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Author for correspondence:
P. A. Martin
e-mail: pamartin@mines.edu

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The pulsating orb: solving the wave equation outside a ball

P. A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401, USA

PAM, 0000-0002-0384-3885

Transient acoustic waves are generated by the oscillations of an object or are scattered by the object. This leads to initial-boundary value problems (IBVPs) for the wave equation. Basic properties of this equation are reviewed, with emphasis on characteristics, wavefronts and compatibility conditions. IBVPs are formulated and their properties reviewed, with emphasis on weak solutions and the constraints imposed by the underlying continuum mechanics. The use of the Laplace transform to treat the IBVPs is also reviewed, with emphasis on situations where the solution is discontinuous across wavefronts. All these notions are made explicit by solving simple IBVPs for a sphere in some detail.

1. Introduction

The three-dimensional wave equation has been studied since the nineteenth century: recall the familiar contributions of Poisson, Kirchhoff, Rayleigh and Hadamard, for example. There are at least two reasons for such studies. First, the wave equation arises in the context of physical applications, especially in acoustics where sound waves propagate and are scattered by obstacles. Second, the wave equation is the prototypical hyperbolic partial differential equation (PDE). In the theory of such PDEs, a prominent role is played by characteristics (§3). These are related to scattering problems because the wave field can be discontinuous across moving surfaces, and these surfaces (wavefronts) are closely related to characteristics.

The usual way to handle discontinuities is to seek weak solutions; this idea goes back to Courant & Hilbert [1] in 1937. Thus, classical (twice differentiable) solutions are weak solutions, but not all weak solutions are classical solutions. The theory predicts that the jump in the solution across wavefronts evolves according to the transport equation (§4c). On the other hand, the balance laws of continuum mechanics imply constraints on the same jump (§5), and these can conflict with the implications of seeking weak solutions.

After reviewing properties of the wave equation and its solutions, we turn to the formulation of initial-boundary value problems (IBVPs) in §6. Typical scattering problems involve one boundary condition on the surface of the scatterer and two initial conditions. Various possibilities and combinations are encountered in the literature, conveniently separated into two groups, those with zero boundary conditions and those with zero initial conditions. The first group is simpler mathematically (because of energy conservation), but the second group is more common in applications.

Initial conditions are specified everywhere outside the scatterer at time $t=0$, whereas the boundary condition is specified on the scatterer for all $t>0$. The smoothness of the solution is largely determined by what happens at the 'space–time corner', where the initial conditions meet the boundary condition. Consistency conditions arise, and their effects are examined, especially as they affect the jump conditions across wavefronts.

There are several good ways to solve IBVPs numerically. One way, currently receiving a lot of attention, is to derive and solve a time-domain boundary integral equation. Another way is to use Laplace transforms, thus converting the wave equation into the modified Helmholtz equation (an elliptic PDE). We discuss this second way, paying attention to the effects of wavefronts and their associated jump discontinuities. As might have been anticipated, these jumps can be ignored when applying the Laplace transform if one is satisfied with a weak solution—but solutions obtained in this way will have to be examined *a posteriori* to ensure that physical constraints across wavefronts are satisfied.

To illustrate many of these phenomena, spherically symmetric IBVPs for a sphere are solved explicitly. These solutions have independent interest because they can be used as benchmarks.

In summary, the purpose of the paper is to review the formulation of acoustic scattering problems in the time domain. Extensions to electromagnetic and elastodynamic problems are envisaged. We emphasize characteristics, wavefronts and compatibility conditions. We compare and contrast jump properties of weak solutions across wavefronts with those mandated by the underlying continuum mechanics. We observe that some recent analyses (and numerical analyses of associated computational methods) assume more smoothness of the wave field than is expected in realistic applications. A good example is the scattering of a step pressure pulse by an obstacle: such an incident field is discontinuous, and so it is desirable that numerical methods can accommodate restricted classes of non-smooth data. We hope that this review will encourage further work on time-domain scattering problems.

2. Acoustic scattering

Linear acoustics in a homogeneous compressible fluid is governed by the wave equation. In three dimensions, this equation is

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{2.1}$$

where x, y and z are Cartesian coordinates, t is time and c is the (positive) constant speed of sound. We always consider u to be a velocity potential, so that the velocity and (excess) pressure in the fluid are given by

$$v = \operatorname{grad} u \quad \text{and} \quad p = -\rho \frac{\partial u}{\partial t}, \tag{2.2}$$

respectively, where ρ is the constant ambient density of the fluid. Evidently, p and each Cartesian component of v all satisfy (2.1). Solutions of (2.1) are called *wave functions*. For details on the derivation of the governing equations, see ch. 1 of [2] or ch. 1 of [3].

One branch of scattering theory concerns time-harmonic wave functions. These are of the form $u(x,y,z,t) = \mathrm{Re}\{U(x,y,z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, where U is complex valued and ω is the circular frequency: solutions are constructed in the *frequency domain*. From (2.1), U satisfies the Helmholtz equation,

$\nabla^2 U + (\omega/c)^2 U = 0$. We want to retain the unspecified dependence on time: we work in the *time domain*. Of course, the two domains are linked via Fourier or Laplace transforms.

In the time domain, we usually specify both a boundary condition and initial conditions. To fix ideas, consider a bounded obstacle B with smooth boundary S. Denote the unbounded region exterior to S by B_e. The problem is to find a wave function $u(P, t)$, where $P(x, y, z)$ is a typical point in B_e, subject to a boundary condition when $P \in S$ and initial conditions at $t = 0$. All these will be discussed and defined later (§6).

By way of comparison, in the frequency domain, we usually require that $U(P)$ satisfies a boundary condition when $P \in S$ and a 'condition at infinity' as P recedes away from B. The latter is usually taken as the Sommerfeld radiation condition; it ensures that waves generated or scattered by B travel away from B.

3. Characteristics and discontinuities

Solutions of the wave equation can be discontinuous. This means that the exterior domain B_e may be partitioned into subdomains by moving surfaces, with discontinuities across them. These surfaces are known as *wavefronts*, and they are closely related to *characteristics*. To motivate their study, we can hardly do better than quote Courant & Hilbert:

> The relevant fact, of great importance for wave propagation, is: Physically meaningful discontinuities of solutions occur only across characteristic surfaces (hence in this context such discontinuities are called wave fronts) and are propagated in these characteristics along bicharacteristic rays. This propagation is governed by a simple ordinary differential equation.
>
> Courant & Hilbert [4, p. 570]

This quotation is a little misleading because characteristics are not 'surfaces', they are three-dimensional objects (hypersurfaces) in four-dimensional space–time, as we shall see.

(a) Characteristics

Denote a typical point in space by $\boldsymbol{r} = (x, y, z) = (x_1, x_2, x_3)$. Denote a typical point in space–time by $\mathbf{x} = (x_0, x_1, x_2, x_3) = (ct, \boldsymbol{r})$. We use two summation conventions. Repeated lower case subscripts or superscripts are summed from 1 to 3, whereas repeated upper case subscripts or superscripts are summed from 0 to 3. Thus, $|\boldsymbol{r}|^2 = x_i x_i$ and $|\mathbf{x}|^2 = x_I x_I$.

Write the wave equation as a *first-order symmetric hyperbolic system*,

$$A_K \frac{\partial \mathbf{u}}{\partial x_K} = \frac{1}{c}\frac{\partial \mathbf{u}}{\partial t} + A_k \frac{\partial \mathbf{u}}{\partial x_k} = \mathbf{0}, \tag{3.1}$$

where $\mathbf{u} = (u_0, u_1, u_2, u_3) = (c^{-1}\partial u/\partial t,\ \partial u/\partial x_1,\ \partial u/\partial x_2,\ \partial u/\partial x_3)$ and A_K is a symmetric 4×4 matrix, $K = 0, 1, 2, 3$: $A_0 = I_4$, the 4×4 identity matrix,

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

If we denote the entries in A_K by a^K_{IJ}, we can write (3.1) as

$$a^K_{IJ} \frac{\partial u_J}{\partial x_K} = 0, \quad I = 0, 1, 2, 3. \tag{3.2}$$

Writing hyperbolic PDEs as the first-order systems such as (3.1) is standard practice; see, for example, §7.3 of [5] or Appendix 2.I of [6].

In the next few paragraphs, we follow [7], where further details can be found; see also §5.9 of [8] and [9]. We take the defining property of characteristics to be that $\mathbf{u}$ need not be smooth across a characteristic, even though it satisfies the wave equation (3.1) elsewhere. To be more precise, write the equation of a characteristic $\mathcal{C}$ as $F(\mathbf{x}) = 0$. Denote a normal to $\mathcal{C}$ by $\mathbf{N} = (N_0, N_1, N_2, N_3)$; this 4-vector is parallel to the gradient of F, so that $N_I = \alpha\, \partial F/\partial x_I$ for some α. We seek those $\mathbf{N}(\mathbf{x})$ for which $\mathbf{u}(\mathbf{x})$ is not constrained to be smooth across $\mathcal{C}$.

Let $\mathbf{q} = (q_0, q_1, q_2, q_3)$ be an arbitrary 4-vector. Multiply (3.2) by q_I and sum over I, giving

$$q_I a^K_{IJ} \frac{\partial u_J}{\partial x_K} = 0. \tag{3.3}$$

This states that the directional derivative of u_J in the direction of the 4-vector

$$\mathbf{d}_J = (q_I a^0_{IJ}, q_I a^1_{IJ}, q_I a^2_{IJ}, q_I a^3_{IJ}) \tag{3.4}$$

is zero. We have four directions, one for each J. Can we arrange that all four are perpendicular to a single direction $\mathbf{N}$? For this to happen, we require

$$q_I a^K_{IJ} N_K = 0, \quad J = 0, 1, 2, 3. \tag{3.5}$$

A consequence would be that the four 4-vectors $\mathbf{d}_J$ become linearly dependent.

At this stage, $\mathbf{q}$ is arbitrary. Our question reduces to seeking a non-zero $\mathbf{q}$ such that (3.5) holds, and this will be possible if the determinant of the 4×4 matrix with entries $(a^K_{IJ} N_K)$ is zero. Some calculation gives

$$N_0^2\{N_0^2 - (N_1^2 + N_2^2 + N_3^2)\} = 0. \tag{3.6}$$

One solution of (3.6) is $N_0 = 0$. Then (3.5) gives $q_0 = 0$ and $N_i q_i = 0$. As there is no time dependence, such characteristics are hypercylinders in space–time corresponding to arbitrary static surfaces in space. Alternatively, from (3.6) we have

$$N_0^2 - (N_1^2 + N_2^2 + N_3^2) = 0, \tag{3.7}$$

which is equation (2.3.1) in [10], for example. In detail, system (3.5) is $N_0 q_0 = N_i q_i$, $N_0 q_1 = N_1 q_0$, $N_0 q_2 = N_2 q_0$ and $N_0 q_3 = N_3 q_0$. Then, ignoring constant factors, we can express the directions $\mathbf{d}_J$ in terms of the components of $\mathbf{N}$, $\mathbf{d}_0 = (N_0, -N_1, -N_2, -N_3)$, $\mathbf{d}_1 = (N_1, -N_0, 0, 0)$, $\mathbf{d}_2 = (N_2, 0, -N_0, 0)$ and $\mathbf{d}_3 = (N_3, 0, 0, -N_0)$. We note that, using (3.7), $N_0 \mathbf{d}_0 - N_i \mathbf{d}_i = \mathbf{0}$: we do have linear dependence, as expected.

(b) Eikonal equations

As $N_I = \alpha\, \partial F/\partial x_I$, (3.7) gives the *eikonal equation* (see p. 153 of [6])

$$\frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_i} = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial F}{\partial t}\right)^2. \tag{3.8}$$

However, it is perhaps clearer and more usual to write $F(\mathbf{x}) = x_0/c - \tau(x_1, x_2, x_3) = t - \tau(\boldsymbol{r})$, so that a characteristic $\mathcal{C}$ is defined by

$$t = \tau(\boldsymbol{r}), \quad \boldsymbol{r} \in \mathcal{H} \subset \mathbb{R}^3, \tag{3.9}$$

for some function τ and some spatial domain $\mathcal{H}$. Then, for fixed t, $\tau(\boldsymbol{r}) = t$ defines a surface in space. As t varies, this surface moves through space: it is called a *wavefront*.

Substituting (3.9) in (3.8) gives

$$\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i} = \frac{1}{c^2} \tag{3.10}$$

or, equivalently, $|\mathrm{grad}\, \tau| = c^{-1}$. These are also known as eikonal equations; see eqn (2.3.3) in [10] or p. 93 in [5].

Solving the eikonal equation (3.10) as a PDE generates characteristics. A simple solution is

$$\tau(\boldsymbol{r}) = t' + \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{r} - \boldsymbol{r}'}{c}, \tag{3.11}$$

where $(ct', \boldsymbol{r}')$ is an arbitrary fixed point in space–time and $\hat{\boldsymbol{k}}$ is a constant unit 3-vector. This solution represents a plane moving through space at speed c in the direction of $\hat{\boldsymbol{k}}$. Further solutions of (3.10) are suggested by its evident spherical spatial symmetry. For example,

$$\tau(\boldsymbol{r}) = t' + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c} \tag{3.12}$$

represents an expanding sphere, centred at $\boldsymbol{r}'$; the sphere has radius $c(t - t')$ at time $t > t'$. In space–time, $t = \tau(\boldsymbol{r})$ gives a cone-like structure, a *conoid*.

(c) Discontinuous solutions

The calculations in §3a show that certain second-order derivatives of u, namely

$$N_I \frac{\partial u_J}{\partial x_I}, \quad J = 0, 1, 2, 3,$$

are unrestricted across characteristics: they can be discontinuous. John [11, §3.5] gives an alternative analysis of the same situation. However, it is more interesting to ask if u or first derivatives of u can be discontinuous across characteristics.

Before doing that, let us clarify what we mean by 'discontinuous across a characteristic'. For a characteristic $\mathcal{C}$ defined by $F(\mathbf{x}) = 0$, we can consider the level sets of F defined by $F(\mathbf{x}) = F_0$, where F_0 is a constant. Then we can say that solutions for $\mathbf{x}$ when $F_0 > 0$ define one 'side' of $\mathcal{C}$, whereas solutions for $\mathbf{x}$ when $F_0 < 0$ define the other side. (Recall that $\mathcal{C}$ is a hypersurface in space–time.)

The situation is clearer when we define $\mathcal{C}$ by (3.9). The wavefront $t = \tau(\boldsymbol{r})$ defines a surface $\Gamma(t)$ moving through space as t varies. Denote the two sides of $\Gamma(t)$ by Γ^+ and Γ^-. The discontinuity (or jump) in some quantity g across $\Gamma(t)$ is denoted and defined by

$$[\![g]\!](\boldsymbol{r}, t) = g^+ - g^-, \quad \boldsymbol{r} \in \Gamma(t), \tag{3.13}$$

where $g^{\pm}$ is the limiting value of g when $\boldsymbol{r} \in \Gamma(t)$ is approached from the $\pm$ side.

As t varies, $[\![g]\!]$ varies, giving

$$[\![g]\!](\boldsymbol{r}, \tau(\boldsymbol{r})) \equiv [\![g]\!](\boldsymbol{r}), \quad \boldsymbol{r} \in \mathcal{H} \subset \mathbb{R}^3, \tag{3.14}$$

with a slight abuse of notation. Thus, (3.13) gives the jump across a single surface $\Gamma(t)$, whereas (3.14) gives the jump across a characteristic.

A normal to Γ is grad τ, so that a unit normal $\boldsymbol{n}(\boldsymbol{r}, t) = (n_1, n_2, n_3)$ is given by

$$n_i = \frac{\pm 1}{|\mathrm{grad}\,\tau|} \frac{\partial \tau}{\partial x_i} = \pm c \frac{\partial \tau}{\partial x_i},$$

using (3.10). Also, differentiating $t = \tau$ with respect to t gives

$$1 = \frac{\partial \tau}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t} = \pm \frac{1}{c} n_i \frac{\mathrm{d}x_i}{\mathrm{d}t}.$$

Thus, the normal velocity of Γ is $\pm c$; the sign depends on the chosen direction of $\boldsymbol{n}$. This result was obtained by Love [12, §9]. He noted that his 'rather intricate analysis ... constitutes an abstract proof of the proposition that the velocity with which the wave-boundary advances is the velocity c .... From a physical point of view, this conclusion might be perhaps assumed' [12].

Let us fix definitions. For a moving wavefront $\Gamma(t)$, we choose $\boldsymbol{n}$ so that it points in the direction of motion and into the region denoted by Γ^- above (3.13). Thus, (3.13) becomes

$$[\![g]\!] = (\text{value of } g \text{ just behind } \Gamma) - (\text{value of } g \text{ just ahead of } \Gamma). \tag{3.15}$$

This definition is convenient because in many applications the second term on the right-hand side of (3.15) is zero. Indeed, Love [12, §8] considered such a problem with Γ advancing into a quiescent region so that $u = 0$ ahead of Γ. He assumed further that u is continuous across Γ, $[\![u]\!] = 0$, but first derivatives of u may be discontinuous. As $u = 0$ at $\Gamma(t)$, we have

$$0 = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial u}{\partial t} + cn_i\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial n} \quad \text{on } \Gamma(t). \tag{3.16}$$

One year later, Love allowed motion ahead of Γ but with $[\![u]\!] = \text{const}$. He obtained [13, eqn (5)]

$$\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] + \frac{1}{c}\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right] = 0 \quad \text{on } \Gamma(t). \tag{3.17}$$

For a careful derivation of many such compatibility conditions, see Chadwick & Powdrill [14], where it is also pointed out that (3.17) was derived by Hadamard [15, p. 101, eqn (48)] in his 1903 book on wave propagation.

In terms of pressure and velocity, (2.2) and (3.17) give

$$[\![p]\!] = \rho c\boldsymbol{n} \cdot [\![\boldsymbol{v}]\!] \quad \text{on } \Gamma(t). \tag{3.18}$$

This jump condition will appear later, in §5.

4. Weak solutions and discontinuities

(a) Green's formula

In order to study jumps across characteristics, we will use a four-dimensional version of Green's formula. To derive this, let Σ be a bounded region of space–time bounded by a hypersurface $\mathcal{S}$. The divergence theorem for such a region is

$$\int_\Sigma \frac{\partial V_I}{\partial x_I}\,\mathrm{d}\Sigma = \int_\mathcal{S} V_I N_I\,\mathrm{d}\mathcal{S}, \tag{4.1}$$

where $\mathbf{N} = (N_0, N_1, N_2, N_3)$ is a unit normal 4-vector to $\mathcal{S}$ pointing out of Σ and $\mathbf{V} = (V_0, V_1, V_2, V_3)$ is a continuously differentiable 4-vector field. Also the element of integration $\mathrm{d}\mathcal{S}$ is given by $N_0\,\mathrm{d}\mathcal{S} = \mathrm{d}V(\boldsymbol{r}) = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$ (e.g. eqn (2.2.3) in [10]).

Apply (4.1) with $\mathbf{V} = (u\,\partial v/\partial t, 0, 0, 0)$, $\mathbf{V} = (0, u\partial v/\partial x_1, u\partial v/\partial x_2, u\partial v/\partial x_3)$ and again with u and v interchanged. Subtracting the results gives Green's formula [10, eqn (3.2.3)]

$$\int_\Sigma (u\,\Box^2 v - v\,\Box^2 u)\,\mathrm{d}\Sigma = \int_\mathcal{S}\left(u\frac{\partial v}{\partial T} - v\frac{\partial u}{\partial T}\right)\mathrm{d}\mathcal{S}, \tag{4.2}$$

where $\Box^2 u \equiv \nabla^2 u - c^{-2}\partial^2 u/\partial t^2$ defines the wave operator and $\partial u/\partial T$ denotes a transverse derivative of u, defined by

$$\frac{\partial u}{\partial T} = N_i\frac{\partial u}{\partial x_i} - \frac{N_0}{c}\frac{\partial u}{\partial t} = T_I\frac{\partial u}{\partial x_I}, \tag{4.3}$$

say, where $\mathbf{T} = (T_0, T_1, T_2, T_3) = (-N_0, N_1, N_2, N_3)$. We have $N_I T_I = N_i N_i - N_0^2$, so that $\mathbf{T}$ is tangential to $\mathcal{S}$ if, and only if, $\mathcal{S}$ is a characteristic [10, §3.2]; here we have used (3.7) to define a characteristic.

(b) Weak solutions of the wave equation

Classical solutions of the wave equation are twice differentiable, but we expect that discontinuous solutions can occur and, in fact, such solutions are physically interesting. To handle them, we generalize the notion of 'solution'.

Following Friedlander [10, pp. 42–45] and Lax [16], we say that u is a *weak solution* of the wave equation $\square^2 u = 0$ in the space–time region Σ if

$$\int_{\Sigma} u \,\square^2 v \,\mathrm{d}\Sigma = 0 \quad \text{for all } v \in V^0_{\Sigma}, \tag{4.4}$$

where V^0_{Σ} is the set of all smooth test functions v with compact support contained in Σ (implying that $v \equiv 0$ in the vicinity of $\mathcal{S}$).

If u is twice differentiable in Σ, we can use (4.2) to infer that $\int_{\Sigma} v \,\square^2 u \,\mathrm{d}\Sigma = 0$ for all $v \in V^0_{\Sigma}$, whence $\square^2 u = 0$ in Σ. In other words, smooth weak solutions are classical solutions.

(c) Discontinuous weak solutions

Suppose that u is discontinuous across some hypersurface, and that this hypersurface intersects a bounded space–time region Σ, splitting it into two sub-regions, Σ_1 and Σ_2. Let $\mathcal{S}$ denote the piece of the hypersurface inside Σ; it is the 'interface' between Σ_1 and Σ_2.

Clearly, if u is a weak solution of $\square^2 u = 0$ in Σ, then it is also a classical solution in Σ_1 and Σ_2. To handle the interface, we write the definition (4.4) as

$$\int_{\Sigma_1} u \,\square^2 v \,\mathrm{d}\Sigma + \int_{\Sigma_2} u \,\square^2 v \,\mathrm{d}\Sigma = 0 \quad \text{for all } v \in V^0_{\Sigma}.$$

We use (4.2) twice, once in Σ_1 and once in Σ_2. As $\square^2 u = 0$ in $\Sigma_1 \cup \Sigma_2$ and $v \equiv 0$ near the boundary of Σ (but not near $\mathcal{S}$), we are left with an integral over $\mathcal{S}$,

$$\int_{\mathcal{S}} \left([\![u]\!] \frac{\partial v}{\partial T} - v \left[\!\!\left[\frac{\partial u}{\partial T} \right]\!\!\right] \right) \mathrm{d}\mathcal{S} = 0 \quad \text{for all } v \in V^0_{\Sigma}, \tag{4.5}$$

where $[\![w]\!]$ is the jump in w across $\mathcal{S}$, that is, the difference in the values of w as a point in $\mathcal{S}$ is approached from Σ_1 and from Σ_2; this difference arises from the opposite directions of the outward pointing normals for Σ_1 and Σ_2.

Suppose first that $\mathcal{S}$ is not a characteristic. Then the transverse derivative, $\partial/\partial T$, is not a tangential derivative. It follows that v and $\partial v/\partial T$ on $\mathcal{S}$ are independent. We conclude from (4.5) that $[\![u]\!] = 0$: thus, jump discontinuities can only occur across characteristics.

So, suppose next that $\mathcal{S}$ is a piece of a characteristic, $\mathcal{C}$. Then the transverse derivative is a tangential derivative. This means that $\partial v/\partial T$ on $\mathcal{C}$ can be calculated as a certain tangential derivative of v on $\mathcal{C}$. In more detail, define $\mathcal{C}$ by $t = \tau(\boldsymbol{r})$ for $\boldsymbol{r} \in \mathcal{H}$, where τ satisfies the eikonal equation (3.10). Then, given a function $w(\mathbf{x}) = w(\boldsymbol{r}, t)$, its values on $\mathcal{C}$ are $w(\boldsymbol{r}, \tau(\boldsymbol{r})) = \{w\}$, say, with $\boldsymbol{r} \in \mathcal{H}$; we use { } to indicate evaluation on $\mathcal{C}$.

The chain rule gives

$$\frac{\partial \{w\}}{\partial x_i} = \left\{ \frac{\partial w}{\partial x_i} \right\} + \left\{ \frac{\partial w}{\partial t} \right\} \frac{\partial \tau}{\partial x_i}. \tag{4.6}$$

Hence definition (4.3) gives

$$\begin{aligned}
\left\{ \frac{\partial v}{\partial T} \right\} \mathrm{d}\mathcal{S} &= \left\{ N_i \frac{\partial v}{\partial x_i} - \frac{N_0}{c} \frac{\partial v}{\partial t} \right\} \mathrm{d}\mathcal{S} = -cN_0 \left\{ \frac{\partial \tau}{\partial x_i} \frac{\partial v}{\partial x_i} + \frac{1}{c^2} \frac{\partial v}{\partial t} \right\} \mathrm{d}\mathcal{S} \\
&= -c \left(\frac{\partial \tau}{\partial x_i} \left\{ \frac{\partial v}{\partial x_i} \right\} + \frac{1}{c^2} \left\{ \frac{\partial v}{\partial t} \right\} \right) N_0 \,\mathrm{d}\mathcal{S} \\
&= -c \left(\frac{\partial \tau}{\partial x_i} \left(\frac{\partial \{v\}}{\partial x_i} - \left\{ \frac{\partial v}{\partial t} \right\} \frac{\partial \tau}{\partial x_i} \right) + \frac{1}{c^2} \left\{ \frac{\partial v}{\partial t} \right\} \right) \mathrm{d}V = -c \frac{\partial \tau}{\partial x_i} \frac{\partial \{v\}}{\partial x_i} \,\mathrm{d}V(\boldsymbol{r}),
\end{aligned}$$

using $N_0 \,\mathrm{d}\mathcal{S} = \mathrm{d}V$ and the eikonal equation (3.10). A similar calculation gives

$$\left[\!\!\left[\frac{\partial u}{\partial T} \right]\!\!\right] \mathrm{d}\mathcal{S} = -c \frac{\partial \tau}{\partial x_i} \frac{\partial [\![u]\!]}{\partial x_i} \,\mathrm{d}V(\boldsymbol{r}).$$

Hence (4.5) becomes

$$\int_{\mathcal{H}} \left([\![u]\!] \frac{\partial \tau}{\partial x_i} \frac{\partial \{v\}}{\partial x_i} - \{v\} \frac{\partial \tau}{\partial x_i} \frac{\partial [\![u]\!]}{\partial x_i} \right) \mathrm{d}V(\boldsymbol{r}) = 0 \tag{4.7}$$

for all smooth functions $\{v\}$ with support in $\mathcal{H}$. For such functions, the divergence theorem gives

$$\int_{\mathcal{H}} [\![u]\!] \frac{\partial \tau}{\partial x_i} \frac{\partial \{v\}}{\partial x_i} \, \mathrm{d}V = -\int_{\mathcal{H}} \{v\} \frac{\partial}{\partial x_i} \left([\![u]\!] \frac{\partial \tau}{\partial x_i} \right) \mathrm{d}V.$$

Hence (4.7) gives

$$0 = \frac{\partial}{\partial x_i} \left([\![u]\!] \frac{\partial \tau}{\partial x_i} \right) + \frac{\partial \tau}{\partial x_i} \frac{\partial [\![u]\!]}{\partial x_i} = 2 \frac{\partial \tau}{\partial x_i} \frac{\partial [\![u]\!]}{\partial x_i} + [\![u]\!] \nabla^2 \tau. \tag{4.8}$$

'This equation is called, following Luneburg, the *transport equation* associated with the wavefronts' (3.9) [10, p. 45]; see [17] and eqn (3.3.7) in [10].

Equation (4.8) is a first-order PDE for $[\![u]\!]$ on $\mathcal{C}$. It is the differential equation mentioned in the quotation from Courant & Hilbert [4] at the beginning of §3.

To investigate the consequences of (4.8), we choose a function $\tau(\boldsymbol{r})$ that satisfies the eikonal equation. One simple choice is the plane-wave function (3.11); in particular, if we choose the unit vector $\hat{\boldsymbol{k}}$ in the x_3-direction, we have

$$\tau(\boldsymbol{r}) = t' + \frac{x_3 - x_3'}{c}, \quad \frac{\partial \tau}{\partial x_i} = \frac{1}{c} \delta_{i3}, \quad \nabla^2 \tau = 0.$$

For this choice, (4.8) reduces to $\partial [\![u]\!] / \partial x_3 = 0$: $[\![u]\!]$ can be an arbitrary function of the in-plane variables $x = x_1$ and $y = x_2$ but it cannot depend on the out-of-plane variable $z = x_3$. To labour the point, we construct $[\![u]\!]$ from $u(x, y, z, t)$ with $t = \tau(x, y, z)$, giving $[\![u]\!]$ as a function of x, y and z; for a plane characteristic moving in the z-direction, $[\![u]\!]$ cannot change with z.

Next, suppose we have a spherical wavefront with τ given by (3.12). Then, if we put $R = |\boldsymbol{r} - \boldsymbol{r}'|$, we obtain grad $\tau = (\boldsymbol{r} - \boldsymbol{r}')/(cR)$, $\nabla^2 \tau = 2/(cR)$ and (4.8) reduces to

$$(x_i - x_i') \frac{\partial [\![u]\!]}{\partial x_i} + [\![u]\!] = 0. \tag{4.9}$$

If we define spherical polar coordinates by $x_1 - x_1' = R \sin\theta \cos\phi$, $x_2 - x_2' = R \sin\theta \sin\phi$ and $x_3 - x_3' = R \cos\theta$, (4.9) simplifies to

$$R \frac{\partial [\![u]\!]}{\partial R} + [\![u]\!] = 0 \quad \text{whence } u = \frac{A(\theta, \phi)}{R}, \tag{4.10}$$

where A is an arbitrary function of θ and ϕ. Note that the wavefront is defined by $t = \tau$ whence $R = c(t - t')$ in (4.10).

We remark that the results of §4, including (4.10), are valid for any weak solutions of the wave equation such as the potential u, the pressure p and any Cartesian component of the velocity $\boldsymbol{v}$.

5. Jump relations in continuum mechanics

Let us return to the quotation from Courant & Hilbert [4] in §3. It refers to 'physically meaningful discontinuities': what does this mean? Love [12, p. 53] remarked that there may be conditions on the solution 'imposed by the constitution of the medium or the nature of the disturbance, if it is to represent waves of a specified type transmitted through a specified medium' with a footnote: 'The importance of these conditions was emphasized by Dr. Larmor at the meeting at which the paper was communicated' (in January 1903).

The conditions mentioned are derived within the general theory of continuum mechanics. For a thorough development, see Part C of [18]. For the special case of a compressible inviscid fluid, see, for example, [19] or [20]. In this context, there are two exact jump relations across a moving surface $\Gamma(t)$,

$$[\![\rho_{\mathrm{ex}}(V - \boldsymbol{n} \cdot \boldsymbol{v}_{\mathrm{ex}})]\!] = 0 \quad \text{and} \quad [\![\rho_{\mathrm{ex}}(V - \boldsymbol{n} \cdot \boldsymbol{v}_{\mathrm{ex}})\boldsymbol{v}_{\mathrm{ex}} - p_{\mathrm{ex}} \boldsymbol{n}]\!] = \boldsymbol{0}, \tag{5.1}$$

where ρ_{ex}, $\boldsymbol{v}_{\mathrm{ex}}$ and p_{ex} denote the exact density, velocity and pressure, respectively, V is the normal velocity of Γ, and the unit normal to Γ, $\boldsymbol{n}$, points in the direction of motion. For a characteristic $\mathcal{C}$, Γ is defined by (3.9) and the normal velocity is $V = c$.

Now, in linear acoustics, we have $\rho_{\text{ex}} \simeq \rho + \varepsilon\rho_1$, $p_{\text{ex}} \simeq p_0 + \varepsilon p_1$ and $\boldsymbol{v}_{\text{ex}} \simeq \varepsilon\boldsymbol{v}_1$, where ρ and p_0 are constants and ε is a small parameter. Then we see that (5.1) is satisfied exactly at leading order in ε, whereas at first order in ε we obtain

$$c[\![\rho_1]\!] = \rho\boldsymbol{n}\cdot[\![\boldsymbol{v}_1]\!] \quad \text{and} \quad \rho c[\![\boldsymbol{v}_1]\!] = [\![p_1]\!]\boldsymbol{n}. \tag{5.2}$$

The (excess) pressure p is defined by $p_{\text{ex}} = p_0 + p$. Therefore, in linear acoustics, we have $p = \varepsilon p_1 = \varepsilon c^2\rho_1$, where the second equality comes from the definition of c. We also define the velocity $\boldsymbol{v}$ by $\boldsymbol{v} = \varepsilon\boldsymbol{v}_1$. Then the first of (5.2) gives

$$[\![p]\!] = \rho c\,\boldsymbol{n}\cdot[\![\boldsymbol{v}]\!] \tag{5.3}$$

and the second gives

$$[\![p]\!]\,\boldsymbol{n} = \rho c\,[\![\boldsymbol{v}]\!]. \tag{5.4}$$

Evidently, (5.3) is the normal component of the vector equation (5.4). It is also equivalent to the condition found by Hadamard [15] and by Love [13]; see (3.18).

Let $\boldsymbol{t}$ be a tangent vector to Γ at $P \in \Gamma$. Then (5.4) gives $[\![\boldsymbol{v}]\!]\cdot\boldsymbol{t} = 0$: tangential components of the velocity are continuous across a wavefront Γ, whereas, by (5.3), the normal component of $\boldsymbol{v}$ has a jump proportional to the jump in the pressure across Γ. The wavefront Γ is sometimes called a *weak acoustic shock*.

These results go back to Christoffel [21] (see also [22,23, 10, p. 45]).

The jump relation (5.4) holds as $\Gamma(t)$ evolves. If we introduce the velocity potential u, so that, from (2.2), $\boldsymbol{v} = \operatorname{grad} u$ and $p = -\rho\,(\partial u/\partial t)$, (5.4) becomes

$$\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right]\operatorname{grad}\tau = -[\![\operatorname{grad} u]\!], \quad \boldsymbol{r}\in\mathcal{H},$$

where we have used $\boldsymbol{n} = c\operatorname{grad}\tau$. But, from (4.6),

$$\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right]\operatorname{grad}\tau = \operatorname{grad}[\![u]\!] - [\![\operatorname{grad} u]\!], \quad \boldsymbol{r}\in\mathcal{H}.$$

Comparing these two equations, we infer that $\operatorname{grad}[\![u]\!] = \boldsymbol{0}$ in $\mathcal{H}$, whence

$$[\![u]\!](\boldsymbol{r}) = \text{const.}, \quad \boldsymbol{r}\in\mathcal{H}. \tag{5.5}$$

As all physical quantities are obtained from appropriate derivatives of u, we expect that we can take the constant to be zero. This would mean that the velocity potential is continuous across wavefronts, a result stated by Friedlander [10, p. 45], and then we could use formula (3.16) at a wavefront. Also, when u is continuous across wavefronts, it can be advantageous to work with the potential u rather than the pressure p, for example. In general, p is not continuous across wavefronts; in fact, discontinuous pressure pulses are of great interest.

If we compare with the discussion of discontinuous weak solutions in §4c, we see that we must have $A = 0$ in (4.10) when u is required to be continuous across a wavefront. Similar formulae hold for p and the Cartesian components of $\boldsymbol{v}$, but then the coefficients corresponding to A will not vanish, in general.

The constant in (5.5) might be determined by boundary and initial conditions. If it is not zero, we see a mismatch between the physically justified jump condition (5.5) and the jump condition (4.10) obtained by seeking weak solutions.

6. Initial-boundary value problems

Consider a bounded obstacle B with smooth boundary S. Denote the unbounded region exterior to S by B_{e}. We are interested in solving the wave equation in B_{e} subject to initial and boundary conditions. This leads to a variety of IBVPs.

Formally, we seek a wave function $u(\mathbf{x})$ for $\mathbf{x} = (ct, \boldsymbol{r}) \in \Sigma = \mathbb{R}^+ \cup B_{\text{e}}$, where $\mathbb{R}^+$ denotes positive real numbers. Thus, Σ is a semi-infinite hypercylinder in space–time. There is a boundary condition on the lateral boundary of Σ, $\mathbb{R}^+ \cup S$, and there are two initial conditions on the 'base' of Σ, $\{0\} \cup B_e$. Of some interest will be the intersection of the lateral boundary and the base, $\{0\} \cup S = \mathcal{E}$, say (the boundary S at $t = 0$), because this is where the boundary condition and the initial conditions may be in conflict. We call $\mathcal{E}$ the *space–time edge* or *corner* of Σ.

Returning to the formulation of IBVPs, suppose that, as usual, u is a velocity potential. Thus, from (2.2), the pressure $p = -\rho\, \partial u/\partial t$ and the velocity $\boldsymbol{v} = \operatorname{grad} u$. We seek a wave function $u(\mathbf{x}) \equiv u(P, t)$, where $P(x, y, z)$ is a typical point in B_e, subject to a boundary condition when $P \in S$ and initial conditions at $t = 0$. We take the latter as

$$u(P, 0) = u_0(P) \quad \text{and} \quad \frac{\partial u}{\partial t}(P, 0) = u_1(P) \ \text{ for all } P \in B_e, \tag{6.1}$$

where u_0 and u_1 are given functions.

We shall consider three choices for the boundary condition (although other choices could be made). The first choice is the *Dirichlet condition*,

$$u(P, t) = d(P, t) \quad \text{for } P \in S \text{ and } t > 0, \tag{6.2}$$

where d is a given function. The second choice is the *Neumann condition*,

$$\frac{\partial u}{\partial n}(P, t) = v(P, t) \quad \text{for } P \in S \text{ and } t > 0, \tag{6.3}$$

where v is a given function and $\partial/\partial n$ denotes normal differentiation.

The Neumann condition (6.3) is physically realizable: the normal velocity is prescribed on S. However, the Dirichlet condition (6.2) does not have an obvious physical interpretation. For this reason, we also consider the 'pressure condition',

$$p(P, t) = -\rho \frac{\partial u}{\partial t}(P, t) = p_S(P, t) \quad \text{for } P \in S \text{ and } t > 0, \tag{6.4}$$

where p_S is the prescribed pressure on S. At first sight, it appears that the solution satisfying (6.4) can be obtained from a time-derivative of the solution to the Dirichlet problem, but we will see that the situation is not so straightforward. Note that p itself is a wave function, so IBVPs could be formulated directly in terms of p.

(a) Six basic problems

By linearity, the IBVPs outlined above can be reduced to problems with homogeneous boundary conditions or homogeneous initial conditions. This leads to six simpler problems, listed in table 1.

The first three problems in table 1 have zero boundary conditions, the last three have zero initial conditions. The literature is divided, with most authors preferring to work with zero initial conditions. In fact, there is no loss of generality in doing this, because we can construct a wave function satisfying the inhomogeneous initial conditions (6.1), and then subtract this wave function in order to obtain a new IBVP with zero initial conditions. The relevant construction is due to Poisson (see [24, §287, 10, §1.6, 4, pp. 201–202] or [8, §7.6]).

Much of the earlier mathematical literature assumes zero boundary conditions. For example, the whole of the famous book on *Scattering Theory* by Lax & Phillips [25] is concerned with Problem ID_0. Wilcox [26,27] allows non-smooth S and he also considers Problem IN_0. The Wilcox and Lax–Phillips theories are abstract: their goal is not to actually construct solutions. Moreover, although different, it is known that their theories are equivalent [28]. There is also the work of Ladyzhenskaya, which goes back to the early 1950s. In her 1985 book, she discusses Problem ID_0 [29, §IV.3] and also the Robin problem with boundary condition $\partial u/\partial n + \sigma u = 0$ on S where $\sigma(P, t)$ is a prescribed function [29, §IV.5].

Prior to the work of Lax, Phillips and Wilcox, there is the book by Friedlander [10] on *Sound Pulses*. He also considers problems with zero boundary conditions but, in addition, he discusses scattering problems where a specified wave field (an incident 'sound pulse') interacts with B; this leads to Problems DI_0 and NI_0 [10, pp. 8–9], with zero initial conditions and an inhomogeneous boundary condition. Ladyzhenskaya [29, p. 149] also mentions Problem DI_0 and a related Robin

Table 1. Six initial-boundary value problems.

problem	initial conditions		boundary condition on S
ID_0	$u(P,0)=u_0(P)$	$(\partial u/\partial t)(P,0)=u_1(P)$	$u(P,t)=0$
IP_0	$u(P,0)=u_0(P)$	$(\partial u/\partial t)(P,0)=u_1(P)$	$\partial u/\partial t=0$
IN_0	$u(P,0)=u_0(P)$	$(\partial u/\partial t)(P,0)=u_1(P)$	$\partial u/\partial n=0$
DI_0	$u(P,0)=0$	$(\partial u/\partial t)(P,0)=0$	$u(P,t)=d(P,t)$
PI_0	$u(P,0)=0$	$(\partial u/\partial t)(P,0)=0$	$-\rho\,\partial u/\partial t=p_S(P,t)$
NI_0	$u(P,0)=0$	$(\partial u/\partial t)(P,0)=0$	$\partial u/\partial n=v(P,t)$

problem. She remarks that such IBVPs 'are more difficult, but are more frequently encountered in applications'. It is these problems, together with Problem PI_0, that will be our main concern.

One advantage of assuming zero boundary conditions is that we have conservation of energy: energy is input via the initial conditions and there is no subsequent energy flux through the boundary S. Thus, writing in 1970, Morawetz could state that 'for conservative systems we can prove rather easily the existence and uniqueness of solutions' [30]. For problems in acoustics, where p and v are specified at $t=0$, see ch. 7 in [31].

Existence and uniqueness results for problems with zero initial conditions (such as Problems DI_0 and NI_0) are more recent. The basic results are due to Bamberger & Ha Duong [32,33]. Thus, for Problem DI_0, we have [32, §3], with summaries in [34, §2.3, 35, §1]. With our notation, Lubich [34, p. 370] states: 'For smooth compatible boundary data $d(P,t)$ there exists a unique smooth solution u with $u(\cdot,t)\in H^1(B_\mathrm{e})$ for all t'. Here, H^1 is a Sobolev space and 'compatible' means $d(P,0)=0$ for all $P\in S$; additional smoothness comes by requiring that partial derivatives of d with respect to t vanish at $t=0$ for all $P\in S$, thus permitting a smoother extension of $d(P,t)$ from $t>0$ to $t\le 0$. We will say more about compatibility conditions in §6b.

For Problem NI_0, we have [33, §2.1], with summaries in [36, §2, 37, §1]. Again, it is assumed that we have 'compatible data'; in particular [37, eqn (2.1)] 'at least the following conditions' are assumed: $v=\partial v/\partial t=\partial^2 v/\partial t^2=0$ at $t=0$ for all $P\in S$.

(b) Compatibility conditions

> The solution to an elliptic [PDE] is weakly singular in the corners of the domain unless the forcing and boundary data are special. These 'corner' singularities are well known … It is less well known that hyperbolic [PDEs] are equally prone to singularities in the corners of the space–time domain—that is to say, at the spatial boundaries at the initial time, $t=0$. Unless the initial conditions, boundary data and forcing satisfy [certain] 'compatibility' conditions, the kth spatial derivative of u will be unbounded at the spatial boundary for some finite order k. … In the absence of damping, [these] weak singularities propagate away from the boundary and persist forever. Boyd & Flyer [38, p. 281]

The compatibility conditions mentioned have been well studied; Boyd & Flyer [38] refer to the early work of Ladyzhenskaya (see p. 165 in [29]) and give additional references.

For simple examples, consider Problems ID_0 and DI_0, and examine the behaviour near the space–time corner $\mathcal{E}$, where $t=0$ and $P\in S$. For Problem ID_0, we have the boundary condition $u(P,t)=0$ for $P\in S$ and the initial condition $u(P,0)=u_0(P)$ for $P\in B_\mathrm{e}$. If we want both of these to hold at $\mathcal{E}$, then $u_0(P)=0$ for $P\in S$; this is a compatibility condition. Similarly, for Problem DI_0, the simplest compatibility condition is $d(P,0)=0$, $P\in S$. If these conditions are not satisfied, u will be discontinuous across characteristics passing through $\mathcal{E}$, in general.

Higher order compatibility conditions have been worked out. For example, Boyd & Flyer [38, Theorem 1] give formulae for Problems ID_0 and IN_0. The underlying question is often: What do we have to do at $\mathcal{E}$ to make the solution smoother? In our applications, we have some kind of incident wave (such as a pressure step pulse), with known properties, and we want to calculate the scattered waves: we do not have the luxury of being able to make the solution smoother by adjusting the conditions at $\mathcal{E}$. In addition, we also have to ensure that physical constraints across wavefronts are satisfied.

7. Problems with spherical symmetry

Let us consider IBVPs for a sphere of radius a. We assume that the forcing is such that the waves generated have spherical symmetry, which means that u depends on $r = |\boldsymbol{r}|$ and t only. This is a strong assumption but it permits exact solutions, and these solutions are revealing.

Because of the assumed spherical symmetry, the wave equation simplifies to

$$\frac{\partial^2(ru)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(ru)}{\partial t^2}, \tag{7.1}$$

and then we can write down the general solution. It is

$$u(r,t) = r^{-1}\{f(r-ct) + g(r+ct)\}, \tag{7.2}$$

where f and g are arbitrary smooth functions. These are to be determined using initial and boundary conditions. Specifically, we seek $u(r,t)$ in $\mathcal{Q}$, a quarter of the (r,t)-plane where $r > a$ and $t > 0$. There is a boundary condition at $r = a$ for $t > 0$, and two initial conditions at $t = 0$ for $r > a$. There is a characteristic of the PDE (7.1) emanating from the corner at $(r,t) = (a,0)$ along the straight line $r = a + ct$, $t > 0$. Denote this line by $\mathcal{W}$ as it corresponds to the wavefront; solutions may be discontinuous across $\mathcal{W}$. In three-dimensional space, there is a spherical wavefront at $r = a + ct$, $\Gamma(t)$. The space–time edge or corner $\mathcal{E}$ is at $r = a$, $t = 0$; it is $\Gamma(0)$.

We use the wavefront $\mathcal{W}$ to partition the quadrant $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^- \cup \mathcal{W}$, where

$$\mathcal{Q}^+ = \{(r,t) : a < r < a + ct,\ t > 0\} \quad \text{and} \quad \mathcal{Q}^- = \{(r,t) : r > a + ct,\ t > 0\}; \tag{7.3}$$

thus $\mathcal{Q}^+$ is the region behind the wavefront and $\mathcal{Q}^-$ is the region ahead of the wavefront.

The initial conditions are (6.1), which become

$$u(r,0) = u_0(r) \quad \text{and} \quad \frac{\partial u}{\partial t}(r,0) = u_1(r), \quad r > a,$$

where u_0 and u_1 are given. When combined with (7.2), they give

$$f(r) + g(r) = ru_0(r) \quad \text{and} \quad -f'(r) + g'(r) = \frac{r}{c}u_1(r), \quad r > a$$

whence, for $r > a$,

$$f(r) = \frac{1}{2}ru_0(r) - \frac{1}{2c}\int_a^r \xi u_1(\xi)\,\mathrm{d}\xi + C \tag{7.4}$$

and

$$g(r) = \frac{1}{2}ru_0(r) + \frac{1}{2c}\int_a^r \xi u_1(\xi)\,\mathrm{d}\xi - C, \tag{7.5}$$

where C is an arbitrary constant. Hence (7.2) gives

$$u(r,t) = \frac{1}{2r}\{(r+ct)u_0(r+ct) + (r-ct)u_0(r-ct)\} + \frac{1}{2cr}\int_{r-ct}^{r+ct} \xi u_1(\xi)\,\mathrm{d}\xi, \tag{7.6}$$

for $r > a + ct$: the solution in $\mathcal{Q}^-$ does not depend on the boundary condition because information from the boundary at $r = a$ first reaches the wavefront $\Gamma(t)$ after travelling at speed c for time t.

The solution behind the wavefront in $\mathcal{Q}^+$ is determined using both the boundary condition and the initial conditions, and it splits as

$$u(r,t) = u_{\mathrm{bc}}(r,t) + u_{\mathrm{ic}}(r,t), \quad a < r < a + ct. \tag{7.7}$$

We calculate u for Dirichlet, pressure and Neumann boundary conditions separately.

(a) Dirichlet boundary condition

If we take the Dirichlet condition (6.2),

$$u(a,t) = d(t), \quad t > 0, \tag{7.8}$$

where $d(t) \equiv d(P,t)$ is given when P is on the sphere, and use it in (7.2), we obtain

$$ad(t) = f(a - ct) + g(a + ct), \quad t > 0, \tag{7.9}$$

whence $f(\xi) = ad([a - \xi]/c) - g(2a - \xi)$ for $\xi < a$. Hence (7.2) and (7.5) give (7.7) in which

$$u_{\mathrm{bc}}(r,t) = \frac{a}{r} d\left(t - \frac{[r-a]}{c}\right), \tag{7.10}$$

and

$$u_{\mathrm{ic}}(r,t) = \frac{1}{2r}\{(r+ct)u_0(r+ct) - (2a - r + ct)u_0(2a - r + ct)\} + \frac{1}{2cr}\int_{2a-r+ct}^{r+ct} \xi u_1(\xi)\,\mathrm{d}\xi. \tag{7.11}$$

Note that $u_{\mathrm{bc}}(a,t) = d(t)$ and $u_{\mathrm{ic}}(a,t) = 0$, thus verifying (7.8).

Across the wavefront at $r = a + ct$, we find that the jump in u (defined by (3.15)) is

$$[\![u]\!] = \frac{a}{r}\{d(0) - u_0(a)\},$$

where we have used (7.6), (7.7), (7.10) and (7.11). We note that this result, $[\![u]\!] = A/r$, is consistent with the known jump behaviour of weak solutions, (4.10). However, the physical constraint is much stronger: it requires that $[\![u]\!]$ be constant as the wavefront evolves; see (5.5). Thus, we must impose the consistency condition

$$u_0(a) = d(0).$$

In fact, this condition gives more, namely $[\![u]\!] = 0$.

A similar argument shows that $d(t)$ must be continuous for $t > 0$, otherwise unphysical discontinuities will be induced in u.

(b) Pressure boundary condition

Next, consider the pressure condition (6.4),

$$p(a,t) = -\rho \frac{\partial u}{\partial t}(a,t) = p_a(t), \quad t > 0, \tag{7.12}$$

where $p_a(t) \equiv p_S(P,t)$ is given when P is on the sphere. Use of (7.2) gives

$$ap_a(t) = \rho c\{f'(a - ct) - g'(a + ct)\}, \quad t > 0.$$

An integration then gives

$$f(\xi) = -\frac{a}{\rho}\int_0^{(a-\xi)/c} p_a(\eta)\,\mathrm{d}\eta - g(2a - \xi) + aA, \quad \xi < a,$$

where A is an arbitrary constant. Hence (7.2) and (7.5) give (7.7) in which $u_{\mathrm{ic}}(r,t)$ is given by (7.11) again and

$$u_{\mathrm{bc}}(r,t) = -\frac{a}{r\rho}\int_0^{t-(r-a)/c} p_a(\eta)\,\mathrm{d}\eta + \frac{a}{r}A, \quad a < r < a + ct. \tag{7.13}$$

Note that $-\rho\,\partial u_{\mathrm{bc}}/\partial t = p_a(t)$ and $\partial u_{\mathrm{ic}}/\partial t = 0$ when $r = a$, which confirms (7.12).

Across the wavefront at $r = a + ct$, we obtain

$$[\![u]\!] = \frac{a}{r}\{A - u_0(a)\}.$$

For this to be independent of r, in accordance with the physical constraint (5.5), we must take the constant $A = u_0(a)$, whence $[\![u]\!] = 0$.

We also find $[\![p]\!] = (a/r)\{\rho u_1(a) + p_a(0)\}$. Thus, we obtain $[\![p]\!] = 0$ if $p_a(0) = -\rho u_1(a)$. If this condition is not satisfied, the pressure will jump across the wavefront even though the potential does not. Similarly, if $p_a(t)$ is not continuous for $t > 0$, points of discontinuity in $p_a(t)$ will induce admissible discontinuities in $p(r, t)$ but no discontinuities in $u(r, t)$.

Of course, jumps in p across wavefronts are what we expect in more general cases, such as when a plane pressure pulse is scattered by a sphere.

(c) Neumann boundary condition

Instead of (7.8) or (7.12), we can take the Neumann boundary condition (6.3),

$$\frac{\partial u}{\partial r}(a, t) = v(t), \quad t > 0, \tag{7.14}$$

where $v(t) \equiv v(P, t)$ is given. Substitution of (7.2) in (7.14) gives a first-order differential equation for f, $f'(\xi) - a^{-1}f(\xi) = h(\xi)$ for $\xi < a$, where

$$h(\xi) = av\left(\frac{a-\xi}{c}\right) - g'(2a - \xi) + a^{-1}g(2a - \xi)$$

and g is given by (7.5). Solving gives

$$f(\xi)\,\mathrm{e}^{-\xi/a} = A_1 - \int_\xi^a h(\eta)\,\mathrm{e}^{-\eta/a}\,\mathrm{d}\eta, \quad \xi < a, \tag{7.15}$$

where A_1 is an arbitrary constant. The piece of h containing v generates u_{bc} in (7.7):

$$u_{\mathrm{bc}}(r, t) = -\frac{a}{r}\mathrm{e}^{(r-ct)/a}\int_{r-ct}^a \mathrm{e}^{-\eta/a}v\left(\frac{a-\eta}{c}\right)\mathrm{d}\eta, \quad a < r < a + ct; \tag{7.16}$$

one can check that u_{bc} satisfies (7.14). Substituting the other piece of h in (7.15) gives

$$\begin{aligned} f(\xi)\,\mathrm{e}^{-\xi/a} &= A_1 + \int_\xi^a g'(2a-\eta)\,\mathrm{e}^{-\eta/a}\,\mathrm{d}\eta - \frac{1}{a}\int_\xi^a g(2a-\eta)\,\mathrm{e}^{-\eta/a}\,\mathrm{d}\eta \\ &= A_2 + g(2a-\xi)\,\mathrm{e}^{-\xi/a} - \frac{2}{a}\int_\xi^a g(2a-\eta)\,\mathrm{e}^{-\eta/a}\,\mathrm{d}\eta, \end{aligned} \tag{7.17}$$

after an integration by parts; A_2 ($=A_1 - \mathrm{e}^{-1}g(a)$) is an arbitrary constant. We use (7.17) with $\xi = r - ct$, and then (7.2) and (7.7) give

$$u_{\mathrm{ic}}(r, t) = \frac{A_2}{r}\,\mathrm{e}^{(r-ct)/a} + \frac{1}{r}\{g(r+ct) + g(2a - r + ct)\} - \frac{2}{ar}\mathrm{e}^{(r-ct)/a}\int_{r-ct}^a g(2a-\eta)\,\mathrm{e}^{-\eta/a}\,\mathrm{d}\eta. \tag{7.18}$$

The last term can be simplified after using (7.5). Then, using (7.5) again, (7.18) gives

$$\begin{aligned} u_{\mathrm{ic}}(r, t) &= \frac{A}{r}\,\mathrm{e}^{(r-ct)/a} + \frac{1}{2r}\{(r+ct)u_0(r+ct) + (2a - r + ct)u_0(2a - r + ct)\} \\ &\quad + \frac{1}{2cr}\int_{2a-r+ct}^{r+ct}\xi u_1(\xi)\,\mathrm{d}\xi - \frac{\mathrm{e}^{(r-ct)/a}}{r\mathrm{e}^2}\int_a^{2a-r+ct}\xi\,\mathrm{e}^{\xi/a}\left(\frac{u_0(\xi)}{a} - \frac{u_1(\xi)}{c}\right)\mathrm{d}\xi \end{aligned} \tag{7.19}$$

for $a < r < a + ct$, where A ($=A_2 - 2C/\mathrm{e}$) is an arbitrary constant. Direct calculation verifies that $\partial u_{\mathrm{ic}}/\partial r = 0$ on $r = a$.

Note that the first term in (7.19), $(A/r)\mathrm{e}^{(r-ct)/a}$, is a wave function that satisfies the homogeneous boundary condition, $\partial u/\partial r = 0$ on $r = a$.

Table 2. The six IBVPs in table 1 for a sphere of radius a. The wavefront Γ is at $r = a + ct$.

problem	behind Γ, in $\mathcal{Q}^+$	ahead of Γ, in $\mathcal{Q}^-$	physical constraint
ID_0	$u = u_{\mathrm{ic}}$, defined by (7.11)	u is given by (7.6)	$u_0(a) = 0$
IP_0	$u = u_{\mathrm{ic}} + aA/r$, u_{ic} from (7.11)	u is given by (7.6)	$u_0(a) = A$
IN_0	$u = u_{\mathrm{ic}}$ defined by (7.19)	u is given by (7.6)	$A = 0$ in (7.19)
DI_0	$u = u_{\mathrm{bc}}$, defined by (7.10)	$u \equiv 0$	$d(0) = 0$
PI_0	$u = u_{\mathrm{bc}}$, defined by (7.13)	$u \equiv 0$	$A = 0$ in (7.13)
NI_0	$u = u_{\mathrm{bc}} + (A/r)\mathrm{e}^{(r-ct)/a}$, with u_{bc} defined by (7.16)	$u \equiv 0$	$A = 0$

Across the wavefront, we find that $[\![u]\!] = A\mathrm{e}/r$, after using (7.6), (7.7), (7.16) and (7.19). Then the physical constraint (5.5) implies that we must take $A = 0$, whence $[\![u]\!] = 0$. Note that seeking a weak solution would lead to non-uniqueness because the term involving A is admissible.

(d) Discussion and special cases

In table 2, we collect the results for the six simpler problems listed in table 1. In each case, there is a wavefront $\Gamma(t)$ at $r = a + ct$, and the physical constraint (5.5) implies that $[\![u]\!] = 0$ across Γ.

The pressure and velocity fields are easily calculated. For example, we calculate the pressure jump across the wavefront when there are zero initial conditions. For Problem DI_0, (7.10) gives $[\![p]\!] = -\rho(a/r)d'(0)$, so that $[\![p]\!] = 0$ if $d'(0) = 0$ in addition to $d(0) = 0$. For Problem PI_0, $[\![p]\!] = (a/r)p_a(0)$. For Problem NI_0, (7.16) gives $[\![p]\!] = -\rho(a/r)v(0)$.

The literature on spherically symmetric pulsations of a sphere, using the general solution (7.2), is extensive, going back to Lord Rayleigh [39]. See also Love [12,13], Lamb [24, §286], John [11, p. 15], Whitham [8, §7.3], Lighthill [2, §1.11] and Pierce [3, §4-1].

8. Use of Laplace transforms

It is natural to use Laplace transforms to solve initial-value problems. Their use converts the wave equation into the modified Helmholtz equation. The earliest use of transform methods for a pulsating sphere is probably that in Jeffreys' 1927 book [40] on *Operational methods*.

However, there is a difficulty: what is the effect of possible discontinuities across wavefronts? Can we ignore such discontinuities?

(a) Laplace transform of discontinuous functions

Define the Laplace transform of $u(\boldsymbol{r}, t)$ with respect to t by

$$U(\boldsymbol{r}, s) = \mathcal{L}\{u\} = \int_0^\infty u(\boldsymbol{r}, t)\,\mathrm{e}^{-st}\,\mathrm{d}t.$$

Suppose that there is a surface of discontinuity at $t = \tau(\boldsymbol{r})$. Then, splitting the range of integration at $t = \tau(\boldsymbol{r})$ followed by differentiation, we obtain

$$\begin{aligned}\frac{\partial}{\partial x_i}\mathcal{L}\{u\} &= \int_0^\tau \frac{\partial u}{\partial x_i}\,\mathrm{e}^{-st}\,\mathrm{d}t + u(\boldsymbol{r}, \tau-)\,\mathrm{e}^{-s\tau}\frac{\partial \tau}{\partial x_i} + \int_\tau^\infty \frac{\partial u}{\partial x_i}\,\mathrm{e}^{-st}\,\mathrm{d}t - u(\boldsymbol{r}, \tau+)\,\mathrm{e}^{-s\tau}\frac{\partial \tau}{\partial x_i} \\ &= \mathcal{L}\left\{\frac{\partial u}{\partial x_i}\right\} + [\![u]\!]\,\mathrm{e}^{-s\tau}\frac{\partial \tau}{\partial x_i},\end{aligned} \tag{8.1}$$

where $u(\boldsymbol{r},\tau\pm)=\lim_{\varepsilon\to 0} u(\boldsymbol{r},\tau\pm\varepsilon^2)$ and $[\![u]\!]=u(\boldsymbol{r},\tau-)-u(\boldsymbol{r},\tau+)$, in agreement with (3.15). Hence, replacing u by $\partial u/\partial x_i$,

$$\mathcal{L}\{\nabla^2 u\}=\frac{\partial}{\partial x_i}\mathcal{L}\left\{\frac{\partial u}{\partial x_i}\right\}-\left[\!\!\left[\frac{\partial u}{\partial x_i}\right]\!\!\right]\frac{\partial \tau}{\partial x_i}\,\mathrm{e}^{-s\tau}=\frac{\partial}{\partial x_i}\left(\frac{\partial U}{\partial x_i}-[\![u]\!]\,\mathrm{e}^{-s\tau}\frac{\partial \tau}{\partial x_i}\right)-\left[\!\!\left[\frac{\partial u}{\partial x_i}\right]\!\!\right]\frac{\partial \tau}{\partial x_i}\,\mathrm{e}^{-s\tau}$$
$$=\nabla^2 U-\left(\frac{\partial[\![u]\!]}{\partial x_i}\frac{\partial \tau}{\partial x_i}+[\![u]\!]\nabla^2\tau-s[\![u]\!]\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i}+\left[\!\!\left[\frac{\partial u}{\partial x_i}\right]\!\!\right]\frac{\partial \tau}{\partial x_i}\right)\mathrm{e}^{-s\tau}. \tag{8.2}$$

For time derivatives, similar calculations give

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\}=s\mathcal{L}\{u\}-u_0+[\![u]\!]\,\mathrm{e}^{-s\tau} \tag{8.3}$$

and

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}=s^2\mathcal{L}\{u\}-su_0-u_1+\left(s[\![u]\!]+\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right]\right)\mathrm{e}^{-s\tau}, \tag{8.4}$$

where we have used the initial conditions (6.1).

Equations (8.1)–(8.4) can be found in a paper by Chadwick & Powdrill [41]. They show the effects of discontinuities across $t=\tau(\boldsymbol{r})$; if there are no discontinuities, all the terms involving $[\![\cdot]\!]$ are absent, and we obtain standard formulae.

(b) Laplace transform of wave functions

Suppose that u is a wave function. Then taking the Laplace transform of (2.1) gives

$$\nabla^2 U-\left(\frac{s}{c}\right)^2 U=f_{\mathrm{ic}}(\boldsymbol{r},s)+f(\boldsymbol{r},s)\,\mathrm{e}^{-s\tau(\boldsymbol{r})} \tag{8.5}$$

where the initial conditions are contained in $f_{\mathrm{ic}}=-[su_0(\boldsymbol{r})+u_1(\boldsymbol{r})]/c^2$ and

$$f=\frac{\partial[\![u]\!]}{\partial x_i}\frac{\partial \tau}{\partial x_i}+[\![u]\!]\,\nabla^2\tau-s[\![u]\!]\left(\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i}-\frac{1}{c^2}\right)+\left[\!\!\left[\frac{\partial u}{\partial x_i}\right]\!\!\right]\frac{\partial \tau}{\partial x_i}+\frac{1}{c^2}\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right].$$

The chain rule gives (4.6), which implies

$$\frac{\partial[\![u]\!]}{\partial x_i}=\left[\!\!\left[\frac{\partial u}{\partial x_i}\right]\!\!\right]+\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right]\frac{\partial \tau}{\partial x_i}.$$

Using this to eliminate $[\![\partial u/\partial x_i]\!]$, we obtain

$$f=2\frac{\partial[\![u]\!]}{\partial x_i}\frac{\partial \tau}{\partial x_i}+[\![u]\!]\nabla^2\tau-\left(s[\![u]\!]+\left[\!\!\left[\frac{\partial u}{\partial t}\right]\!\!\right]\right)\left(\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i}-\frac{1}{c^2}\right).$$

Now, we know that discontinuities can only occur across characteristics. It follows that τ satisfies the eikonal equation (3.10) thus simplifying f to

$$f=2\frac{\partial[\![u]\!]}{\partial x_i}\frac{\partial \tau}{\partial x_i}+[\![u]\!]\nabla^2\tau, \tag{8.6}$$

which does not depend on the transform variable s.

If we are satisfied with a weak solution, then we know that $[\![u]\!]$ satisfies the transport equation (4.8). This immediately yields $f\equiv 0$: for weak solutions, we can ignore the presence of discontinuities. On the other hand, we know that weak solutions need not respect the physical constraints across wavefronts (see §5).

The physical constraints lead to (5.5), $[\![u]\!]=\text{const.}$, which reduces (8.6) to $f=[\![u]\!]\,\nabla^2\tau$, which is non-zero in general. However, if we can arrange that $[\![u]\!]=0$, then $f\equiv 0$ and, again, we can ignore the presence of discontinuities. Note that although u is continuous, p, $\boldsymbol{v}$ and derivatives of u can be discontinuous.

(c) Laplace transform of boundary conditions

Write $u(\boldsymbol{r}, t) \equiv u(P, t)$ and $U(\boldsymbol{r}, s) = U(P, s)$. As in §6, we impose a boundary condition at $P \in S$. The simplest is the Dirichlet condition (6.2): $u(P, t) = d(P, t)$ for $P \in S$ and $t > 0$. Hence

$$U(P, s) = D(P, s), \quad P \in S, \tag{8.7}$$

where $D = \mathcal{L}\{d\}$ and we have assumed that d is a continuous function of t.

The pressure condition (6.4) requires that $-\rho(\partial u/\partial t) = p_S(P, t)$ for $P \in S$ and $t > 0$. Applying $\mathcal{L}$, using (8.3), gives

$$sU(P, s) - u(P, 0) + [\![u]\!]\, \mathrm{e}^{-s\tau(P)} = -\rho^{-1} P_S(P, s), \quad P \in S, \tag{8.8}$$

where $P_S(P, s) = \mathcal{L}\{p_S\}$ and $\tau(P) = \tau(\boldsymbol{r})$. We are free to choose $u(P, 0) = u_0(P)$ for $P \in S$. Doing this ensures that $[\![u]\!] \equiv 0$ and then (8.8) simplifies.

For the Neumann condition (6.3), $\partial u/\partial n(P, t) = v(P, t)$, we use (8.1) and obtain

$$\frac{\partial U}{\partial n} = V(P, s) + [\![u]\!]\, \mathrm{e}^{-s\tau(P)} \frac{\partial \tau}{\partial n}, \quad P \in S, \tag{8.9}$$

where $V = \mathcal{L}\{v\}$ and we have assumed that v is a continuous function of t.

9. Laplace transforms and problems with spherical symmetry

Problems with spherical symmetry were discussed at length in §7. The wave equation is to be solved exterior to a sphere of radius a. The solution $u(r, t)$ depends on r and t only, where r is a spherical polar coordinate and the sphere is $r = a$.

For simplicity, let us take zero initial conditions so that $f_{\mathrm{ic}} = 0$ in (8.5). We expect a wavefront at $t = \tau(r) = (r - a)/c$, with $[\![u]\!]$ constant. From (8.5), (8.6) and the spherically symmetric form $\nabla^2 u = r^{-1}(\partial^2/\partial r^2)(ru)$, we obtain

$$\frac{\partial^2 (rU)}{\partial r^2} - \frac{s^2}{c^2}(rU) = [\![u]\!] \frac{\partial^2 (r\tau)}{\partial r^2} = \frac{2}{c}[\![u]\!]. \tag{9.1}$$

Solving this equation, $rU(r, s) = \mathcal{A}\, \mathrm{e}^{-sr/c} + \mathcal{B}\, \mathrm{e}^{sr/c} - (2c/s^2)[\![u]\!]$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary. We take $\mathcal{B} = 0$ because we do not want solutions that grow exponentially with r, whence

$$rU(r, s) = \mathcal{A}(s)\, \mathrm{e}^{-sr/c} - \frac{2c}{s^2}[\![u]\!]. \tag{9.2}$$

(a) Dirichlet boundary condition

For Problem DI_0, we apply the Laplace transform of the Dirichlet boundary condition (7.8), namely (8.7) at $r = a$, whence

$$rU(r, s) = aD(s)\mathrm{e}^{-s(r-a)/c} + \frac{2c}{s^2}(\mathrm{e}^{-s(r-a)/c} - 1)[\![u]\!].$$

Inverting, using $\mathcal{L}\{f(t - b)H(t - b)\} = \mathrm{e}^{-sb}\mathcal{L}\{f(t)\}$, gives

$$u(r, t) = \left(\frac{a}{r}\right) d\left(t - \frac{[r-a]}{c}\right) H\left(t - \frac{[r-a]}{c}\right) + \frac{2c}{r}\left\{\left(t - \frac{[r-a]}{c}\right) H\left(t - \frac{[r-a]}{c}\right) - t\right\} [\![u]\!]. \tag{9.3}$$

There is a wavefront at $r = a + ct$. Behind the wavefront (in $\mathcal{Q}^+$, see (7.3)),

$$u(r, t) = \frac{a}{r} d\left(t - \frac{[r-a]}{c}\right) + \frac{2}{r}(a - r)[\![u]\!].$$

Ahead of the wavefront (in $\mathcal{Q}^-$), $u(r, t) = -(2ct/r)[\![u]\!]$. Combining these two equations so as to calculate $[\![u]\!]$ gives $[\![u]\!] = (a/r)d(0)$. But $[\![u]\!]$ is required to be constant, whence $d(0) = [\![u]\!] = 0$.

Thus, (9.3) reduces to

$$u(r,t) = \frac{a}{r} d\left(t - \frac{[r-a]}{c}\right) H\left(t - \frac{[r-a]}{c}\right),$$

in agreement with the solution of Problem DI_0 obtained in §7d (table 2). We observe that, for this problem, the term on the right-hand side of (9.1) is, in fact, zero.

(b) Pressure boundary condition

Next, consider Problem PI_0. The Laplace transform of the pressure boundary condition (7.12) gives (8.8), which becomes $sU(a,s) = -\rho^{-1}P_a(s)$ with $P_a = \mathcal{L}\{p_a\}$; recall that, for this problem, we can assume that $[\![u]\!] = 0$. Then (9.2) gives $s\mathcal{A}\mathrm{e}^{-sa/c} = -(a/\rho)P_a$ and

$$rU(r,s) = -\frac{a}{\rho s} P_a(s)\,\mathrm{e}^{-s(r-a)/c} = -\frac{a}{\rho}\mathrm{e}^{-s(r-a)/c}\mathcal{L}\left\{\int_0^t p_a(\eta)\,\mathrm{d}\eta\right\}.$$

Thus, inverting,

$$u(r,t) = -\frac{a}{r\rho} H\left(t - \frac{[r-a]}{c}\right)\int_0^{t-(r-a)/c} p_a(\eta)\,\mathrm{d}\eta,$$

in agreement with the solution of Problem PI_0 obtained in §7d (table 2).

(c) Neumann boundary condition

For Problem NI_0, we apply the Laplace transform of the Neumann boundary condition (7.14), namely (8.9), which becomes $\partial U/\partial r = V + [\![u]\!]/c$ at $r = a$. Using this in (9.2) gives

$$\mathcal{A}(s) = \frac{\mathrm{e}^{sa/c}}{sa + c}\left(\frac{2c^2}{s^2}[\![u]\!] - a^2[\![u]\!] - a^2 cV(s)\right),$$

whence

$$\begin{aligned} rU(r,s) &= -\frac{a^2 cV(s)}{sa+c}\,\mathrm{e}^{-s(r-a)/c} + [\![u]\!]\left(\frac{2c^2 - s^2a^2}{(sa+c)s^2}\,\mathrm{e}^{-s(r-a)/c} - \frac{2c}{s^2}\right) \\ &= -acV(s)\mathcal{L}\{\mathrm{e}^{-ct/a}\}\,\mathrm{e}^{-s(r-a)/c} - 2[\![u]\!]\mathcal{L}\{ct\} + [\![u]\!]\mathcal{L}\{a\mathrm{e}^{-ct/a} + 2ct - 2a\}\,\mathrm{e}^{-s(r-a)/c}. \end{aligned}$$

The convolution theorem gives $V(s)\mathcal{L}\{\mathrm{e}^{-ct/a}\} = \mathcal{L}\{\int_0^t v(\eta)\,\mathrm{e}^{-c(t-\eta)/a}\,\mathrm{d}\eta\}$. Hence, inverting,

$$\begin{aligned} u(r,t) = &-\frac{ac}{r} H\left(t - \frac{[r-a]}{c}\right)\mathrm{e}^{(r-ct)/a}\int_0^{t-(r-a)/c} v(\eta)\,\mathrm{e}^{(c\eta-a)/a}\,\mathrm{d}\eta \\ &+ r^{-1}\left\{(a\mathrm{e}^{(r-a-ct)/a} + 2ct - 2r)H\left(t - \frac{[r-a]}{c}\right) - 2ct\right\}[\![u]\!]. \end{aligned}$$

Behind the wavefront at $r = a + ct$, this solution simplifies to

$$u(r,t) = -\frac{ac}{r}\,\mathrm{e}^{(r-ct)/a}\int_0^{t-(r-a)/c} v(\eta)\,\mathrm{e}^{(c\eta-a)/a}\,\mathrm{d}\eta + \frac{1}{r}\{a\mathrm{e}^{(r-a-ct)/a} - 2r\}[\![u]\!],$$

whereas ahead of the wavefront $u(r,t) = -(2ct/r)[\![u]\!]$. Combining these two equations so as to calculate $[\![u]\!]$ gives $[\![u]\!] = 0$. Hence we find agreement with the solution of Problem NI_0 obtained in §7d (see table 2 and (7.16)).

(d) Discussion

One observation from the calculations above is that the presence of all the $[\![u]\!]$ terms causes a lot of complications. Ultimately, these disappear because it turns out that $[\![u]\!] = 0$! This suggests that we should start by seeking a weak solution (implying that all the $[\![u]\!]$ terms are set to zero). Indeed, this is what is done in the literature. However, having found such a solution, we must then check the physical constraints across wavefronts. This is seldom done, but it can lead to non-trivial consequences such as necessary consistency conditions on the data.

There are many papers in which Laplace transforms are combined with separation of variables in spherical polar coordinates so as to solve a variety of IBVPs for a sphere. The earliest application to problems that are not spherically symmetric seems to be that by Brillouin in 1950 [42]; see also [43]. Friedlander [10, pp. 166–174] constructed Green's function for a hard sphere (Neumann problem). We also mention papers by Barakat [44], Tupholme [45] and Huang & Gaunaurd [46]. Finally, there is a paper by Greengard *et al.* [47] on scattering by a soft sphere (Dirichlet problem).

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