approximation is excellent.

4.5. A semi-infinite inhomogeneous solid

It is of interest to consider another special case, that of a semi-infinite inhomogeneous solid, obtained by letting $h \to \infty$ (there is no longer a substrate). Thus, $\zeta_h \to 0$ and the limit $z \to \infty$ corresponds to $\zeta \to 0$. In this limit, $\mu \to \mu_0 + \mu_1 \equiv \mu_\infty$, say, the shear modulus far from the fluid–solid interface. Now, from Eq. (28),

$$\zeta^{\pm\delta} = \zeta^{\pm\mathrm{i}\theta_0} = \zeta_0^{\pm\mathrm{i}\theta_0}\mathrm{e}^{\mp\mathrm{i}k_\infty z},$$

where $k_\infty = \alpha\theta_0 = \omega\sqrt{\rho_0/\mu_\infty}$ is the shear wavenumber as $z \to \infty$. Thus, as we want waves that propagate in the $+z$ direction, we take $B = 0$ in Eq. (34), giving

$$U(\zeta) = A\zeta^{-\delta}F(-\delta, 1-\delta; 1-2\delta; \zeta), \quad \delta = \mathrm{i}k_\infty/\alpha.$$

Then the boundary condition, $U'(\zeta_0) = 0$, gives (see Eqs. (35) and (39))

$$\mathcal{F}(-\delta; \zeta_0) = 0. \tag{47}$$

We are interested in finding (complex) values of ω for which Eq. (47) is satisfied. Note that δ is proportional to ω, and that $\zeta_0 = 1 - \mu_0/\mu_\infty$ is fixed. To proceed, let $\omega = (\sigma - \mathrm{i}\tau)c_\infty\alpha$, where σ and τ are real, and $c_\infty = \omega/k_\infty = \sqrt{\mu_\infty/\rho_0}$ is the shear wave speed as $z \to \infty$. Then $\delta = \mathrm{i}k_\infty/\alpha = \tau + \mathrm{i}\sigma$.

A numerical investigation shows that all solutions for δ are real and positive, meaning that $\mathrm{Re}\,\omega = 0$. For example, if $\zeta_0 = 0.99$, we find that the smallest value of δ is about 0.9797. The fact that we do not find solutions with $\mathrm{Re}\,\omega \neq 0$ implies that we do not expect to find shear-wave resonances with an inhomogeneous half-space: the interface at $z = h$ plays an important role.

5. Discussion

We have shown how to construct exact solutions of the shear problem in certain special situations. It is clear that we could solve the analogous "compressional problem" (see Section 2.2), for p in the water, w in the layer, and w_s in the substrate. To do this, we would assume that

$$\frac{\mu(z)}{\lambda(z)} = \frac{1-2\nu}{2\nu}$$

is constant; here, ν is Poisson's ratio. (In the applications discussed in [2,3], $0 < \mu/\lambda \ll 1$, so that $\nu \simeq \frac{1}{2}$: the inhomogeneous layer is almost incompressible.)

For waves at oblique incidence, the full coupled problem (Section 2.1) would have to be tackled. Solutions can be constructed by the method of Frobenius. This has been done by Vrettos [6] for Rayleigh-like waves at the surface of a semi-infinite inhomogeneous solid. His approach could be adapted to the present problem.

Acknowledgment

I thank Jon Collis for sparking my interest in the problem discussed.

Appendix. Some asymptotics

The Sturm–Liouville problem solved in Section 4.3 led to a determinant, $\Delta_0(\delta)$, given in terms of hypergeometric functions. Zeros of Δ_0 were found numerically. Here, we are going to find a simple approximation for these zeros by estimating $\Delta_0(\delta)$ as $|\delta| \to \infty$. To do this, we use an asymptotic approximation due to G.N. Watson (see [19, p. 77, Eq. (16)]):

$$\begin{aligned}&(2/[z-1])^{\delta+a} F(a+\delta, a-c+1+\delta; a-b+1+2\delta; 2/[1-z]) \\ &\quad \sim \frac{2^{a+b}\sqrt{\pi}\,\Gamma(a-b+1+2\delta)}{\delta^{1/2}\Gamma(a-c+1+\delta)\,\Gamma(c-b+\delta)} \frac{\mathrm{e}^{-(\delta+a)\xi}(1-\mathrm{e}^{-\xi})^{-c+1/2}}{(1+\mathrm{e}^{-\xi})^{a+b-c+1/2}}\end{aligned} \tag{A.1}$$

as $|\delta| \to \infty$, $|\arg\delta| < \pi$. (For a related expansion, see [14, Eq. (36)].) The quantity ξ is defined by $z \pm (z^2-1)^{1/2} = \mathrm{e}^{\pm\xi}$. To simplify the calculations, we first make a preliminary transformation, using one of Kummer's relations [19, p. 105, Eq. (3)],

$$F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; z/[z-1]). \tag{A.2}$$

Let us start with $\mathcal{F}(\delta;\zeta)$, defined by Eq. (39). We have

$$\begin{aligned}\mathcal{F}(\delta;\zeta) &= F(1+\delta, 1+\delta; 1+2\delta; \zeta) \\ &= (1-\zeta)^{-1-\delta} F(\delta, 1+\delta; 1+2\delta; \zeta/[\zeta-1]) \\ &\sim \Lambda(\delta) P(\delta;\zeta) \quad \text{as } |\delta| \to \infty,\end{aligned}$$

where

$$\Lambda(\delta) = \frac{\sqrt{\pi}\,\Gamma(1+2\delta)}{\delta^{1/2}\Gamma(1+\delta)\,\Gamma(\delta)}, \qquad P(\delta;\zeta) = \frac{\mathrm{e}^{-\delta\xi}}{(1-\zeta)\zeta^{\delta}} \frac{(1-\mathrm{e}^{-\xi})^{1/2}}{(1+\mathrm{e}^{-\xi})^{1/2}},$$

we used Eq. (A.1) with $a = b = c = 0$ and $2/(1-z) = \zeta/(\zeta-1)$, whence $z = (2/\zeta) - 1$ and $\xi(\zeta)$ is defined by

$$\mathrm{e}^{-\xi(\zeta)} = \zeta^{-1}\left(2-\zeta-2\sqrt{1-\zeta}\right).$$

Similarly, from Eq. (40),

$$\begin{aligned}\mathcal{G}(\delta;\zeta) &= F(1+\delta, \delta; 1+2\delta; \zeta) \\ &= (1-\zeta)^{-1-\delta} F(1+\delta, 1+\delta; 1+2\delta; \zeta/[\zeta-1]) \\ &\sim \Lambda(\delta) Q(\delta;\zeta) \quad \text{as } |\delta| \to \infty,\end{aligned}$$

where we used Eq. (A.1) with $a = b = c = 1$ and

$$Q(\delta;\zeta) = \frac{4}{\zeta^{\delta+1}} \frac{\mathrm{e}^{-(\delta+1)\xi}}{(1-\mathrm{e}^{-\xi})^{1/2}(1+\mathrm{e}^{-\xi})^{3/2}}.$$

Next, from Eq. (43),

$$\begin{aligned}\Delta_0(\delta) &= \zeta_h^{2\delta}\mathcal{F}(-\delta;\zeta_0)\mathcal{G}(\delta;\zeta_h) + \zeta_0^{2\delta}\mathcal{F}(\delta;\zeta_0)\mathcal{G}(-\delta;\zeta_h)\\ &\sim \Lambda(\delta)\Lambda(-\delta)\left\{\zeta_h^{2\delta}Q(\delta;\zeta_h)P(-\delta;\zeta_0) + \zeta_0^{2\delta}P(\delta;\zeta_0)Q(-\delta;\zeta_h)\right\}\end{aligned} \tag{A.3}$$

as $|\delta| \to \infty$. But, as $\pm\delta = \theta_0\,e^{\pm i\pi/2}$,

$$\begin{aligned}\Lambda(\delta)\Lambda(-\delta) &= \frac{\pi\,\Gamma(1+2\delta)\,\Gamma(1-2\delta)}{\theta_0\,\Gamma(1+\delta)\,\Gamma(1-\delta)\,\Gamma(\delta)\,\Gamma(-\delta)}\\ &= \frac{\pi}{\theta_0}\,\frac{2\delta\,\Gamma(2\delta)\,\Gamma(1-2\delta)}{\Gamma(1-\delta)\,\Gamma(\delta)\,\Gamma(1-[-\delta])\,\Gamma(-\delta)}\\ &= \frac{2\pi\delta}{\theta_0}\,\frac{\pi}{\sin 2\pi\delta}\,\frac{\sin\pi\delta}{\pi}\,\frac{\sin(-\pi\delta)}{\pi}\\ &= -i\tan\pi\delta = \tanh\pi\theta_0,\end{aligned}$$

using $\Gamma(z)\Gamma(1-z) = \pi/(\sin\pi z)$ thrice. Then, substitution in Eq. (A.3) gives

$$\Delta_0(\delta) \sim \frac{8(1-e^{-\xi_0})^{1/2}\,e^{-\xi_h}\tanh(\pi\theta_0)(\zeta_0\zeta_h)^{\delta}\cosh(\delta[\xi_h-\xi_0])}{\zeta_h(1-\zeta_0)(1+e^{-\xi_0})^{1/2}(1-e^{-\xi_h})^{1/2}(1+e^{-\xi_h})^{3/2}}, \tag{A.4}$$

where $\xi_0 = \xi(\zeta_0)$ and $\xi_h = \xi(\zeta_h)$. Thus, for large θ_0, there are zeros of $\Delta_0(\delta)$ at $\delta = i\theta_0$ where $\cos(\theta_0[\xi_h - \xi_0]) = 0$. This estimate is compared with direct numerical computations in Section 4.3; see Eq. (38).

For the full shear problem, with layer bonded to substrate, we can derive a similar estimate for Δ_1, defined by (45). Thus

$$\begin{aligned}\Delta_1(\delta) &= \zeta_h^{2\delta}\mathcal{F}(-\delta;\zeta_0)\mathcal{F}(\delta;\zeta_h) - \zeta_0^{2\delta}\mathcal{F}(\delta;\zeta_0)\mathcal{F}(-\delta;\zeta_h)\\ &\sim \Lambda(\delta)\Lambda(-\delta)\left\{\zeta_h^{2\delta}P(\delta;\zeta_h)P(-\delta;\zeta_0) - \zeta_0^{2\delta}P(\delta;\zeta_0)P(-\delta;\zeta_h)\right\}\\ &= \frac{2\tanh(\pi\theta_0)}{(1-\zeta_0)(1-\zeta_h)}\sqrt{\frac{(1-e^{-\xi_0})(1-e^{-\xi_h})}{(1+e^{-\xi_0})(1+e^{-\xi_h})}}(\zeta_0\zeta_h)^{\delta}\sinh(\delta[\xi_0-\xi_h]).\end{aligned}$$

Then, if δ_0 solves $\Delta_0(\delta_0) = 0$, and we write the estimate in Eq. (A.4) as $\Delta_0(\delta) \sim \mathcal{A}(\delta)\cosh(\delta[\xi_h - \xi_0])$, we infer that

$$\Delta_0'(\delta_0) \sim (\xi_h - \xi_0)\mathcal{A}(\delta_0)\sinh(\delta_0[\xi_h - \xi_0]).$$

Direct calculation then gives

$$\frac{\Delta_1(\delta_0)}{\Delta_0'(\delta_0)} \sim \frac{\zeta_h\sinh\xi_h}{2(\xi_0-\xi_h)(1-\zeta_h)} = \frac{1}{(\xi_0-\xi_h)\sqrt{1-\zeta_h}}. \tag{A.5}$$

This estimate is used in Section 4.4.

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