Wave Motion 67 (2016) 68–80

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Acoustic scattering by a sphere in the time domain

P.A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887, USA

HIGHLIGHTS

- Classical approach using Laplace transforms is reviewed.
- Integral representation using Legendre polynomials and similarity variables.
- Volterra integral equations derived and solved.
- Asymptotic approximations developed.

ARTICLE INFO

Article history:
Received 5 May 2016
Received in revised form 19 July 2016
Accepted 20 July 2016
Available online 29 July 2016

Keywords:
Wave equation
Similarity variables
Legendre polynomial
Laplace transform

ABSTRACT

A sound pulse is scattered by a sphere leading to an initial–boundary value problem for the wave equation. A method for solving this problem is developed using integral representations involving Legendre polynomials in a similarity variable and Volterra integral equations. The method is compared and contrasted with the classical method, which uses Laplace transforms in time combined with separation of variables in spherical polar coordinates.

1. Introduction

We consider the scattering of a sound pulse by a sphere. This is a canonical problem in time-domain scattering theory, a problem with an extensive literature. Nevertheless, the problem continues to attract attention; we shall cite references later.

The incident field is a sound pulse, which means that there is a propagating wavefront with no disturbance ahead of the wavefront. With t denoting time, we define $t = 0$ as the time when the wavefront first touches the boundary of the spherical scatterer. Consequently, the scattered field solves an initial–boundary value problem (IBVP), with zero initial conditions and a boundary condition.

The standard method for solving an IBVP for a sphere is to combine the Laplace transform with separation of variables in spherical polar coordinates. This method was first used in the 1950s; see Section 3.4 for references. The relevant separated solutions are recalled in Section 2.1 and the method itself is developed in Section 3; connections with the literature on Bessel polynomials are made.

An alternative method is developed in Section 4. It uses integral representations involving similarity variables, building on an old solution found by Bateman in 1938; this solution has the form $P_n(ct/r)Y_n^m$ where P_n is a Legendre polynomial and Y_n^m is a spherical harmonic. These solutions (and a few others) are derived in Sections 2.2 and 2.3.

E-mail address: pamartin@mines.edu.

http://dx.doi.org/10.1016/j.wavemoti.2016.07.007

Both methods assume that the desired solution is continuous. This means that it is appropriate to solve for a velocity potential; doing this does not exclude interesting solutions exhibiting pressure step-pulses, for example. This limitation on the use of Laplace-transform techniques for hyperbolic problems usually goes unremarked.

We apply both methods to IBVPs with three different boundary conditions: Dirichlet condition (velocity potential specified), pressure condition (this is the sound-soft case) and Neumann condition (rigid, or sound-hard case). The Dirichlet problem is discussed in most detail because it has already received the most attention in the literature (even though, physically, it is perhaps the least interesting problem). Our discussion includes some asymptotic analysis (correcting some previous work and relegated to the Appendix).

Some concluding remarks can be found in Section 5. For more background, see [1] and references therein.

2. Some solutions of the wave equation

Small-amplitude acoustic disturbances are governed by the wave equation,

$$\nabla^2 u = c^{-2}\partial^2 u/\partial t^2 = 0, \tag{1}$$

where c is the constant speed of sound. In what follows, we always assume that u is the velocity potential. Thus the excess pressure p and the fluid velocity $\mathbf{v}$ are given by

$$p = -\rho\frac{\partial u}{\partial t} \quad \text{and} \quad \mathbf{v} = \operatorname{grad} u,$$

where ρ is the constant fluid density in the absence of motion. Any solution of the wave equation is called a *wavefunction*.

Using spherical polar coordinates, (r, θ, ϕ), Eq. (1) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2}$$

For spherically symmetric solutions (with no dependence on θ and ϕ), we have

$$u(r, t) = r^{-1}\left\{f_1(r - ct) + f_2(r + ct)\right\}, \tag{3}$$

where f_1 and f_2 are arbitrary (piecewise smooth) functions. These functions can be determined explicitly from initial and boundary conditions. See [1] for details, examples and references.

More generally, let us seek solutions of Eq. (2) in the form

$$u(\mathbf{r}, t) \equiv u(r, \theta, \phi, t) = u_n(r, t)\, Y_n^m(\theta, \phi),$$

where Y_n^m is a spherical harmonic. We find that u_n satisfies the partial differential equation (PDE)

$$\frac{1}{c^2}\frac{\partial^2(ru_n)}{\partial t^2} - \frac{\partial^2(ru_n)}{\partial r^2} + n(n+1)\frac{u_n}{r} = 0. \tag{4}$$

When $n = 0$, u is spherically symmetric and Eq. (4) leads back to Eq. (3).

Various solutions of Eq. (4) are available. We describe some of these, separated solutions in Section 2.1 and solutions built with a similarity variable in Sections 2.2 and 2.3.

2.1. Separated solutions

The standard procedure for solving Eq. (4) is to look for separated solutions, $u_n(r, t) = R(r)\, e^{st}$, where s is a parameter. We use the letter s because, later, we will use Laplace transforms with s as the transform variable. The differential equation for $R(r)$ can be solved. Thus (as is well known) separated solutions of Eq. (4) are

$$u_n(r, t) = \{A\, i_n(sr/c) + B\, k_n(sr/c)\}\, e^{st}, \tag{5}$$

where i_n and k_n are modified spherical Bessel functions and A and B are arbitrary constants.

Properties of i_n and k_n can be found in [2, Chapter 10] (where our i_n is denoted by $\mathrm{i}_n^{(1)}$). For example, $i_n(x)$ is bounded at $x = 0$ but exponentially large as $x \to \infty$ whereas $k_n(x)$ is unbounded at $x = 0$ but exponentially small as $x \to \infty$. Also, from [2, 10.49.12], we have

$$e^x\, k_n(x) = \frac{\pi}{2x}\sum_{j=0}^{n}\frac{(n+j)!}{(2x)^j j!\,(n-j)!}. \tag{6}$$

Thus $xe^x k_n(x)$ is a polynomial in $1/x$ of degree n. Equivalently,

$$(2/\pi)x^{n+1}e^x k_n(x) = \theta_n(x), \tag{7}$$

a polynomial in x of degree n known as a *reverse Bessel polynomial*; see Eq. (29). Similarly, from [2, 10.49.8],

$$2(-x)^{n+1} i_n(x) = e^{-x}\theta_n(x) - e^x\theta_n(-x). \tag{8}$$

2.2. Similarity solutions

Making the substitution $u_n(r,t) = r^{-1}V(r,t)$ in Eq. (4) gives

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial r^2} + \frac{n(n+1)}{r^2}V = 0. \tag{9}$$

Next, introduce a dimensionless similarity variable, $\zeta(r,t) = ct/r$, and look for a solution of Eq. (9) in the form $V(r,t) = v(\zeta)$. After some calculation, we find that $v(\zeta)$ satisfies

$$(1-\zeta^2)v''(\zeta) - 2\zeta v'(\zeta) + n(n+1)v(\zeta) = 0, \tag{10}$$

which is Legendre's equation. Solutions are $P_n(\zeta)$ and $Q_n(\zeta)$, where P_n is a Legendre polynomial and Q_n is a Legendre function. Thus

$$u_n(r,t) = r^{-1}\left\{AP_n(ct/r) + BQ_n(ct/r)\right\}. \tag{11}$$

The solution with $B = 0$, $u_n(r,t) = r^{-1}P_n(ct/r)$, was found by Bateman in 1938 [3]; for other occurrences, see [4, p. 63], [5, §IV] and [6, Eq. (6)].

Evidently, we can replace ζ by $c(t-\tau)/r$, where τ is a parameter, and then construct more solutions by integrating with respect to τ. For example,

$$u_n(r,t) = \frac{1}{r}\int_{t_1}^{t_2} f(\tau)P_n(c[t+t_0-\tau]/r)\,\mathrm{d}\tau$$

is a wavefunction, where f is an arbitrary function and t_0, t_1 and t_2 are constants.

Solutions can also be constructed with variable limits of integration,

$$u_n(r,t) = \frac{1}{r}\int_{t_1}^{t+t_0-r/c} f(\tau)P_n(c[t+t_0-\tau]/r)\,\mathrm{d}\tau. \tag{12}$$

This can be verified by direct calculation. (The upper integration limit can be replaced by $t + t_0 + r/c$, but wavefunctions with the retarded argument $t - r/c$ are more useful.) It is interesting to note that if we replace P_n in Eq. (12) by the Legendre function Q_n, then we do not obtain a wavefunction.

2.3. A Bateman-like wavefunction

There is a useful generalization of Bateman's similarity solution. Instead of $u_n = r^{-1}P_n(ct/r)$, look for a solution of Eq. (4) in the form

$$u_n(r,t) = r^{-1}P_n(\varphi/r), \tag{13}$$

where $\varphi(r,t)$ is to be found. Proceeding as in Section 2.2, put $V = P_n(\zeta)$ in Eq. (9) and compare with the differential equation satisfied by $P_n(\zeta)$, Eq. (10). Doing this gives

$$\frac{r^2}{c^2}\left(\frac{\partial\zeta}{\partial t}\right)^2 - r^2\left(\frac{\partial\zeta}{\partial r}\right)^2 = 1-\zeta^2 \quad\text{and}\quad \frac{r^2}{c^2}\frac{\partial^2\zeta}{\partial t^2} - r^2\frac{\partial^2\zeta}{\partial r^2} = -2\zeta.$$

The substitution $\zeta = \varphi/r$ then gives

$$\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{2\varphi}{r}\frac{\partial\varphi}{\partial r} - 1 = 0, \tag{14}$$

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} = 0. \tag{15}$$

We seek $\varphi(r,t)$ satisfying both of these PDEs. To do this, we look for solutions of the linear homogeneous PDE, Eq. (15), and then see if any of these satisfy Eq. (14). It turns out that interesting solutions are obtained by writing $\varphi(r,t) = R(r) + T(t)$ in Eq. (15), resulting in $\varphi = A(r^2 - c^2t^2) + Bct + C + Dr^3$. This function also solves Eq. (14) provided $B^2 + 4AC = 1$ and $D = 0$. Thus Eq. (13) gives a wavefunction when

$$\varphi(r,t) = A(r^2 - c^2t^2) + Bct + C \quad \text{with } B^2 + 4AC = 1. \tag{16}$$

Dimensionally, because φ/r is dimensionless, the constant A is an inverse length, B is dimensionless and C is a length. As an example, the choices $A = \frac{1}{2}r_0^{-1}$, $B = 0$ and $C = \frac{1}{2}r_0$, where r_0 is a constant (length), give the solution

$$u_n(r,t) = \frac{1}{r}P_n\left(\frac{r^2 + r_0^2 - c^2t^2}{2rr_0}\right). \tag{17}$$

This solution can be found in Copson's 1958 survey of known *Riemann functions* [7, p. 341]; for some later occurrences, see [8, Eq. (12)], [5, Eq. (8)], [9, Eq. (14)] and [10, Eq. (23)].

3. Scattering by a sphere: use of Laplace transforms

Suppose that the sphere has radius a. Our scattering problem reduces to an IBVP for the scattered field $u(r, \theta, \phi, t)$ for $r > a$ and $t > 0$. For an incident sound pulse, we can take zero initial conditions on u at $t = 0$. There is also a boundary condition at $r = a$.

In general, we expect discontinuities across wavefronts. However, it is simplest if we assume that the velocity potential u is continuous. This assumption does not preclude discontinuities in the pressure or velocity across wavefronts, and such discontinuities are of great interest physically. Moreover, for most boundary conditions, this assumption can be made without any loss of generality; for an exception, see Section 3.1.

It is natural to use Laplace transforms to solve IBVPs. Thus, suppose that $u(\mathbf{r}, t)$ satisfies Eq. (1). Define the Laplace transform of $u(\mathbf{r}, t)$ with respect to t by

$$U(\mathbf{r}, s) = \mathcal{L}\{u\} = \int_0^\infty u(\mathbf{r}, t)\, \mathrm{e}^{-st}\, \mathrm{d}t. \tag{18}$$

Formally applying $\mathcal{L}$ to Eq. (1) gives

$$\nabla^2 U - (s/c)^2 U = 0, \tag{19}$$

where we have used the zero initial conditions. We have also used the continuity of u across possible wavefronts: if u is not continuous, Eq. (19) acquires an additional term on its right-hand side involving the (unknown) discontinuity in u. For more details, see [1].

Eq. (19) is the modified Helmholtz equation, an elliptic PDE. If it can be solved together with the Laplace-transformed boundary condition, we can then invert to obtain u from U,

$$u(\mathbf{r}, t) = \mathcal{L}^{-1}\{U\} = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} U(\mathbf{r}, s)\, \mathrm{e}^{st}\, \mathrm{d}s, \tag{20}$$

where Br is the Bromwich contour in the complex s-plane.

The method outlined above is the standard way to solve IBVPs for a sphere. Separating variables in spherical polar coordinates, (r, θ, ϕ), we find solutions of Eq. (19),

$$i_n(sr/c)\, Y_n^m(\theta, \phi) \quad \text{and} \quad k_n(sr/c)\, Y_n^m(\theta, \phi);$$

see Section 2.1. We discard the solutions containing $i_n(sr/c)$ because of their exponential growth with r. For simplicity, we also assume that the incident wave is axisymmetric about the z-axis. Then we can write

$$U(\mathbf{r}, s) = U(r, \theta, s) = \sum_{n=0}^{\infty} B_n(s)\, k_n(sr/c)\, P_n(\cos\theta), \tag{21}$$

where the functions $B_n(s)$ $(n = 0, 1, 2, \ldots)$ are to be determined from the boundary condition on the sphere. Then, inverting $\mathcal{L}$,

$$u(r, \theta, t) = \sum_{n=0}^{\infty} u_n(r, t)\, P_n(\cos\theta), \tag{22}$$

where

$$u_n(r, t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} B_n(s)\, k_n(sr/c)\, \mathrm{e}^{st}\, \mathrm{d}s. \tag{23}$$

3.1. Dirichlet boundary condition

For the axisymmetric Dirichlet problem, the boundary condition is

$$u(a, \theta, t) = d(\theta, t), \quad 0 \le \theta \le \pi,\ t > 0. \tag{24}$$

To impose it, let us expand d similarly to Eq. (22),

$$d(\theta, t) = \sum_{n=0}^{\infty} d_n(t)\, P_n(\cos\theta); \tag{25}$$

orthogonality of Legendre polynomials gives

$$d_n(t) = \frac{2n+1}{2} \int_0^\pi d(\theta, t) P_n(\cos\theta) \sin\theta\, \mathrm{d}\theta. \tag{26}$$

We assume that

$$d_n(t) \text{ is continuous for } t > 0 \text{ with } d_n(0) = 0, \quad n = 0, 1, 2, \ldots . \tag{27}$$

These conditions ensure that $u(\mathbf{r}, t)$ is continuous so that we can use Laplace transforms directly.

Applying $\mathscr{L}$ to the boundary condition $u_n(a, t) = d_n(t)$ gives $B_n(s)\, k_n(sa/c) = D_n(s)$, where $D_n(s) = \mathscr{L}\{d_n\}$. Hence Eqs. (22) and (23) give $u(r, \theta, t)$ with

$$u_n(r, t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} D_n(s)\, \frac{k_n(sr/c)}{k_n(sa/c)}\, \mathrm{e}^{st}\, \mathrm{d}s. \tag{28}$$

Recall that $k_n(x)$ can be written in terms of the reverse Bessel polynomial, $\theta_n(x)$ (see Eq. (7)),

$$\frac{2}{\pi}\, x^{n+1} \mathrm{e}^x\, k_n(x) = \theta_n(x) = \sum_{j=0}^{n} \frac{(2n-j)!\, x^j}{2^{n-j} j!\, (n-j)!}. \tag{29}$$

For example, $\theta_0(x) = 1$, $\theta_1(x) = x + 1$ and $\theta_2(x) = x^2 + 3x + 3$. Much is known about these polynomials; see [11, §4.10], [2, §18.34] and the book by Grosswald [12]. For example, all zeros of $\theta_n(x)$ are simple [12, p. 75] and they are in the left half of the complex x-plane. We denote them by $\beta_{n,m}$ with $m = 1, 2, \ldots, n$: $\theta_n(\beta_{n,m}) = 0$ with $\operatorname{Re} \beta_{n,m} < 0$. Asymptotic properties of $\beta_{n,m}$ for large n are described in the Appendix.

In terms of θ_n, Eq. (28) becomes

$$u_n(r, t) = \frac{1}{2\pi \mathrm{i}} \left(\frac{a}{r}\right)^{n+1} \int_{\mathrm{Br}} D_n(s)\, \frac{\theta_n(sr/c)}{\theta_n(sa/c)}\, \mathrm{e}^{s(t-[r-a]/c)}\, \mathrm{d}s. \tag{30}$$

In particular, as $\theta_0 = 1$ and $\mathscr{L}\{d_0(t-b)H(t-b)\} = \mathrm{e}^{-sb} D_0(s)$, the spherically symmetric component is

$$u_0(r, t) = (a/r)\, d_0(T) H(T) \quad \text{with } T = t - (r-a)/c, \tag{31}$$

where $H(t)$ is the Heaviside unit function. Eq. (31) is known [1, §7(a)].

Motivated by Eqs. (30) and (31), define $\Psi_n(r, s)$ by

$$\Psi_n(r, s) = \frac{a^n \theta_n(sr/c)}{r^n \theta_n(sa/c)} - 1 \tag{32}$$

so that Eq. (30) becomes

$$u_n(r, t) = (a/r)\{d_n(T)H(T) + w_n(r, t)\} \tag{33}$$

with

$$w_n(r, t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} D_n(s)\, \Psi_n(r, s)\, \mathrm{e}^{sT}\, \mathrm{d}s. \tag{34}$$

Now, as $x^{-n}\theta_n(x) \to 1$ as $x \to \infty$ (see Eq. (29)), $\Psi_n(r, s) \to 0$ as $s \to \infty$, implying that there is a function $\psi_n(r, t)$ with $\mathscr{L}\{\psi_n\} = \Psi_n$. Then, by the convolution theorem,

$$D_n(s) \Psi_n(r, s) = \mathscr{L}\left\{ \int_0^t d_n(t')\, \psi_n(r, t - t')\, \mathrm{d}t' \right\}$$

and hence Eq. (33) becomes

$$u_n(r, t) = \frac{a}{r} \left\{ d_n(T) + \int_0^T d_n(t')\, \psi_n(r, T - t')\, \mathrm{d}t' \right\} H(T). \tag{35}$$

This is [13, Eq. (15)]. To use Eq. (35), we need $d_n(t)$ (which is defined by Eq. (26) as an integral of the boundary data $d(\theta, t)$) and $\psi_n(r, t)$ (which is defined by inverting Eq. (32)).

To obtain ψ_n from Ψ_n, we can use a partial-fraction expansion or, equivalently, a residue calculation. We know that $\theta_n(sa/c)$ has simple zeros at $s = (c/a)\beta_{n,j}$, $j = 1, 2, \ldots, n$, $n \geq 1$. Hence

$$\psi_n(r, t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} \Psi_n(r, s)\, \mathrm{e}^{st}\, \mathrm{d}s = \frac{c}{a} \sum_{j=1}^{n} a_{n,j}(r/a) \exp(\beta_{n,j}\, ct/a), \tag{36}$$

where

$$a_{n,j}(\varrho) = \frac{\theta_n(\varrho \beta_{n,j})}{\varrho^n\, \theta_n'(\beta_{n,j})} = \frac{\varrho\, k_n(\varrho \beta_{n,j})}{k_n'(\beta_{n,j})} \exp\{(\varrho - 1)\beta_{n,j}\} \tag{37}$$

and the second form comes by differentiating Eq. (29):

$$\theta_n'(\beta) = (2/\pi)\beta^{n+1}\mathrm{e}^{\beta}\, k_n'(\beta) \quad \text{when } k_n(\beta) = 0. \tag{38}$$

We conclude that $\psi_n(r,t)$ is a linear combination of n exponential functions of t with coefficients that are rational functions of r. As the dependence on r and t is separated, the integral term in Eq. (35) becomes

$$\frac{c}{a}\sum_{j=1}^{n} a_{n,j}(r/a)\exp(\beta_{n,j}\, cT/a)\int_0^T d_n(t')\exp(-\beta_{n,j}\, ct'/a)\,\mathrm{d}t'.$$

When this is used in Eq. (35), we obtain a formula stated by Wilcox [14] in 1959; see also [13, Eq. (13)]. Computationally, this formula is not useful 'due to catastrophic cancellation in carrying out the summation' [13, p. 193]: the coefficients $a_{n,j}(r/a)$ grow exponentially with n, and the rate increases with r. These facts are derived in the Appendix. Greengard et al. [13] advocate using a recursive version of Eq. (35).

Although the convolution form of Eq. (35) is attractive, we could evaluate $w_n(r,t)$ directly from the contour integral in Eq. (34). The integrand has n simple poles coming from $\Psi_n(r,s)$ (at the zeros of $\theta_n(sa/c)$) and additional poles coming from $D_n(s)$. To examine the latter, let us consider scattering of a plane sound pulse, defined by

$$u_{\mathrm{inc}}(z,t) = w_{\mathrm{inc}}(t-[z+a]/c)\, H(t-[z+a]/c), \tag{39}$$

where H is the Heaviside unit function and $w_{\mathrm{inc}}(t)$ is specified. The total field is $u+u_{\mathrm{inc}}$. As the incident pulse does not reach the sphere until $t = 0$, u satisfies zero initial conditions. The Dirichlet boundary condition for scattering by the sphere, $u+u_{\mathrm{inc}} = 0$ at $r = a$, gives $d = -u_{\mathrm{inc}}$ with

$$d(\theta,t) = -w_{\mathrm{inc}}(t-[1+\cos\theta]a/c)\, H(t-[1+\cos\theta]a/c).$$

Then $D = \mathcal{L}\{d\}$ is given by

$$\begin{aligned} D(\theta,s) &= -\int_{(1+\cos\theta)a/c}^{\infty} w_{\mathrm{inc}}(t-[1+\cos\theta]a/c)\,\mathrm{e}^{-st}\,\mathrm{d}t \\ &= -\mathrm{e}^{-sa/c}\mathrm{e}^{-(sa/c)\cos\theta}\, W_{\mathrm{inc}}(s) \\ &= \mathrm{e}^{-sa/c}W_{\mathrm{inc}}(s)\sum_{n=0}^{\infty}(-1)^{n+1}(2n+1)\, i_n(sa/c)P_n(\cos\theta) \end{aligned}$$

where $W_{\mathrm{inc}}(s) = \mathcal{L}\{w_{\mathrm{inc}}\}$ and we have used [2, 10.60.9]

$$\mathrm{e}^{-w\cos\theta} = \sum_{n=0}^{\infty}(-1)^n(2n+1)\, i_n(w)P_n(\cos\theta).$$

Comparison with Eq. (25) gives

$$D_n(s) = (2n+1)(-1)^{n+1}\mathrm{e}^{-sa/c}\, i_n(sa/c)W_{\mathrm{inc}}(s), \tag{40}$$

which shows that D_n inherits its singularities from those of W_{inc}. For example, if $w_{\mathrm{inc}}(t) = \sin\omega_0 t$, $W_{\mathrm{inc}}(s) = \omega_0/(s^2+\omega_0^2)$, which has simple poles at $s = \pm\mathrm{i}\omega_0$.

From Eq. (8), $2(-x)^{n+1}i_n(x) = \mathrm{e}^{-x}\theta_n(x) - \mathrm{e}^{x}\theta_n(-x)$. Hence, if β is a zero of θ_n, Eq. (40) gives

$$2D_n(\beta c/a) = -(2n+1)\beta^{-n-1}\theta_n(-\beta)W_{\mathrm{inc}}(\beta c/a).$$

This is useful when evaluating D_n at the poles of Ψ_n.

3.2. Pressure boundary condition: sound-soft scatterers

The Dirichlet problem discussed in Section 3.1 does not correspond to a physical problem. For a realistic problem, we can consider the sphere to be sound-soft, which means the total (excess) pressure is zero at $r = a$. (For frequency-domain problems, the Dirichlet problem is equivalent to the sound-soft problem.) Thus, the axisymmetric pressure condition is $-\rho\,\partial u/\partial t = p_a(\theta,t)$ on $r = a$ for $0 \le \theta \le \pi$ and $t > 0$, where p_a is specified. If we write

$$p_a(\theta,t) = -\rho\sum_{n=0}^{\infty} q_n(t)P_n(\cos\theta), \tag{41}$$

we find that B_n in Eq. (23) is given by $sB_n(s)\, k_n(sa/c) = Q_n(s)$, where $Q_n(s) = \mathcal{L}\{q_n\}$. Hence, as in Section 3.1, Eqs. (22) and (28) give $u(r,\theta,t)$ with

$$u_n(r,t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Br}}\frac{Q_n(s)}{s}\,\frac{k_n(sr/c)}{k_n(sa/c)}\,\mathrm{e}^{st}\,\mathrm{d}s. \tag{42}$$

As $s^{-1}Q_n(s) = \mathcal{L}\int_0^t q_n(\eta)\,\mathrm{d}\eta$, we can write Eq. (42) as Eq. (35) with $d_n(t)$ replaced by $\int_0^t q_n(\eta)\,\mathrm{d}\eta$,

$$u_n(r,t) = \frac{a}{r}\left\{\int_0^T q_n(\eta)\,\mathrm{d}\eta + \int_0^T \psi_n(r,T-t')\int_0^{t'} q_n(\eta)\,\mathrm{d}\eta\,\mathrm{d}t'\right\}H(T)$$
$$= \frac{a}{r}\int_0^T q_n(\eta)\left\{1 + \int_\eta^T \psi_n(r,T-t')\,\mathrm{d}t'\right\}\mathrm{d}\eta\,H(T). \tag{43}$$

When $n = 0$, this agrees with a known result for spherically symmetric solutions [1, §7(b)].

3.3. Neumann boundary condition: sound-hard scatterers

For the axisymmetric Neumann problem, the boundary condition is $\partial u/\partial r = v(\theta,t)$ on $r = a$ for $0 \le \theta \le \pi$ and $t > 0$, where v is specified. This boundary condition is appropriate for rigid scatterers. Expanding $v(\theta,t) = \sum_{n=0}^{\infty} v_n(t)P_n(\cos\theta)$ and differentiating Eq. (23) with respect to r, we obtain

$$(s/c)\,k_n'(sa/c)B_n(s) = V_n(s) = \mathcal{L}\{v_n\}.$$

Hence Eqs. (22) and (23) give $u(r,\theta,t)$ with

$$u_n(r,t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Br}} V_n(s)\,\frac{k_n(sr/c)}{(s/c)\,k_n'(sa/c)}\,\mathrm{e}^{st}\,\mathrm{d}s. \tag{44}$$

From [2, 10.51.5], $xk_n'(x) = nk_n(x) - xk_{n+1}(x)$. Combining this with Eq. (29) gives

$$(2/\pi)xk_n'(x) = x^{-n-1}\mathrm{e}^{-x}\phi_n(x) \quad \text{with } \phi_n(x) = n\theta_n(x) - \theta_{n+1}(x),$$

which shows that the denominator in Eq. (44) has $n+1$ zeros. Hence Eq. (44) becomes

$$u_n(r,t) = \frac{a}{2\pi\mathrm{i}}\left(\frac{a}{r}\right)^{n+1}\int_{\mathrm{Br}} V_n(s)\,\frac{\theta_n(sr/c)}{\phi_n(sa/c)}\,\mathrm{e}^{sT}\,\mathrm{d}s, \tag{45}$$

with $T = t - (r-a)/c$, as previously; see Eq. (31).

Denote the zeros of $\phi_n(x)$ by $\beta_{n,m}'$: $\phi_n(\beta_{n,m}') = 0$ for $m = 1, 2, \ldots, n+1$. They are the zeros of $k_n'(x)$ and they have the same qualitative properties as the zeros of $\theta_n(x)$. They are tabulated in [15, Table 1] for $n \le 25$.

As $\theta_0 = 1$ and $\phi_0(x) = -\theta_1(x) = -(1+x)$, we can confirm that the spherically symmetric component, u_0, agrees with the known result [1, §7(c)]. More generally, define

$$\Lambda_n(r,s) = \frac{a^{n+1}\theta_n(sr/c)}{c\,r^n\,\phi_n(sa/c)}. \tag{46}$$

As $\Lambda_n(r,s) \to 0$ as $s \to \infty$, there is a function $\lambda_n(r,t)$ with $\mathcal{L}\{\lambda_n\} = \Lambda_n$. Hence

$$u_n(r,t) = \frac{ac}{r}\,H(T)\int_0^T v_n(\eta)\lambda_n(r,T-\eta)\,\mathrm{d}\eta. \tag{47}$$

We could now calculate λ_n in the same way as we calculated ψ_n in Section 3.1, making use of the zeros of $\phi_n(sa/c)$.

3.4. Literature

There are many papers where Laplace transforms are used to solve the wave equation exterior to a sphere. For problems that are not spherically symmetric (see [1] for those), the earliest work is by Brillouin [16]; his long two-part French paper was given a detailed exposition by Hanish [17, §§2.1 & 7.4]. A variety of Neumann radiation problems are solved.

Friedlander [18, pp. 166–174] constructed Green's function for a hard sphere (Neumann boundary condition); the incident field is generated by a simple source at a point on the z-axis outside the sphere. At about the same time, Wilcox [14] published a 'preliminary report' on the Dirichlet problem with zero initial conditions; his short note is discussed in [13].

In 1960, Barakat [19] discussed the scattering of a plane pulse, $u_{\mathrm{inc}} = H(z-ct)\mathrm{e}^{\mathrm{i}k(z-ct)}$, by both Dirichlet and sound-hard spheres. Cohen and Handelman [20] considered other incident plane pulses. The Neumann problem, with various forcings, has been studied in other papers from the 1960s [21–23]. For example, Tupholme [23] gave a detailed study when only a cap of the sphere moves.

Huang and Gaunaurd [24] consider acoustic scattering of a plane step pulse in pressure by a hard sphere with emphasis on calculating p at $r = a$. The series expansion of this quantity, using spherical harmonics, is not uniformly convergent: this is an example of Gibbs' phenomenon [25]. To compensate for this phenomenon, the authors use Cesàro summation, extending previous work by others [26,27]. Better remedies are available [28,25], but these do not seem to have been used for transient scattering problems.

Hamilton and Astley [15] solved the Neumann problem with various forcings. Greengard et al. [13] gave results for Dirichlet and Robin problems. For the latter problem, see [29].

Similar computations have been made for transient electromagnetic and elastodynamic problems but we do not give references here.

4. Scattering by a sphere: use of an integral representation

As an alternative to using Laplace transforms, let us use the representation Eq. (12) for the wave field generated in the exterior of a sphere of radius a. We take $t_0 = a/c$ and $t_1 = 0$, giving

$$u_n(r,t) = \frac{a}{r}\int_0^{t-(r-a)/c} f_n(\tau)P_n\left(\frac{c}{r}\left[t-\tau+\frac{a}{c}\right]\right)\mathrm{d}\tau. \tag{48}$$

Zero initial conditions, $u_n = 0$ and $\partial u_n/\partial t = 0$ at $t = 0$ for $r > a$, are enforced by requiring that $f_n(\tau) = 0$ for $\tau < 0$.

Before using Eq. (48), we confirm that it is equivalent to the more familiar representation obtained by combining a Laplace transform with separation of variables (Section 3). Let $T = t - (r-a)/c$ so that Eq. (48) becomes

$$u_n(r,t) = \frac{a}{r}\int_0^{T} f_n(\tau)P_n\left(\frac{c}{r}\left[T-\tau+\frac{r}{c}\right]\right)\mathrm{d}\tau, \quad T > 0. \tag{49}$$

The right-hand side is a Laplace convolution, so we take the Laplace transform with respect to T,

$$\int_0^\infty u_n(r,t)\,\mathrm{e}^{-sT}\,\mathrm{d}T = \frac{a}{r}F_n(s)K(s;r), \tag{50}$$

where $F_n(s) = \mathcal{L}\{f_n\}$ is the Laplace transform of f_n,

$$K(s;r) = \int_0^\infty \mathrm{e}^{-st}P_n\left(\frac{c}{r}\left[t+\frac{r}{c}\right]\right)\mathrm{d}t = \frac{2r}{\pi c}\mathrm{e}^{sr/c}k_n(sr/c) \tag{51}$$

and we have used [30, 7.143.1]. Once F_n has been determined using the boundary condition (see below), we invert the Laplace transform, which means we multiply Eq. (50) by e^{sT} and integrate over the Bromwich contour, Br. This gives

$$u_n(r,t) = \frac{a}{\pi^2 c\,\mathrm{i}}\int_{\mathrm{Br}} F_n(s)k_n(sr/c)\,\mathrm{e}^{s(t+a/c)}\,\mathrm{d}s, \tag{52}$$

which has the same form as Eq. (23). Alternatively, we could invert $F_n(s)$ giving $f_n(t)$, and then Eq. (48) provides a different representation for $u_n(r,t)$, one whose properties have not been fully investigated.

4.1. Dirichlet boundary condition

For the Dirichlet boundary condition, use of $u_n(a,t) = d_n(t)$ in Eq. (48) gives

$$\int_0^t f_n(\tau)P_n\left(\frac{c}{a}\left[t-\tau+\frac{a}{c}\right]\right)\mathrm{d}\tau = d_n(t), \quad t > 0, \tag{53}$$

a Volterra integral equation of the first kind for f_n. Implicit in this equation is the constraint $d_n(0) = 0$, which is consistent with Eq. (27). Once f_n has been found, u_n is given by Eq. (48).

As a check, we can solve Eq. (53) by taking its Laplace transform. Doing this gives

$$F_n(s)K(s;a) = D_n(s), \tag{54}$$

where $D_n = \mathcal{L}\{d_n\}$ and K is defined by Eq. (51). Solving for F_n followed by substitution in Eq. (52) gives precisely Eq. (28).

Let us go further. From Eqs. (29), (51) and (54), we have

$$F_n(s) = \frac{D_n(s)}{K(s;a)} = \frac{(sa/c)^n}{\theta_n(sa/c)}\,sD_n(s) = (1+X_n(s))\,sD_n(s), \tag{55}$$

say, where

$$X_n(s) = \frac{(sa/c)^n - \theta_n(sa/c)}{\theta_n(sa/c)} = \mathcal{L}\{\chi_n\}$$

for some function $\chi_n(t)$. This function can be found by inverting X_n:

$$\chi_n(t) = \frac{c}{a}\sum_{j=1}^{n} b_{n,j}\exp(\beta_{n,j}ct/a), \tag{56}$$

where

$$b_{n,j} = \frac{(\beta_{n,j})^n}{\theta_n'(\beta_{n,j})} = \frac{\pi\exp(-\beta_{n,j})}{2\beta_{n,j}\,k_n'(\beta_{n,j})}, \tag{57}$$

and we have used Eq. (38). It turns out that, just like $a_{n,j}(r/a)$ in Eq. (36), $b_{n,j}$ grows exponentially with n (see the Appendix). However, there is no dependence on r.

We have $sD_n(s) = \mathcal{L}\{d_n'\}$ (recall that $d_n(0) = 0$) and $X_n(s) = \mathcal{L}\{\chi_n\}$. Hence, inverting Eq. (55),

$$f_n(t) = d_n'(t) + \int_0^t d_n'(\eta)\, \chi_n(t-\eta)\, \mathrm{d}\eta.$$

Substituting in Eq. (49) gives

$$\begin{aligned} u_n(r,t) &= \frac{a}{r}\int_0^T f_n(\tau)\mathcal{Q}_n(r, T-\tau)\,\mathrm{d}\tau \\ &= \frac{a}{r}\int_0^T d_n'(\eta)\mathcal{Q}_n(r, T-\eta)\,\mathrm{d}\eta + \frac{a}{r}\int_0^T\int_0^\tau d_n'(\eta)\,\chi_n(\tau-\eta)\,\mathcal{Q}_n(r, T-\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau \\ &= \frac{a}{r}\int_0^T d_n'(\eta)\left\{\mathcal{Q}_n(r, T-\eta) + \int_\eta^T \chi_n(\tau-\eta)\,\mathcal{Q}_n(r, T-\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\eta \\ &= \frac{a}{r}\int_0^T d_n'(\eta) L_n(r, T-\eta)\,\mathrm{d}\eta, \quad T = t-(r-a)/c > 0, \end{aligned} \tag{58}$$

where $\mathcal{Q}_n(r,t) = P_n(1+ct/r)$ and

$$L_n(r,t) = \mathcal{Q}_n(r,t) + \int_0^t \chi_n(\sigma)\mathcal{Q}_n(r, t-\sigma)\,\mathrm{d}\sigma.$$

Notice that $\mathcal{Q}_n$ depends on r but χ_n does not. Eq. (58) is an exact formula for the solution of the Dirichlet problem with zero initial conditions.

Let us make another observation concerning the Volterra integral equation of the first kind for f_n, Eq. (53). Make the substitution $f_n(t) = g_n'(t)$ with $g_n(0) = 0$. After an integration by parts, we find that g_n satisfies

$$g_n(t) + \frac{c}{a}\int_0^t g_n(\tau)\,P_n'(1+[t-\tau]c/a)\,\mathrm{d}\tau = d_n(t), \quad t > 0, \tag{59}$$

a Volterra integral equation of the second kind. Such equations are attractive because they can always be solved by iteration. Similar equations (with different right-hand sides) can be found in [4, Eq. (3.12)] and [31, Eq. (A.6)], and below as Eq. (61). In terms of g_n, Eq. (49) gives

$$u_n(r,t) = \frac{a}{r}g_n(T) + \frac{ac}{r^2}\int_0^T g_n(\tau)\,P_n'\left(\frac{c}{r}\left[t-\tau+\frac{a}{c}\right]\right)\mathrm{d}\tau, \quad T = t-(r-a)/c > 0. \tag{60}$$

4.2. Pressure boundary condition

For the pressure boundary condition, we differentiate Eq. (48) with respect to t,

$$\frac{\partial u_n}{\partial t} = \frac{a}{r}f_n(T) + \frac{ac}{r^2}\int_0^T f_n(\tau)P_n'\left(\frac{c}{r}\left[t-\tau+\frac{a}{c}\right]\right)\mathrm{d}\tau,$$

using $P_n(1) = 1$. The boundary condition, $\partial u_n/\partial t = q_n(t)$ at $r = a$ (see Eq. (41)), gives

$$f_n(t) + \frac{c}{a}\int_0^t f_n(\tau)P_n'(1+[t-\tau]/t_0)\,\mathrm{d}\tau = q_n(t), \quad t > 0, \tag{61}$$

where $t_0 = a/c$. Eq. (61) has the same form as Eq. (59). We can solve it by applying $\mathcal{L}$. From Eq. (51), we have

$$\mathcal{L}\{P_n(1+t/t_0)\} = (2/\pi)t_0 \mathrm{e}^{st_0} k_n(st_0) = K(s; ct_0), \tag{62}$$

whence

$$\mathcal{L}\{P_n'(1+t/t_0)\} = t_0(sK-1) \tag{63}$$

and Eq. (61) gives $sK(s;a)F_n(s) = Q_n(s)$. Substitution for F_n in Eq. (52) leads back to Eq. (42).

4.3. Neumann boundary condition

For the Neumann boundary condition, $\partial u_n/\partial r = v_n(t)$ on $r = a$, we differentiate Eq. (48) and obtain

$$\frac{\partial u_n}{\partial r} = -\frac{u_n}{r} - \frac{a}{rc}f_n(T) - \frac{ac}{r^3}\int_0^T f_n(\tau)\left(t-\tau+\frac{a}{c}\right)P_n'\left(\frac{c}{r}\left[t-\tau+\frac{a}{c}\right]\right)\mathrm{d}\tau.$$

Applying the boundary condition, we find that

$$f_n(t) + \frac{c}{a}\int_0^t f_n(\tau)\mathcal{K}_n(t-\tau)\,\mathrm{d}\tau = -c v_n(t), \quad t > 0, \tag{64}$$

where

$$\mathcal{K}_n(t) = P_n(1+t/t_0) + (1+t/t_0)P_n'(1+t/t_0).$$

Eq. (64) is a Volterra integral equation of the second kind for f_n. Again, to solve it, we apply $\mathcal{L}$. Using Eqs. (62), (63) and

$$\mathcal{L}\{tP_n'(1+t/t_0)\} = -t_0^2\frac{\partial K}{\partial t_0} = -t_0^2\left\{(s+t_0^{-1})K + (2/\pi)st_0\mathrm{e}^{st_0}k_n'(st_0)\right\},$$

we obtain

$$\begin{aligned}\mathcal{L}\{\mathcal{K}(t)\} &= K + t_0(sK-1) - t_0\left\{(s+t_0^{-1})K + (2/\pi)st_0\mathrm{e}^{st_0}k_n'(st_0)\right\}\\ &= -t_0 - (2/\pi)st_0^2\mathrm{e}^{st_0}k_n'(st_0),\end{aligned}$$

and then Eq. (64) gives

$$(2/\pi)st_0\mathrm{e}^{st_0}k_n'(st_0)F_n(s) = cV_n(s). \tag{65}$$

Solving for F_n and substitution in Eq. (52) gives precisely Eq. (44).

5. Discussion

We have solved IBVPs for a sphere using the integral representation for $u_n(r,t)$, Eq. (48), containing Legendre polynomials in a similarity variable, $P_n(ct/r)$. The density function in the integral representation, f_n, solves a Volterra integral equation; there is a different integral equation depending on the choice of boundary condition on the sphere. If these integral equations are solved by Laplace transforms, we recover the equations that would have been obtained if the IBVPs had been treated by the classical approach: Laplace transform in t combined with separation of variables. However, we could avoid using Laplace transforms and solve the Volterra integral equation directly. For each IBVP, we obtained an integral equation of the second kind, so convergent iterative schemes are available. Good numerical methods are also available [32]. We leave investigations in this direction for future work.

Scattering by a sphere is a relatively simple problem. Recently, there has been increased activity on scattering by objects of other shapes, using time-domain boundary integral equations; see the reviews by Ha-Duong [33] and Costabel [34] and the book by Sayas [35]. Nevertheless, we anticipate that benchmark solutions for scattering by a sphere will continue to be valuable.

Appendix. Some asymptotics

Recall the definition of the reverse Bessel polynomial, Eq. (29),

$$\theta_n(z) = (2/\pi)\,z^{n+1}\mathrm{e}^z\,k_n(z) = (2z/\pi)^{1/2}\,z^n\,\mathrm{e}^z\,K_{n+1/2}(z),$$

where K_ν is a modified Bessel function. Let β denote any zero of θ_n: $\theta_n(\beta) = k_n(\beta) = 0$. It is known that all zeros satisfy $\mathrm{Re}\,\beta < 0$.

We are interested in estimating

$$a_n(\rho) = \frac{\theta_n(\beta\rho)}{\rho^n\,\theta_n'(\beta)} \quad \text{and} \quad b_n = \frac{\beta^n}{\theta_n'(\beta)} \tag{A.1}$$

as $n\to\infty$, where $\rho > 1$; see Eqs. (37) and (57). It is known that, asymptotically, the zeros of θ_n lie on a certain convex arc, symmetric about the real axis, meeting the imaginary axis at $\pm\mathrm{i}n$ and crossing the real axis at $-n\zeta_0$ with $\zeta_0 \simeq 0.66$:

$$\beta \sim n\zeta\,\mathrm{e}^{\mathrm{i}\pi} \text{ as } n\to\infty \text{ for some } \zeta \text{ with } \zeta_0 < |\zeta| \le 1 \text{ and } |\arg\zeta| < \frac{1}{2}\pi. \tag{A.2}$$

Thus, both n and $|\beta|$ are large. The number ζ_0 is defined by

$$\zeta_0 = \sqrt{t_0^2 - 1} \text{ where } t_0 \text{ is the positive real root of } \coth t_0 = t_0. \tag{A.3}$$

These results are due to Olver [36, p. 354]; for the convex arc, rotate any of the following figures clockwise by $\pi/2$: [36, Fig. 15], [37, Fig. 9.6], [2, Fig. 10.21.6]. For a plot of the zeros of θ_{10} and θ_{11}, see [13, Fig. 5]. For an early tabulation of $\beta_{n,m}$, $n \le 7$, see [38, Table I].

Use of standard asymptotic methods

Suppose ζ in Eq. (A.2) is real. In this special case, we can use integral representations combined with *Laplace's method* [39, §5.1], [40, Chapter 3, §7]. From [37, 9.6.23], we have the integral representation

$$K_\nu(z) = \frac{\sqrt{\pi}\,(z/2)^\nu}{\Gamma(\nu+1/2)} \int_1^\infty \mathrm{e}^{-z\xi}\,\left(\xi^2-1\right)^{\nu-1/2}\,\mathrm{d}\xi,$$

valid for $\operatorname{Re}\nu > -\frac{1}{2}$ and $\operatorname{Re}z > 0$. As $k_n(z) = \sqrt{\pi/(2z)}\,K_{n+1/2}(z)$,

$$k_n(z) = \frac{\pi}{2}\,\frac{(z/2)^n}{n!} \int_1^\infty \mathrm{e}^{-z\xi}\,\left(\xi^2-1\right)^n\,\mathrm{d}\xi, \quad \operatorname{Re}z > 0.$$

Then, using the definition $\theta_n(z) = (2/\pi)\,z^{n+1}\mathrm{e}^z\,k_n(z)$ and the substitution $\xi = t+1$,

$$\theta_n(z) = \frac{z^{2n+1}}{2^n\,n!} \int_0^\infty \mathrm{e}^{-zt}\,t^n(2+t)^n\,\mathrm{d}t, \quad \operatorname{Re}z > 0.$$

Differentiating,

$$\theta_n'(z) = \frac{2n+1}{z}\theta_n(z) - \frac{z^{2n+1}}{2^n\,n!} \int_0^\infty \mathrm{e}^{-zt}\,t^{n+1}(2+t)^n\,\mathrm{d}t, \quad \operatorname{Re}z > 0.$$

These integral representations are valid for $\operatorname{Re}z > 0$ whereas we want to evaluate $\theta_n(z)$ and $\theta_n'(z)$ when $\operatorname{Re}z \le 0$. To effect the analytic continuation into this half of the z-plane, we rotate the contour of integration in the complex t-plane [40, Chapter 4, §1]. Once the contour is on the negative real t-axis, we obtain

$$\theta_n(z) = \frac{z^{2n+1}}{2^n\,n!}(-1)^{n+1} \int_0^\infty \mathrm{e}^{zt}\,t^n(2-t)^n\,\mathrm{d}t, \quad \operatorname{Re}z < 0$$

and

$$\theta_n'(z) = \frac{2n+1}{z}\theta_n(z) - \frac{z^{2n+1}}{2^n\,n!}(-1)^n \int_0^\infty \mathrm{e}^{zt}\,t^{n+1}(2-t)^n\,\mathrm{d}t, \quad \operatorname{Re}z < 0.$$

Suppose that $\beta = -n\zeta$ with ζ real and positive. Then

$$\theta_n'(\beta) = \frac{\beta^{2n+1}}{2^n\,n!}(-1)^{n+1} \int_0^\infty \mathrm{e}^{-n\zeta t}\,\{t(2-t)\}^n\,t\,\mathrm{d}t. \tag{A.4}$$

Splitting the range of integration at $t = 2$, write the integral as

$$\int_0^2 \mathrm{e}^{nh_1(t)}\,t\,\mathrm{d}t + (-1)^n \int_2^\infty \mathrm{e}^{nh_2(t)}\,t\,\mathrm{d}t$$

where

$$h_1(t) = -\zeta t + \log t + \log(2-t) \quad \text{and} \quad h_2(t) = -\zeta t + \log t + \log(t-2).$$

We estimate the integrals for large n using Laplace's method. We have

$$h_1'(t) = h_2'(t) = -\zeta + t^{-1} + (t-2)^{-1}.$$

Thus $h_j'(t) = 0$ at $t = t_j$ $(j = 1, 2)$, where

$$t_1(\zeta) = \zeta^{-1}\bigl(\zeta + 1 - \sqrt{\zeta^2+1}\bigr) \quad \text{and} \quad t_2(\zeta) = \zeta^{-1}\bigl(\zeta + 1 + \sqrt{\zeta^2+1}\bigr).$$

Note that $0 < t_1(\zeta) < 1$ and $t_2(\zeta) > 2$ for all $\zeta > 0$. Then Laplace's method shows that the integral in Eq. (A.4) is asymptotically proportional to

$$\exp\{n\max[h_1(t_1), h_2(t_2)]\} \quad \text{as } n \to \infty.$$

Some calculation gives

$$h_1(t_1) = \log(2/\zeta) - 1 - \zeta + \eta(\zeta) \quad \text{and} \quad h_2(t_2) = \log(2/\zeta) - 1 - \zeta - \eta(\zeta)$$

where

$$\eta(\zeta) = (1+\zeta^2)^{1/2} + \log\{\zeta/[1+(1+\zeta^2)^{1/2}]\}. \tag{A.5}$$

We note that $\eta(\zeta_0) = 0$ where $\zeta_0 \simeq 0.66$ is defined by Eq. (A.3). This is relevant because there is a zero of θ_n close to $-n\zeta_0$. (For further discussion of η, see [40, p. 375].)

We have $h_1(t_1) - h_2(t_2) = 2\eta(\zeta)$. As $\eta'(\zeta) = \zeta^{-1}(1+\zeta^2)^{1/2}$, we see that $h_1(t_1) > h_2(t_2)$ for $\zeta > \zeta_0$, whence the dominant contribution comes from $h_1(t_1)$ for such ζ. Then, from Eq. (A.4),

$$\frac{\theta_n'(\beta)}{\beta^n} \simeq \frac{(-\beta)^n}{2^n\, n!}\left(\frac{2}{\zeta}\right)^n \mathrm{e}^{-n(1+\zeta)}\, \mathrm{e}^{n\eta(\zeta)} = \frac{n^n}{\mathrm{e}^n\, n!}\, \mathrm{e}^{-n\{\zeta-\eta(\zeta)\}}.$$

Using Stirling's formula, $n^n \simeq \mathrm{e}^n\, n!$, and ignoring algebraic factors, we obtain

$$b_n = \frac{\beta^n}{\theta_n'(\beta)} \simeq \mathrm{e}^{n\{\zeta-\eta(\zeta)\}} \quad \text{as } n \to \infty. \tag{A.6}$$

In particular, b_n grows approximately as $\mathrm{e}^{n\zeta_0}$ when $\beta \simeq -n\zeta_0$.

Turning now to $a_n(\rho)$, a similar calculation gives

$$\begin{aligned}\frac{\theta_n(\beta\rho)}{\beta^n\rho^n} &= \frac{(\beta\rho)^{n+1}}{2^n\, n!}(-1)^{n+1}\int_0^\infty \mathrm{e}^{-n\zeta\rho t}\, t^n(2-t)^n\, \mathrm{d}t \\ &\simeq \frac{(n\zeta\rho)^n}{2^n\, n!}\left(\frac{2}{\zeta\rho}\right)^n \mathrm{e}^{-n(1+\zeta\rho)}\, \mathrm{e}^{n\eta(\zeta\rho)} \simeq \mathrm{e}^{-n\{\zeta\rho-\eta(\zeta\rho)\}}\end{aligned}$$

whence

$$a_n(\rho) \simeq \mathrm{e}^{n\phi(\rho,\zeta)} \quad \text{with } \phi(\rho,\zeta) = -(\rho-1)\zeta + \eta(\zeta\rho) - \eta(\zeta), \tag{A.7}$$

with η defined by Eq. (A.5). For example, when $\rho = 2$ and $\zeta = \zeta_0$, $\phi(2,\zeta_0) = \eta(2\zeta_0) - \zeta_0 \simeq 0.30$. We also note that $\phi(\rho,\zeta)$ is an increasing function of ρ.

When ζ is not real, we can no longer use Laplace's method. Instead, we can use the *method of steepest descent* [39, Chapter 7]. This applies to contour integrals of the form $\int_C f(z)\, \mathrm{e}^{nh(z)}\, \mathrm{d}z$; the contribution from a simple saddle point z_0 (where $h'(z_0) = 0$) is found to be proportional to $\exp\{nh(z_0)\}$ [39, Eq. (7.2.10)] (ignoring multiplicative factors that are algebraic in n).

Use of known asymptotic approximations

From Eq. (A.1), we have

$$a_n(\rho) = \mathrm{e}^{(\rho-1)\beta}\sqrt{\rho}\,\frac{K_\nu(\beta\rho)}{K_\nu'(\beta)} \quad \text{and} \quad b_n = \sqrt{\frac{\pi}{2\beta}}\,\frac{\mathrm{e}^{-\beta}}{K_\nu'(\beta)} \quad \text{with } \nu = n + \frac{1}{2}. \tag{A.8}$$

As β grows with n according to Eq. (A.2), we require the asymptotics of $K_\nu(\nu z)$ for large ν. These are given by [2, 10.41.4] or [37, 9.7.8] but only when $\operatorname{Re} z > 0$ whereas we want asymptotics for $\operatorname{Re} z < 0$. Therefore we first continue K_ν analytically, using [2, 10.34.2] or [37, 9.6.31]:

$$K_\nu(\nu\zeta\,\mathrm{e}^{\mathrm{i}\pi}) = \mathrm{e}^{-\nu\pi\mathrm{i}}K_\nu(\nu\zeta) - \pi\,\mathrm{i}\,I_\nu(\nu\zeta). \tag{A.9}$$

Thus we also need the asymptotics of the other modified Bessel function $I_\nu(\nu z)$, as given in [2, 10.41.3] or [37, 9.7.7]. Hence

$$K_\nu(\nu\zeta\,\mathrm{e}^{\mathrm{i}\pi}) \sim \sqrt{\frac{\pi}{2\nu}}\,\frac{1}{(1+\zeta^2)^{1/4}}\left(\mathrm{e}^{-\nu\pi\mathrm{i}}\mathrm{e}^{-\nu\eta(\zeta)} - \mathrm{i}\,\mathrm{e}^{\nu\eta(\zeta)}\right) \quad \text{as } \nu \to \infty, \tag{A.10}$$

where $\eta(\zeta)$ is defined by Eq. (A.5) and $|\arg\,\zeta| < \frac{1}{2}\pi$.

Similarly, differentiating Eq. (A.9) with respect to ζ gives

$$\begin{aligned}K_\nu'(\nu\zeta\,\mathrm{e}^{\mathrm{i}\pi}) &= -\mathrm{e}^{-\nu\pi\mathrm{i}}K_\nu'(\nu\zeta) + \pi\,\mathrm{i}\,I_\nu'(\nu\zeta) \\ &\sim \sqrt{\frac{\pi}{2\nu}}\,\frac{(1+\zeta^2)^{1/4}}{\zeta}\left(\mathrm{e}^{-\nu\pi\mathrm{i}}\mathrm{e}^{-\nu\eta(\zeta)} + \mathrm{i}\,\mathrm{e}^{\nu\eta(\zeta)}\right) \quad \text{as } \nu \to \infty,\end{aligned} \tag{A.11}$$

after use of [2, 10.41.5 & 6] or [37, 9.7.9 & 10].

Assuming that $\mathrm{e}^{+\nu\eta}$ is the dominant exponential in Eqs. (A.10) and (A.11), we obtain the estimate

$$a_n(\rho) \sim \frac{-\zeta\sqrt{\rho}\,\mathrm{e}^{n\phi(\rho,\zeta)}}{(1+(\rho\zeta)^2)^{1/4}(1+\zeta^2)^{1/4}} \tag{A.12}$$

from Eq. (A.8) using $\beta \sim -n\zeta$ and $\nu \sim n$. This result agrees precisely with Eq. (A.7). Similarly, we find agreement with Eq. (A.6).

Greengard et al. [13] have given a rough argument for the exponential growth of $a_n(\rho)$. Field and Lau [41] have made a more detailed study but their argument is incomplete: they obtain growth as $\mathrm{e}^{n\psi}$ with $\psi(\rho,\zeta) = -(\rho-1)\zeta + \eta(\zeta\rho)$ (see [41, Eqs. (23) & (24)]), which differs from $\phi(\rho,\zeta)$ in Eq. (A.7) by the term $\eta(\zeta)$. This difference vanishes when $\zeta = \zeta_0$.

References

[1] P.A. Martin, The pulsating orb: solving the wave equation outside a ball, Proc. R. Soc. A 472 (2016) 20160037.
[2] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/.
[3] H. Bateman, A partial differential equation associated with Poisson's work on the theory of sound, Amer. J. Math. 60 (1938) 293–296.
[4] F.G. Friedlander, On the radiation field of pulse solutions of the wave equation, Proc. R. Soc. A 269 (1962) 53–65.
[5] O.M. Buyukdura, S.S. Koc, Two alternative expressions for the spherical wave expansion of the time domain scalar free-space Green's function and an application: Scattering by a soft sphere, J. Acoust. Soc. Am. 101 (1997) 87–91.
[6] A. Shlivinski, E. Heyman, A.J. Devaney, Time domain radiation by scalar sources: Plane wave to multipole transform, J. Math. Phys. 42 (2001) 5915–5919.
[7] E.T. Copson, On the Riemann–Green function, Arch. Ration. Mech. Anal. 1 (1958) 324–348.
[8] V.V. Borisov, A.V. Manankova, A.B. Utkin, Spherical harmonic representation of the electromagnetic field produced by a moving pulse of current density, J. Phys. A: Math. Gen. 29 (1996) 4493–4514.
[9] J. Li, D. Dault, B. Shanker, A quasianalytical time domain solution for scattering from a homogeneous sphere, J. Acoust. Soc. Am. 135 (2014) 1676–1685.
[10] J. Li, B. Shanker, Time-dependent Debye–Mie series solutions for electromagnetic scattering, IEEE Trans. Antennas Propag. 63 (2015) 3644–3653.
[11] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge University Press, Cambridge, 2005.
[12] E. Grosswald, Bessel Polynomials, in: Lect. Notes in Math., vol. 698, Springer, Berlin, 1978.
[13] L. Greengard, T. Hagstrom, S. Jiang, The solution of the scalar wave equation in the exterior of a sphere, J. Comput. Phys. 274 (2014) 191–207.
[14] C.H. Wilcox, The initial–boundary value problem for the wave equation in an exterior domain with spherical boundary, Notices Amer. Math. Soc. 6 (1959) 869–870.
[15] J.A. Hamilton, R.J. Astley, Exact solutions for transient spherical radiation, J. Acoust. Soc. Am. 109 (2001) 1848–1858.
[16] J. Brillouin, Rayonnement transitoire des sources sonores et problèmes connexes, Ann. Telecommun. 5 (1950) 160–172. & 179–194.
[17] S. Hanish, A Treatise on Acoustic Radiation, third ed., Naval Research Laboratory, Washington, D.C, 1989.
[18] F.G. Friedlander, Sound Pulses, Cambridge University Press, Cambridge, 1958.
[19] R.G. Barakat, Transient diffraction of scalar waves by a fixed sphere, J. Acoust. Soc. Am. 32 (1960) 61–66.
[20] D.S. Cohen, G.H. Handelman, Scattering of a plane acoustical wave by a spherical obstacle, J. Acoust. Soc. Am. 38 (1965) 827–834.
[21] O.G. Kozina, G.I. Makarov, N.N. Shaposhnikov, Transient processes in the acoustic fields generated by a vibrating spherical segment, Sov. Phys. - Acoust. 8 (1962) 53–57.
[22] M.C. Junger, W. Thompson Jr., Oscillatory acoustic transients radiated by impulsively accelerated bodies, J. Acoust. Soc. Am. 38 (1965) 978–986.
[23] G.E. Tupholme, Generation of an axisymmetrical acoustic pulse by a deformable sphere, Proc. Cambridge Philos. Soc. 63 (1967) 1285–1308.
[24] H. Huang, G.C. Gaunaurd, Transient diffraction of a plane step pressure pulse by a hard sphere: neoclassical solution, J. Acoust. Soc. Am. 104 (1998) 3236–3244.
[25] D. Gottlieb, C.-W. Shu, On the Gibbs phenomenon and its resolution, SIAM Rev. 39 (1997) 644–668.
[26] B.S. Berger, D. Klein, Application of the Cesaro mean to the transient interaction of a spherical acoustic wave and a spherical elastic shell, J. Appl. Mech. 39 (1972) 623–625.
[27] P. Zhang, T.L. Geers, Excitation of a fluid-filled, submerged spherical shell by a transient acoustic wave, J. Acoust. Soc. Am. 93 (1993) 696–705.
[28] A. Gelb, The resolution of the Gibbs phenomenon for spherical harmonics, Math. Comp. 66 (1997) 699–717.
[29] T. Tokita, Exponential decay of solutions for the wave equation in the exterior domain with spherical boundary, J. Math. Kyoto Univ. 12 (1972) 413–430.
[30] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, fourth ed., Academic Press, New York, 1980.
[31] A. Karlsson, G. Kristensson, Wave splitting in the time domain for a radially symmetric geometry, Wave Motion 12 (1990) 197–211.
[32] C.T.H. Baker, A perspective on the numerical treatment of Volterra equations, J. Comput. Appl. Math. 125 (2000) 217–249.
[33] T. Ha-Duong, On retarded potential boundary integral equations and their discretisation, in: M. Ainsworth, P. Davies, D. Duncan, B. Rynne, P. Martin (Eds.), Topics in Computational Wave Propagation: Direct and Inverse Problems, Springer, Berlin, 2003, pp. 301–336.
[34] M. Costabel, Time-dependent problems with the boundary integral equation method, in: E. Stein, R. de Borst, T.J.R. Hughes (Eds.), Encyclopedia of Computational Mechanics, Vol. 1, Wiley, New York, 2004, pp. 703–721.
[35] F.-J. Sayas, Retarded Potentials and Time Domain Boundary Integral Equations: A Road Map, Springer, Switzerland, 2016.
[36] F.W.J. Olver, The asymptotic expansion of Bessel functions of large order, Phil. Trans. R. Soc. A 247 (1954) 328–368.
[37] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1965.
[38] V.H. Weston, Pulse return from a sphere, IRE Trans. Antennas Propag. 7 (1959) S43–S51.
[39] N. Bleistein, R.A. Handelsman, Asymptotic Expansions of Integrals, second ed., Dover, New York, 1986.
[40] F.W.J. Olver, Asymptotics and Special Functions, A.K. Peters, Natick, Massachusetts, 1997.
[41] S.E. Field, S.R. Lau, Fast evaluation of far-field signals for time-domain wave propagation, J. Sci. Comput. 64 (2015) 647–669.