

One-dimensional reflection by a semi-infinite periodic row of scatterers



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HIGHLIGHTS

- A canonical one-dimensional scattering problem is solved exactly in 3 ways.
- First approach: shift semi-infinite row by one period.
- Second approach: solve for finite row and then take limit.
- Third approach: solve semi-infinite problem (discrete Wiener–Hopf).

ARTICLE INFO

Article history:

Received 16 January 2015

Received in revised form 18 June 2015

Accepted 28 June 2015

Available online 3 July 2015

Keywords:

Semi-infinite array

Reflection

Band-gap

Point scatterer

ABSTRACT

Three methods are described in order to solve the canonical problem of the one-dimensional reflection by a semi-infinite periodic row of identical scatterers. The exact reflection coefficient R is determined. The first method is associated with shifting the domain by a single period and subsequently considering two scatterers, one being a single scatterer and the second being the entire semi-infinite array. The second method determines the reflection coefficient R_N associated with a finite array of N scatterers. The limit as $N \rightarrow \infty$ is then taken. In general R_N does *not* converge to R in this limit, although we summarize various arguments that can be made to ensure the *correct* limit is achieved. The third method considers direct approaches. In particular, for point masses, the governing inhomogeneous ordinary differential equation is solved using the discrete Wiener–Hopf technique.

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1. Introduction

The wave bearing properties of periodic inhomogeneous media are important in many application areas in physics, engineering and applied mathematics and have been studied extensively [1]. The standard procedure is to consider unbounded inhomogeneous media, for which Floquet–Bloch conditions are imposed in order to determine their band-gap properties. Ranges of frequencies where the predicted Bloch wavenumber is complex are associated with stopbands where no propagation is permitted in the structure. Of course, in practise, no periodic medium is of infinite extent, and so inspection of the propagation characteristics of such periodic materials involves inclusion of boundary or interface conditions. These are less-well studied models; for example, the wave propagation from a semi-infinite homogeneous material into a semi-infinite or finite-width heterogeneous periodic medium, where the aim is to predict the reflection and transmission coefficients, R and T , respectively. The link between this and the unbounded problem is the expectation that $|R| = 1$ ($T = 0$) for incident

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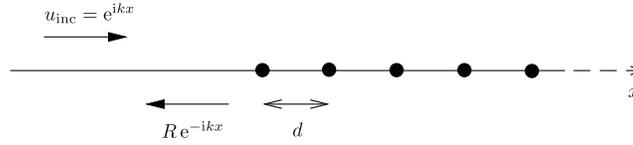


Fig. 1. Depiction of a time-harmonic wave incident on, and reflected from, a semi-infinite array of identical scatterers located at $x = nd$, $n = 0, 1, 2, \dots$. The scatterers are sketched as beads, although other kinds of inhomogeneities are permitted in the analysis.

frequencies that reside in band-gaps, i.e. all the incident energy is reflected. However, the reflection and/or transmission coefficients are measured by experimentalists to infer all the effective properties of the heterogeneous material, not just the band-gaps, and hence it is important to understand more fully the semi-infinite scattering problem. This article offers exact procedures for a simple canonical problem in order to illustrate possible approaches that may be applicable for tackling more complex heterogeneous periodic materials.

Here we consider the problem of steady one-dimensional reflection of waves by a semi-infinite periodic row of identical scatterers. As illustrated in Fig. 1 the scatterers are located at $x = nd$, where $n = 0, 1, 2, \dots$, and $d > 0$ is the spacing. Between the scatterers, the motion is governed by a wave equation for the displacement $U(x, t) = \text{Re} \{u(x)e^{-i\omega t}\}$, and so $u(x)$ satisfies the reduced equation $u''(x) + k^2u(x) = 0$, where k is real and positive. As the field is steady, the time-dependence $e^{-i\omega t}$ is henceforth suppressed. The forcing for the problem is a unit-amplitude incident wave, $u_{\text{inc}} = e^{ikx}$, propagating towards the right (x increasing), the main aim is to calculate the reflected wave, $R e^{-ikx}$, where R is the (complex) reflection coefficient.

The literature on this one-dimensional “reflection problem” is substantial; some of it will be discussed below. The problem itself is interesting for a number of reasons, not least that it can be solved explicitly, and so it is one of our purposes to give this solution. The goal is to express R in terms of the scattering properties (reflection and transmission coefficients) of the constituent scatterers and kd . Inevitably, properties of the periodic structure (passbands and stopbands) will play a role. Once R is known, it can be used for other related problems, such as scattering by additional scatterers to the left of the semi-infinite row, or for studying the effects of defects between two semi-infinite rows (one extending to $x = +\infty$ and one extending to $x = -\infty$).

We describe three ways to solve the reflection problem. The first way is very simple (Section 4). We exploit the consequences of shifting the semi-infinite row by one period (to the right or left). In effect, we regard the semi-infinite row as two scatterers, one of which is another semi-infinite row. This idea goes back to a series of papers by Millar in the 1960s, starting with [2]. He used it for several two-dimensional grating problems. A similar approach was used for layered media by Shenderov [3]. In our one-dimensional context, we obtain a quadratic equation for R ; we show how to select the correct solution. We remark that there has been much recent interest in related two-dimensional waveguide problems; see, for example, [4–6], where the shifting-by-one-period idea is again employed, leading to a quadratic equation for a certain operator.

Another feature of the “solution-by-shifting” is that the Floquet–Bloch relation for periodic media, $u(x + d) = e^{iqd}u(x)$ (see Section 3), is not used. Nevertheless, we show subsequently that, in fact, it is satisfied (Section 4.2). Then, as an example, we show that our formula for R agrees with one obtained by Levine [7] for point scatterers (Section 4.3).

The second way to find R is to start with reflection (and transmission) by a finite periodic row of N scatterers, followed by taking the limit as $N \rightarrow \infty$ (Section 5). Explicit formulas for the reflection and transmission coefficients, R_N and T_N , are known. It turns out that $R_N \rightarrow R$ as $N \rightarrow \infty$, where R comes from the shifting method, but only when we are in a stopband for the periodic structure. In a passband, R_N does not have a well-defined limit as $N \rightarrow \infty$. In the literature, methods have been devised for finding a sensible limiting solution, by introducing some kind of loss into the system: we show that doing this leads to R , as found by the shifting method of Section 4.

The third way is to solve the reflection problem (with a semi-infinite row) directly. Some possible approaches are described in Section 6. After a description of Levine’s work [8] (Section 6.1), we solve the problem using a discrete form of the Wiener–Hopf technique (Section 6.2).

Perhaps inevitably, there are other ways to solve the problem, which we do not discuss. For example, one such approach is developed in [9].

2. One scatterer

Suppose that we have just one scatterer, located at the origin. Assume it is inside a “cell”, $|x| < a$, where $0 < a < \frac{1}{2}d$. The properties of the scatterer are encoded in the cell’s reflection and transmission coefficients, r_{\pm} and t_{\pm} , defined as follows. For a wave incident from the left, we have

$$u(x) \equiv u_+(x) = \begin{cases} e^{ikx} + r_+ e^{-ikx}, & x < -a, \\ t_+ e^{ikx}, & x > a. \end{cases} \quad (1)$$

Similarly, for a wave incident from the right,

$$u(x) \equiv u_-(x) = \begin{cases} t_- e^{-ikx}, & x < -a, \\ r_- e^{ikx} + e^{-ikx}, & x > a. \end{cases} \quad (2)$$

We assume that the scatterers are lossless. Then it is well known that

$$t_+ = t_- \equiv t, \quad 1 - |t|^2 = |r_\pm|^2 \equiv |r|^2, \quad r_+^* t + r_- t^* = 0, \quad (3)$$

where the asterisk denotes complex conjugation. If $t \neq 0$ and $r \neq 0$, and we write

$$r_\pm = |r| e^{i\rho_\pm} \quad \text{and} \quad t = |t| e^{i\tau},$$

the third of Eq. (3) reduces to a constraint on the phases,

$$e^{2i\tau} + e^{i(\rho_+ + \rho_-)} = 0. \quad (4)$$

If the cell is moved from $x = 0$ to $x = b$, it is easy to show that the new reflection coefficients are $r_+ e^{2ikb}$ and $r_- e^{-2ikb}$, whereas the transmission coefficient remains unchanged.

Henceforth, we assume that r_\pm and t are known or, at least, they can be computed. Then our problem reduces to calculating R in terms of r_\pm and t .

For a specific example, we can take a point scatterer at $x = 0$. Then $a = 0$,

$$u(0^+) = u(0^-) \quad \text{and} \quad u'(0^+) - u'(0^-) = Mu(0), \quad (5)$$

where M is a real constant. We find $r_+ = r_- = r$, say, $t = 1 + r$,

$$r = \frac{M}{2ik - M}, \quad t = \frac{2ik}{2ik - M}, \quad e^{i\tau} = |t| \left(1 - \frac{iM}{2k} \right). \quad (6)$$

This example was considered by Levine [7]; his α is our M . For a physical interpretation, the scatterer could be a bead of mass m_0 on a string of linear density ρ_0 , and then $M = -m_0 k^2 / \rho_0$ [8, Eq. (17.4)].

3. Periodic media: brief review

Wave propagation in periodic media is well studied. We review some aspects, because of their relevance to our problem.

Classically, Floquet theory covers waves in periodic media, where the governing ordinary differential equation is $v''(x) + k^2 n(x)v(x) = 0$ and $n(x)$ is a periodic function with period d ; $v(x)$ and $n(x)$ are defined for all x . This may not valid as we do not necessarily assume that the behaviour everywhere inside the cell is governed by a differential equation, e.g. we may allow points of discontinuity.

The main result of Floquet theory is that all solutions have the form $v(x) = e^{iqx} w(x)$ where q is a constant and $w(x)$ is d -periodic. A consequence is that

$$v(x + d) = e^{iqd} v(x). \quad (7)$$

Extensions to other situations, including partial differential equations, are usually associated with the name of Felix Bloch: we shall refer to Eq. (7) as the *Floquet–Bloch relation*.¹

Returning to our problem, there are several ways to proceed. We can connect the fields in adjacent regions (where $u' + k^2 u = 0$; see Section 3.1) across the cells, or we can connect the fields in neighbouring *periods* of the periodic structure (Section 3.2). Conceptually, these are different approaches but, of course, they lead to the same results, including a dispersion relation connecting qd to kd and properties of the constituent scatterers.

3.1. Connect across one cell

Consider the cell at $x = md$. To its left, we can write

$$u(x) = A_m e^{+ikx} + B_m e^{-ikx}, \quad (m-1)d + a < x < md - a,$$

and to its right,

$$u(x) = A_{m+1} e^{+ikx} + B_{m+1} e^{-ikx}, \quad md + a < x < (m+1)d - a.$$

¹ The terminology varies. Floquet [10] studied solutions of systems of first-order ordinary differential equations with periodic coefficients, but he did not consider wave motion. That was done later by Bloch [11] in the context of the three-dimensional Schrödinger equation with a periodic potential. For further discussion, see [1].

Using the cell's transmission and reflection coefficients, t and $r_{\pm}^{(m)} = r_{\pm} e^{\pm 2ikmd}$, we obtain

$$A_{m+1} = tA_m + r_-^{(m)} B_{m+1}, \quad B_m = r_+^{(m)} A_m + tB_{m+1}.$$

Rearranging, using $r_+^{(m)} r_-^{(m)} = r_+ r_-$ and $t - r_+ r_- / t = 1/t^*$, we obtain

$$\begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix} = \begin{pmatrix} 1/t^* & r_-^{(m)}/t \\ -r_+^{(m)}/t & 1/t \end{pmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix}. \quad (8)$$

Now, in a periodic medium, the Floquet–Bloch relation, Eq. (7), says that

$$u(x+d) = e^{iqd} u(x), \quad (9)$$

for some q . This gives $A_{m+1} e^{ikd} = A_m e^{iqd}$ and $B_{m+1} e^{-ikd} = B_m e^{iqd}$. Substituting in Eq. (8) gives an eigenvalue problem for q . Thus, if we put $\lambda = e^{iqd}$, we find that

$$\lambda^2 - 2\xi\lambda + 1 = 0, \quad (10)$$

where ξ is a real parameter defined by

$$\xi = \operatorname{Re} \{ t^{-1} e^{-ikd} \} = |t|^{-1} \cos(kd + \tau). \quad (11)$$

The quadratic equation (10) can be written as $\cos qd = \xi$, giving the familiar interpretation in terms of passbands ($|\xi| < 1$) and stopbands ($|\xi| > 1$). In particular, for point scatterers (see Eq. (6)), we obtain

$$\cos qd = \cos kd + \frac{M}{2k} \sin kd. \quad (12)$$

3.2. Connect across one period

Consider neighbouring cells, at $x = md$ and $x = (m+1)d$. In the interval $X-d < md < X$, for some X with $a < X - md < d - a$, we can write

$$u(x) = \mathcal{A}_m u_+^{(m)}(x) + \mathcal{B}_m u_-^{(m)}(x), \quad (13)$$

where \mathcal{A}_m and \mathcal{B}_m are coefficients, and $u_{\pm}^{(m)}$ are fields defined by Eqs. (1) and (2) when the cell is moved from $x = 0$ to $x = md$; this movement changes r_{\pm} to $r_{\pm}^{(m)} = r_{\pm} e^{\pm 2imkd}$. Similarly, in the next interval, from X to $X+d$ (which contains the cell at $x = (m+1)d$), we can write

$$u(x) = \mathcal{A}_{m+1} u_+^{(m+1)}(x) + \mathcal{B}_{m+1} u_-^{(m+1)}(x). \quad (14)$$

The fields in Eqs. (13) and (14) should match at $x = X$ (as should their derivatives):

$$\mathcal{A}_m t e^{ikX} + \mathcal{B}_m (e^{-ikX} + r_-^{(m)} e^{ikX}) = \mathcal{A}_{m+1} (e^{ikX} + r_+^{(m+1)} e^{-ikX}) + \mathcal{B}_{m+1} t e^{-ikX}.$$

As X can vary, we obtain

$$\mathcal{A}_m t + \mathcal{B}_m r_-^{(m)} = \mathcal{A}_{m+1}, \quad \mathcal{B}_m = \mathcal{A}_{m+1} r_+^{(m+1)} + \mathcal{B}_{m+1} t,$$

which we write as

$$\mathcal{A}_{m+1} = \mathcal{A}_m t + \mathcal{B}_m r_-^{(m)}, \quad \mathcal{B}_{m+1} = \mathcal{B}_m t^{-1} \left(1 - r_-^{(m)} r_+^{(m+1)} \right) - \mathcal{A}_m r_+^{(m+1)}.$$

These determine \mathcal{A}_{m+1} and \mathcal{B}_{m+1} from \mathcal{A}_m and \mathcal{B}_m . In our scattering problem we know \mathcal{A}_0 (from the specified incident wave) but not \mathcal{B}_0 (in fact, $R = r_+ + \mathcal{B}_0 t$). For a finite row of scatterers, a second piece of information comes from the radiation condition: there are only right-going waves to the right of the row. However, for a semi-infinite row, we do not have an obvious radiation condition. We could impose the Floquet–Bloch condition, Eq. (9), as appropriate for periodic media, but it is not clear that this is correct. Doing this is equivalent to requiring that

$$\mathcal{A}_{m+1} e^{ikd} = \lambda \mathcal{A}_m \quad \text{and} \quad \mathcal{B}_{m+1} e^{-ikd} = \lambda \mathcal{B}_m$$

for some λ . Imposing this condition gives an eigenvalue problem for λ ,

$$\mathcal{A}_m (t - \lambda e^{-ikd}) + \mathcal{B}_m r_-^{(m)} = 0, \quad \mathcal{A}_m r_+^{(m+1)} + \mathcal{B}_m \{ \lambda e^{ikd} - t^{-1} (1 - r_+ r_- e^{2ikd}) \} = 0.$$

Setting the determinant of this system to zero gives precisely Eq. (10).

4. First method: move the semi-infinite row

If we move the semi-infinite row by one period to the right, the new reflection coefficient is $\text{Re}^{2ikd} = R_1$, say. But we can also view the original row as comprising a single cell located at $x = 0$ to the left of a semi-infinite row with cells located at $x = nd$, $n = 1, 2, \dots$

Between the cells at $x = 0$ and $x = d$ (in fact, for $a < x < d - a$), we can write $u(x) = A_1 e^{ikx} + B_1 e^{-ikx}$. The wave incident on the cell at $x = d$, $A_1 e^{ikx}$, is reflected as $A_1 R_1 e^{-ikx}$. Thus

$$B_1 = A_1 R_1 = A_1 \text{Re}^{2ikd}. \quad (15)$$

The wave incident on the cell at $x = 0$ consists of e^{ikx} from the left and $B_1 e^{-ikx}$ from the right. Thus

$$R = r_+ + B_1 t \quad \text{and} \quad A_1 = t + B_1 r_-. \quad (16)$$

4.1. Calculation of R

We have three equations for A_1 , B_1 and R , Eqs. (15) and (16). When $t = 0$, they give $R = r_+ = e^{i\rho_+}$, $A_1 = B_1 r_- = B_1 e^{-2ikd}/r_+$, and so $A_1 = B_1 = 0$ unless $kd + (\rho_+ + \rho_-)/2 = n\pi$ for some integer n . In this exceptional case, the field between adjacent scatterers is not uniquely determined because standing waves are allowed. Similarly, when $|r| = 0$, Eqs. (15) and (16) give $A_1 = t$, $R = B_1 t$ and $R = \text{Re}^{2i(kd+\tau)}$, whence $R = 0$ unless $kd + \tau = n\pi$ for some integer n . In this exceptional case, R is not uniquely defined because a wave can pass freely through the row, from right to left.

Suppose now that $t \neq 0$ and $|r| \neq 0$. Eliminating A_1 and B_1 from Eqs. (15) and (16) gives

$$r_- e^{ikd} R^2 + [(t^2 - r_+ r_-) e^{ikd} - e^{-ikd}] R + r_+ e^{-ikd} = 0, \quad (17)$$

a quadratic equation for R . We have

$$t^2 - r_+ r_- = |t|^2 e^{2i\tau} - |r|^2 e^{i(\rho_+ + \rho_-)} = (|t|^2 + |r|^2) e^{2i\tau} = e^{2i\tau},$$

using Eqs. (3)₂ and (4). Thus Eq. (17) becomes

$$r_- e^{ikd} R^2 + 2ie^{i\tau} S R + r_+ e^{-ikd} = 0, \quad (18)$$

where $S = \sin(kd + \tau)$. When $S = 0$, the quadratic gives $R = e^{i\rho_+}$. When $S \neq 0$, define

$$\Delta = \begin{cases} \sqrt{|r|^2 - S^2} = C\sqrt{1 - |t|^2/C^2}, & |r|^2 > S^2, \\ -i\sqrt{S^2 - |r|^2} = -iS\sqrt{1 - |r|^2/S^2}, & |r|^2 < S^2, \end{cases} \quad (19)$$

where $C = \cos(kd + \tau)$. Then, solving the quadratic for R , we take the solution

$$R = -\frac{1}{r_-} e^{i(\tau - kd)} (\Delta + iS), \quad (20)$$

which has the alternative form

$$R = \frac{r_+ e^{-i(kd+\tau)}}{\Delta - iS}, \quad (21)$$

using $e^{i\tau} |r|^2 / r_- = -r_+ e^{-i\tau}$. These expressions have the correct limiting behaviour when $t \rightarrow 0$ and when $|r| \rightarrow 0$. Thus, from Eq. (21), $R \sim \frac{1}{2} i(r_+/S) e^{-i(kd+\tau)}$ as $|r| \rightarrow 0$, a result that also can be seen by balancing the second and third terms in Eq. (18).

From Eq. (20), we have

$$|R|^2 = |r|^{-2} \{ |\Delta|^2 + S^2 - iS(\Delta - \Delta^*) \}.$$

Thus $|R|^2 = 1$ when $|r|^2 > S^2$ (all the incident energy is reflected). However, $|R|^2 < 1$ when $|r|^2 < S^2$ (some of the energy is reflected but some passes through the periodic row); in detail,

$$1 - |R|^2 = 2(S^2/|r|^2) \left\{ 1 - \sqrt{1 - |r|^2/S^2} \right\} \sqrt{1 - |r|^2/S^2} > 0.$$

We note that the change between $|R|^2 = 1$ and $|R|^2 < 1$ occurs when

$$|r|^2 = S^2 = \sin^2(kd + \tau), \quad \text{equivalently} \quad |t|^2 = C^2 = \cos^2(kd + \tau). \quad (22)$$

As the case $t = 0$ is of no interest here (see top of Section 4.1), let us define

$$\xi = |t|^{-1} \cos(kd + \tau) \quad \text{and} \quad \sigma = |t|^{-1} \sin(kd + \tau); \quad (23)$$

the parameter ξ occurred in the Floquet–Bloch analysis for a periodic medium (see Eq. (11)). Then Eq. (20) gives

$$R = -\frac{t}{r_-} e^{-ikd} \left\{ i\sigma + \xi \sqrt{1 - \xi^{-2}} \right\}, \quad \xi^2 > 1, \quad (24)$$

and

$$R = -i\sigma \frac{t}{r_-} e^{-ikd} \left\{ 1 - \sqrt{\frac{1 - \xi^2}{|t|^{-2} - \xi^2}} \right\}, \quad \xi^2 < 1. \quad (25)$$

In these formulas, the square-roots are taken to be positive. Equivalent formulas were obtained by Kriegsmann [12, Section 3.1].

When $\xi^2 > 1$, we are in a stopband for the periodic structure and we have perfect reflection: $|R| = 1$. When $\xi^2 < 1$, we are in a passband, and then $|R| < 1$; we examine this case in more detail in Section 4.3.

4.2. Field between the scatterers

We can calculate the field between the cells: does it satisfy the Floquet–Bloch relation, Eq. (9)? Let us start by calculating the field between the cells at $x = 0$ and $x = d$ in terms of R . We have

$$u(x) = A_1 e^{ikx} + B_1 e^{-ikx} = \frac{R - r_+}{t} \left(\frac{1}{R} e^{-2ikd} e^{ikx} + e^{-ikx} \right), \quad a < x < d - a.$$

Next, consider the field between the cells at $x = d$ and $x = 2d$, and write it as $u(x) = A_2 e^{ikx} + B_2 e^{-ikx}$. The wave incident on the cell at $x = 2d$, $A_2 e^{ikx}$, is reflected as $A_2 R_2 e^{-ikx}$, where $R_2 = R e^{4ikd}$, whence $B_2 = A_2 R_2 = A_2 R e^{4ikd}$. The wave incident on the cell at $x = d$ consists of $A_1 e^{ikx}$ from the left and $B_2 e^{-ikx}$ from the right. Thus

$$B_1 = A_1 r_+ e^{2ikd} + B_2 t \quad \text{and} \quad A_2 = t A_1 + B_2 r_- e^{-2ikd}.$$

These can be rearranged to give

$$\begin{aligned} A_2 &= t A_1 + r_- e^{-2ikd} t^{-1} \{ B_1 - A_1 r_+ e^{2ikd} \} \\ &= A_1 t^{-1} \{ t^2 - r_+ r_- + r_- R \} = A_1 t^{-1} \{ e^{2i\tau} + r_- R \}, \\ B_2 &= t^{-1} \{ B_1 - A_1 r_+ e^{2ikd} \} = B_1 t^{-1} \{ 1 - r_+ R^{-1} \}, \end{aligned}$$

that is, A_2 and B_2 are written proportional to A_1 and B_1 , respectively.

Now, if the Floquet–Bloch relation, Eq. (9), is satisfied, we should have $A_2 e^{ikd} = \lambda A_1$ and $B_2 e^{-ikd} = \lambda B_1$, for some λ (see below Eq. (9)). Using these relations, and then eliminating λ , we obtain

$$e^{ikd} t^{-1} \{ e^{2i\tau} + r_- R \} = e^{-ikd} t^{-1} \{ 1 - r_+ R^{-1} \},$$

which reduces precisely to Eq. (18): the fields inside the semi-infinite row do indeed satisfy the Floquet–Bloch relation.

4.3. Application to point scatterers

Let us consider point scatterers in a passband: we have Eqs. (6) and (12), with qd real. Suppose that S and $\sin qd$ are positive. (Care with signs is needed because of the square-roots.) Then Eqs. (21) and (19) give

$$\begin{aligned} \text{Re}^{ikd} &= \frac{ire^{-i\tau}}{S + \sqrt{S^2 - |r|^2}} = \frac{\mu |t|}{\sin(kd + \tau) + |t| \sin qd} \\ &= \frac{\mu}{\sin kd - \mu \cos kd + \sin qd} \\ &= \frac{\cos qd - \cos kd}{(\sin kd + \sin qd) \sin kd - (\cos qd - \cos kd) \cos kd} \\ &= \frac{\cos qd - \cos kd}{1 - \cos([q + k]d)} = \frac{\sin([k - q]d/2)}{\sin([k + q]d/2)}, \end{aligned} \quad (26)$$

where $\mu = \frac{1}{2}M/k = ir/t$. This elegant formula was found by Levine [7, Eq. (38)], using a different method (see Section 6).

Let us return to the matter of care with signs. We find that

$$\text{Re}^{ikd} = \frac{\cos qd - \cos kd}{1 - \cos qd \cos kd + \sin kd |\sin qd| \text{sgn} S},$$

with $S = |t| \{ \sin kd - \mu \cos kd \}$. Thus, we obtain Eq. (26) when $S \sin qd > 0$ but the fraction on the right-hand side must be inverted when $S \sin qd < 0$.

5. Second method: start with a finite row

Consider a periodic row of N identical cells, centred at $x = nd, n = 0, 1, 2, \dots, N - 1$. We define reflection and transmission coefficients, R_N and T_N , by

$$u(x) = \begin{cases} e^{ikx} + R_N e^{-ikx}, & x < -a, \\ T_N e^{ikx}, & x > (N - 1)d + a. \end{cases}$$

We investigate letting $N \rightarrow \infty$ so as to solve the semi-infinite-row problem.

The quantities R_N and T_N are known exactly,

$$R_N = \frac{r_+ U_{N-1}(\xi)}{U_{N-1}(\xi) - t e^{ikd} U_{N-2}(\xi)}, \quad T_N = \frac{t e^{-i(N-1)kd}}{U_{N-1}(\xi) - t e^{ikd} U_{N-2}(\xi)}, \quad (27)$$

where U_n is a Chebyshev polynomial of the second kind, defined by

$$U_{m-1}(\cos \theta) = \frac{\sin m\theta}{\sin \theta}, \quad m = 0, 1, 2, \dots, \quad (28)$$

and ξ is the real parameter defined by Eq. (11).

5.1. Comments on the formulas for R_N and T_N

The formulas in Eqs. (27) are old. They were obtained by Mauguin in 1936 [13, p. 234]; see also [7, p. 109] and [8, p. 314, problem 3], and can also be derived via powers of a propagation matrix

$$P = \begin{pmatrix} e^{ikd}/t^* & e^{ikd} r_- / t \\ -e^{-ikd} r_+ / t & e^{-ikd}/t \end{pmatrix}, \quad (29)$$

see [14]. Note that the eigenvalues of P are given by Eq. (10).

Direct calculation shows that $|R_N|^2 + |T_N|^2 = 1$, as required (for any ξ). To see this, use the recurrence relation

$$U_N(\xi) = 2\xi U_{N-1}(\xi) - U_{N-2}(\xi), \quad U_N U_{N-2} = U_{N-1}^2 - 1$$

and $2\xi |t|^2 = t e^{ikd} + t^* e^{-ikd}$, giving

$$|T_N|^2 = 1 - |R_N|^2 = \frac{|t|^2}{|t|^2 + |r|^2 U_{N-1}^2(\xi)}. \quad (30)$$

5.2. Letting $N \rightarrow \infty$ when $|\xi| > 1$

Suppose that $\xi > 1$. (The case $\xi < -1$ is similar.) Put $\xi = \cosh \eta$ with $\eta > 0$ giving

$$U_{N-1}(\xi) = \frac{\sinh N\eta}{\sinh \eta} \sim \frac{e^{N\eta}}{2 \sinh \eta} \quad \text{as } N \rightarrow \infty.$$

As U_{N-1} grows exponentially with N , we see from Eq. (30) that $T_N \rightarrow 0$ and $|R_N| \rightarrow 1$ exponentially fast as N increases. In detail,

$$R_N \sim \frac{r_+}{1 - t e^{ikd} e^{-\eta}} \equiv R_\infty \quad \text{as } N \rightarrow \infty,$$

where $|R_\infty| = 1$: no energy passes through the semi-infinite row. This is as expected because we are in a stopband ($|\xi| > 1$).

Let us compare R_∞ with R , defined by Eq. (24). Using $e^{-\eta} = \xi - \sqrt{\xi^2 - 1}$, we have

$$\begin{aligned} R_\infty &= \frac{r_+}{|r|^2} (1 - t^* e^{-ikd} e^{-\eta}) \\ &= \frac{1}{r_-} e^{i(\rho_+ + \rho_-)} \left(1 - |t| e^{-i(kd + \tau)} \left\{ \xi - \sqrt{\xi^2 - 1} \right\} \right) \\ &= -\frac{1}{r_-} e^{2i\tau} \left(1 - e^{-i(kd + \tau)} \cos(kd + \tau) + |t| e^{-i(kd + \tau)} \sqrt{\xi^2 - 1} \right) \\ &= -\frac{1}{r_-} e^{-ikd} \left(i e^{i\tau} \sin(kd + \tau) + t \sqrt{\xi^2 - 1} \right), \end{aligned}$$

which agrees precisely with the expression for R obtained in Section 4.1.

5.3. Letting $N \rightarrow \infty$ when $|\xi| < 1$

When $|\xi| < 1$, we are in a passband. Put $\xi = \cos qd$, whence Eq. (27) gives

$$R_N = \frac{r_+ \sin(Nqd)}{\sin(Nqd) - te^{ikd} \sin([N-1]qd)}, \quad T_N = \frac{te^{-i(N-1)kd} \sin qd}{\sin(Nqd) - te^{ikd} \sin([N-1]qd)}. \quad (31)$$

These formulas do not have well-defined limits as $N \rightarrow \infty$, and so we must seek ways to interpret them.

Sprung et al. [15] write Eq. (30) as

$$\frac{1}{|T_N|^2} = 1 + \left(\frac{|r| \sin(Nqd)}{|t| \sin qd} \right)^2 \quad (32)$$

and then note that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \sin^2(nqd) = \frac{1}{2}.$$

“For very large N , and real ϕ , $\sin^2 N\phi$ will oscillate rapidly around the value $\frac{1}{2}$ ” [15, p. 1121], giving a large- N average estimate for $|T_N|^2$ (replacing $\sin^2(Nqd)$ in Eq. (32) by $\frac{1}{2}$), which in turn gives the estimate

$$|R|^2 \simeq \frac{|r|^2}{2|t|^2 \sin^2(qd) + |r|^2}.$$

This is incorrect: it does not agree with Eq. (21), which gives

$$|R|^2 = \frac{|r|^2}{2|t|^2 \sin^2(qd) + |r|^2 + 2S^2 \sqrt{1 - |r|^2/S^2}}.$$

This appears rather concerning: the obvious “averaging” of the transmitted wave as $N \rightarrow \infty$ does not lead to the “correct” answer obtained in the previous section. Indeed, Markoš and Soukoulis offered the pessimistic remark [16, p. 88] concerning point scatterers: “In no case can we claim that an infinite crystal can be obtained in the limit $N \rightarrow \infty$ of δ -function barriers”. However, we now show a way to overcome this difficulty.

In the context of acoustic waves in a layered half-space, Gilbert [17] and Schoenberg and Sen [18] start by introducing a small amount of loss and then reason as follows. The matrix P , Eq. (29), can be diagonalized using its eigenvalues, λ_1 and λ_2 , so that P^N can be written in terms of λ_1^N and λ_2^N . When $|\xi| < 1$, λ_1 and λ_2 are on the unit circle in the complex plane. Introducing some loss moves one eigenvalue (λ_1 , say) inside the circle and one outside. Eliminating the effect of the growing term, λ_2^N , leads to a formula for R . Thus, the eigenvector $(x_1, x_2)^T$ corresponding to the eigenvalue $\lambda_1 = \xi + i\sqrt{1 - \xi^2}$ satisfies

$$(e^{ikd}/t^* - \lambda_1)x_1 + e^{ikd}(r_-/t)x_2 = 0.$$

Then R is given by

$$\begin{aligned} R &= \frac{x_2}{x_1} = \frac{\lambda_1 - e^{ikd}/t^*}{e^{ikd}(r_-/t)} \\ &= \frac{t}{r_-} e^{-ikd} \left(\xi + i\sqrt{1 - \xi^2} - \frac{1}{t^*} e^{ikd} \right) \\ &= \frac{|t|}{r_-} e^{i(\tau - kd)} \left(\frac{1}{2t} e^{-ikd} - \frac{1}{2t^*} e^{ikd} + i\sqrt{1 - \xi^2} \right) \\ &= -\frac{1}{r_-} e^{i(\tau - kd)} \left(i \sin(kd + \tau) - i|t|\sqrt{1 - \xi^2} \right), \end{aligned}$$

which agrees precisely with Eq. (20). Similar approaches have been used for more complicated problems [19–23]. In fact, the idea of adding some damping so as to extract a physically meaningful solution goes back to Rayleigh; see, for example, [24, Section 1.5] or [25, p. 259 and p. 478].

6. Third method: direct treatments

Another possibility is to tackle the reflection problem directly, with a semi-infinite row of scatterers. The first difficulty is specifying and then enforcing some kind of radiation condition as $x \rightarrow \infty$. Explicit recognition of this difficulty is fairly recent. In a stopband, $u(x) \rightarrow 0$ exponentially as $x \rightarrow \infty$. In a passband, it is not so clear what to do: waves propagate back and forth between each cell, as within a finite row of scatterers. Potel et al. [26] require that the power flux be in the $+x$ direction; Levine [8, Section 53] has calculated this quantity for point scatterers. Again, another option is to introduce some loss, leading to a form of “limiting absorption principle”.

6.1. Levine's approach

Levine has given direct treatments of the problem, first for point scatterers [7] (see Section 4.3 and [8, Section 52]) and then [8, Section 62] for a more general situation governed by

$$u''(x) + k^2(1 + \eta(x))u(x) = 0, \quad x \geq 0, \tag{33}$$

with $u(x) = e^{ikx} + Re^{-ikx}$ for $x \leq 0$; here, η is a given d -periodic function. A special role is played by the function φ [8, Eq. (62.8)], defined by

$$\varphi(\zeta) = \int_0^d e^{i\zeta x} \eta(x) u(x) dx. \tag{34}$$

Eliminating $\eta(x)u(x)$ in Eq. (34) using the differential equation, Eq. (33), followed by two integrations by parts gives [8, Eq. (62.9)],

$$\varphi(\zeta) = \frac{i\zeta}{k^2} (e^{i\zeta d} u(d) - u(0)) - \frac{1}{k^2} (e^{i\zeta d} u'(d) - u'(0)) + \frac{\zeta^2 - k^2}{k^2} \int_0^d e^{i\zeta x} u(x) dx. \tag{35}$$

Assuming that $u(x)$ and $u'(x)$ are continuous across $x = 0$, $u(0) = 1 + R$ and $u'(0) = ik(1 - R)$.

Now, to find u , Levine [8, Eq. (62.5)] starts by writing down a Lippmann–Schwinger equation,

$$u(x) = e^{ikx} + \frac{ik}{2} \int_0^\infty e^{ik|x-t|} \eta(t) u(t) dt, \quad -\infty < x < \infty,$$

but this is suspect within a passband when k is real. Instead, we can use Laplace transforms. Thus, let $U(s) = \mathcal{L}\{u\} = \int_0^\infty u(x)e^{-sx} dx$ be the Laplace transform of $u(x)$. Then Eq. (33) gives

$$(s^2 + k^2)U(s) - su(0) - u'(0) = -k^2 \mathcal{L}\{\eta u\}.$$

Inverting, using the convolution theorem $u(0) = 1 + R$ and $u'(0) = ik(1 - R)$, gives

$$u(x) = e^{ikx} + Re^{-ikx} - k \int_0^x \sin(k[x-t]) \eta(t) u(t) dt, \quad x \geq 0. \tag{36}$$

This is a Volterra integral equation of the second kind for u . For similar treatments, see [27,28].

At this stage, we have not used the d -periodicity of η or the Floquet–Bloch relation, Eq. (9), $u(x + d) = e^{iqd}u(x)$. Levine assumes that this relation holds for $x \geq 0$, whence $u(d) = e^{iqd}u(0)$ and $u'(d) = e^{iqd}u'(0)$. Then Eq. (35) gives

$$\varphi(k) = \frac{2i}{k} (e^{i(q+k)d} - 1) R,$$

which can be used to obtain R in terms of qd , kd and $\varphi(k)$. To make further progress, we need $u(x)$ for $0 < x < d$. Levine uses various Fourier series but does not obtain explicit formulas (except for point scatterers). Another option would be to solve the integral equation, Eq. (36); recall that such equations can always be solved by iteration.

To conclude, we might ask: how did Levine obtain the correct result for point scatterers, as discussed in Section 4.3 and [8, Section 52]? At a certain point in his calculation, see [8, Eq. (52.6)], we see

$$\sum_{m=n+1}^\infty e^{i(q+k)md} = \frac{e^{i(q+k)(n+1)d}}{1 - e^{i(q+k)d}},$$

which is valid provided $\text{Im}(q + k) > 0$, i.e. some loss was introduced without comment!

6.2. A Wiener–Hopf approach

For a semi-infinite periodic row of point scatterers, an exact solution can be obtained using the Wiener–Hopf technique [24]. The governing equation is

$$u''(x) + k^2 u(x) = M \sum_{n=0}^\infty u(x) \delta(x - nd).$$

We write the solution u in terms of a new function v , defined by

$$u(x) = e^{ikx} - e^{-ikx} + v(x) \quad \text{for } x < 0 \quad \text{and} \quad u(x) = v(x) \quad \text{for } x > 0.$$

The continuity conditions on u at each scatterer give continuity of v at $x = nd$ for $n = 0, 1, 2, \dots$; the term $-e^{-ikx}$ is included in the definition of $v(x)$ for $x < 0$ so as to ensure continuity of $v(x)$ at $x = 0$. At the first scatterer, the condition $u'(0^+) - u'(0^-) = Mu(0)$ (see Eq. (5)) gives

$$v'(0^+) - v'(0^-) = Mv(0) + 2ik,$$

with $v'(nd^+) - v'(nd^-) = Mv(nd)$ for $n = 1, 2, \dots$. Using Green's function associated with the homogeneous string, $e^{ik|x-y|}/(2ik)$, we find

$$v(x) = e^{ik|x|} + \frac{M}{2ik} \sum_{n=0}^{\infty} v_n e^{ik|x-nd|}, \quad x \in \mathbb{R}, \tag{37}$$

where $v_n = v(nd)$. This equation cannot be solved directly using Floquet–Bloch theory as the geometry is not periodic in x for all x . The semi-infinite structure suggests using the Wiener–Hopf technique, which we do by employing the z -transform. For other applications of the discrete Wiener–Hopf technique see, for example, [29–35].

First set $x = md$ in Eq. (37) to yield the infinite algebraic system

$$v_m = e^{ikd|m|} + \frac{M}{2ik} \sum_{n=0}^{\infty} v_n e^{ikd|m-n|}, \quad m \in \mathbb{Z}. \tag{38}$$

Note that for $m \geq 0$, Eq. (38) gives a closed system to solve for v_m . For $m < 0$, Eq. (38) defines v_m in terms of v_n with $n \geq 0$.

Now, apply the z -transform, i.e. multiply Eq. (38) by z^m and sum over all $m \in \mathbb{Z}$, to give

$$\sum_{m=-\infty}^{\infty} z^m v_m = \sum_{m=-\infty}^{\infty} z^m e^{ikd|m|} + \frac{M}{2ik} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} z^m v_n e^{ikd|m-n|}.$$

The order of summation can be reversed in the final term on the right-hand side, and hence shifting the counter m to $m' = m + n$, say, yields

$$\sum_{m=-\infty}^{\infty} z^m v_m = F(z) + \frac{M}{2ik} F(z) \sum_{n=0}^{\infty} z^n v_n, \tag{39}$$

where

$$\begin{aligned} F(z) &= \sum_{m=-\infty}^{\infty} z^m e^{ikd|m|} = \sum_{p=1}^{\infty} z^{-p} e^{ikdp} + \sum_{p=0}^{\infty} z^p e^{ikdp} \\ &\equiv F_-(z) + F_+(z). \end{aligned}$$

With k real, we see that the series defining F_+ converges for $|z| < 1$ whereas the series defining F_- converges for $|z| > 1$. In order to obtain a common region (annulus) of the complex z -plane for convergence of these sums we add a small amount of damping, replacing kd by $kd + i\varepsilon = \kappa d$, say, with $0 < \varepsilon \ll 1$. Then the series defining F_+ and F_- converge for $|z|e^{-\varepsilon} < 1$ and $|z|e^{\varepsilon} > 1$, respectively, and as $\varepsilon \ll 1$, the overlap region is approximately $1 - \varepsilon < |z| < 1 + \varepsilon$. Summing the geometric series yields

$$F_+ = \frac{1}{1 - ze^{ikd}} = - \left(\frac{e^{-ikd}}{z - e^{-ikd}} \right)_+$$

where we reiterate that the subscript $+$ indicates that the function is analytic in the domain $|z| < e^{\varepsilon}$. Similarly for F_- we find that

$$F_- = \left(\frac{e^{ikd}}{z - e^{ikd}} \right)_-$$

which is analytic in the domain $|z| > e^{-\varepsilon}$.

Using these results for F_{\pm} , we write the system, Eq. (39), as

$$V_+(z) + V_-(z) = \left(1 + \frac{M}{2i\kappa} V_+(z) \right) \left\{ \left(\frac{e^{ikd}}{z - e^{ikd}} \right)_- - \left(\frac{e^{-ikd}}{z - e^{-ikd}} \right)_+ \right\}, \tag{40}$$

where we have introduced the notation

$$V_+(z) = \sum_{m=0}^{\infty} z^m v_m \quad \text{and} \quad V_-(z) = \sum_{m=-\infty}^{-1} z^m v_m. \tag{41}$$

As for F_+ , function V_+ is analytic in a disc; but here we assume this region is $|z| < e^{\delta_+ \varepsilon}$ for some $\delta_+ > 0$, which contains the unit circle. Similarly, V_- is taken to be analytic outside the disc $|z| > e^{-\delta_- \varepsilon}$ for some $\delta_- > 0$. These assumptions, which will be verified *a posteriori*, mean that Eq. (40) is valid in the annular region $D: \exp(-\min\{\delta_-, 1\}\varepsilon) < |z| < \exp(\min\{\delta_+, 1\}\varepsilon)$. Eq. (40) can be rearranged into the Wiener–Hopf equation

$$V_-(z) + K(z)V_+(z) = \left(\frac{e^{ikd}}{z - e^{ikd}} \right)_- - \left(\frac{e^{-ikd}}{z - e^{-ikd}} \right)_+ = \frac{2i\kappa}{M} (1 - K(z)), \tag{42}$$

where

$$K(z) = 1 - \frac{M}{2i\kappa} \left(\frac{e^{i\kappa d}}{z - e^{i\kappa d}} - \frac{e^{-i\kappa d}}{z - e^{-i\kappa d}} \right) = \frac{(z - e^{-iQd})(z - e^{iQd})}{(z - e^{-i\kappa d})(z - e^{i\kappa d})}, \quad (43)$$

and we have introduced the complexified version of the dispersion relation for the periodic medium, Eq. (12),

$$\cos Qd = \cos \kappa d + \frac{M}{2\kappa} \sin \kappa d,$$

which defines Qd in terms of $\kappa d = kd + i\varepsilon$.

In a passband, qd is real, and as $0 < \varepsilon \ll 1$ we can show that $Qd = qd + i\delta\varepsilon$, for some $\delta > 0$; hence e^{iQd} lies inside the unit circle in the z -plane. In a stopband, qd is purely imaginary and a consistent choice of the branch of Eq. (12) yields $\text{Im}(qd) > 0$, so that e^{iQd} will also lie inside the unit circle. So, we can write $K(z) = K_+(z)K_-(z)$ where

$$K_+(z) = \frac{z - e^{-iQd}}{z - e^{-i\kappa d}} \quad \text{and} \quad K_-(z) = \frac{z - e^{iQd}}{z - e^{i\kappa d}}, \quad (44)$$

which have simple zeros and poles outside/inside the annular domain D , respectively. Now, divide both sides of Eq. (42) by $K_-(z)$ to yield

$$\frac{V_-(z)}{K_-(z)} + K_+(z)V_+(z) = \frac{2i\kappa}{M} \left(\frac{1}{K_-(z)} - K_+(z) \right), \quad (45)$$

which can be rearranged into the form

$$\left(\frac{2i\kappa}{M} - V_-(z) \right) \frac{1}{K_-(z)} = K_+(z)V_+(z) + \frac{2i\kappa}{M} K_+(z) = E(z). \quad (46)$$

The left-hand and right-hand sides analytically continue the function $E(z)$ into the whole of the z -plane, and hence E must be entire. This function is easily evaluated by examining the asymptotic behaviour of the left-hand side. From Eqs. (41) and (44), $V_-(z) = O(z^{-1})$ and $K_-(z) \rightarrow 1$ as $|z| \rightarrow \infty$. Thus, $E(z)$ tends to a constant at infinity, and by the extended form of Liouville's theorem we obtain

$$E(z) \equiv \frac{2i\kappa}{M}.$$

Hence, Eq. (46) yields the explicit solution

$$V_+(z) = \frac{2i\kappa}{M} \left(\frac{1}{K_+(z)} - 1 \right) \quad \text{and} \quad V_-(z) = \frac{2i\kappa}{M} (1 - K_-(z)).$$

We note that $V_+(z)$ has a single simple pole at $z = e^{-iQd}$, whereas $V_-(z)$ has a single simple pole at $z = e^{i\kappa d}$. Thus, we can confirm that $\delta_+ = \delta$ and $\delta_- = 1$ and hence the annulus D has finite width and contains the unit circle as required.

Finally, to determine the solution in the physical domain we apply the inverse z -transform

$$v(md) = v_m = \frac{1}{2\pi i} \oint_C (V_+(z) + V_-(z)) z^{-m-1} dz, \quad m \in \mathbb{Z},$$

where C is the unit circle in D which is traversed in an anticlockwise direction.

For $m \geq 0$ the contour C can be taken off to infinity, picking up the residue contribution from the pole of $V_+(z)$ (zero of $K_+(z)$) at $z = e^{-iQd}$. This yields, after a little algebra and after letting $\varepsilon \rightarrow 0$, the solution

$$v_m = \frac{4k}{M} e^{imqd} e^{i(q-k)d/2} \sin([k-q]d/2) = e^{imqd} \frac{e^{i(q-k)d/2} \sin kd}{\sin([k+q]d/2)}, \quad m \geq 0, \quad (47)$$

where the relation between the first and second forms are established using Eq. (12). Note that $v_{m+1} = e^{iqd} v_m$. Similarly, for $m < 0$, the residue contribution from the pole of $V_-(z)$ at $z = e^{i\kappa d}$ yields

$$v_m = \frac{4k}{M} e^{-imkd} e^{i(q-k)d/2} \sin([k-q]d/2) = e^{-imkd} \frac{e^{i(q-k)d/2} \sin kd}{\sin([k+q]d/2)}, \quad m < 0, \quad (48)$$

which takes a remarkably similar form to Eq. (47). Then, comparing $u(x) = e^{ikx} + \text{Re}^{-ikx}$ with $u(x) = e^{ikx} - e^{-ikx} + v(x)$ in $x < 0$, we obtain

$$R = -1 + v_m e^{imkd} = e^{-ikd} \frac{\sin([k-q]d/2)}{\sin([k+q]d/2)}, \quad (49)$$

in agreement with Levine's formula, Eq. (26).

7. Conclusions

Three methods have been described in order to determine the reflection coefficient R associated with the canonical problem of the one-dimensional reflection by a semi-infinite periodic row of identical scatterers located at $x = nd$, for $n = 0, 1, 2, \dots$. The first approach successfully determined R by shifting the array by one period. The third approach used a direct attack on the governing ordinary differential equation for point scatterers, and also yielded the correct R . Both methods treated directly the problem of a semi-infinite array. On the other hand the second approach considered a finite number (N) of scatterers, determined the associated reflection coefficient R_N , and then took the limit as $N \rightarrow \infty$. This limit gives R correctly when the frequency of the incident wave resides in a stopband. In a passband, methods were described that yield the correct exact result for R ; they require the artificial introduction of a small amount of loss into the problem. (We remark that an approach based on the discrete Wiener–Hopf technique can be used to solve the problem with N point scatterers. This yields the solution in a slightly different form to that given in Section 5, but it also requires a small amount of dissipation to recover the correct limit as $N \rightarrow \infty$.)

Of interest in future work is the consideration of how the time-harmonic problem can be used in the time domain in order to yield the reflected and transmitted fields due to transient incidence such as an incoming pulse. Indeed, very few transient problems associated with heterogeneous media have been considered.

Acknowledgements

The authors gratefully acknowledge funding from the Royal Society for a Wolfson Research Merit Award (2013–2018) for Abrahams and from the EPSRC for the research fellowship of Parnell (EP/L018039/1).

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