Explicit energy calculation for a charged elliptical plate

P. A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887, USA

Abstract

Potential problems for thin elliptical plates are solved exactly with emphasis on computation of the electrostatic energy. Expansions in terms of Jacobi polynomials are used.

Keywords: Charged elliptical plate, Jacobi polynomials

1. Introduction

Let Ω denote a thin flat plate lying in the plane $z = 0$, where $Oxyz$ is a system of Cartesian coordinates. The charge distribution on the plate is $\sigma(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y)$. The potential on the plate is

$$f(\boldsymbol{x}') = \frac{1}{4\pi} \int_\Omega \frac{\sigma(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{x}' \in \Omega. \tag{1}$$

The electrostatic energy, I, is given by

$$I = \int_\Omega f(\boldsymbol{x}') \, \overline{\sigma(\boldsymbol{x}')} \, \mathrm{d}\boldsymbol{x}' = \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\overline{\sigma(\boldsymbol{x}')} \, \sigma(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{x}',$$

where the overbar denotes complex conjugation. In a recent paper, Laurens and Tordeux [1] showed how to calculate I when Ω is an ellipse and $\sigma(x, y)$ is a linear function of x and y. We generalize their result: we allow arbitrary polynomials in x and y, and we incorporate a weight function to represent singular behaviour near the edge of the plate.

2. An elliptical plate

When Ω is elliptical, it is convenient to introduce coordinates ρ and ϕ so that

$$x = a\rho \cos\phi, \quad y = b\rho \sin\phi, \quad 0 < b \leq a. \tag{2}$$

Then, Ω is defined by $\Omega = \{(x, y, z) : 0 \leq \rho < 1, -\pi \leq \phi < \pi, z = 0\}$. Thus, $\rho = 1$ gives the edge of the plate Ω.

Equation (1) can be regarded as an integral equation for σ when f is given [2, 3, 4]. Alternatively, (1) can be regarded as a formula for f when σ is given; this is the view adopted in [1].

When f is given, the function σ is infinite at $\rho = 1$, in general. In fact, there is a general result, known as *Galin's theorem*, asserting that if $f(x, y)$ is a polynomial, then σ is a polynomial of the same degree multiplied by $(1 - \rho^2)^{-1/2}$. In particular, if f is a constant, then σ is a constant multiple of $(1 - \rho^2)^{-1/2}$.

Email address: `pamartin@mines.edu` (P. A. Martin)

Appeared as: *Applied Mathematics Letters* **26** (2013) *893–898*

3. Fourier transforms

We start with a standard Fourier integral representation,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\boldsymbol{\xi}|^{-1}\exp\{\mathrm{i}\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{x}')\}\,\mathrm{d}\boldsymbol{\xi},\tag{3}$$

where $\boldsymbol{\xi}=(\xi,\eta)$. Henceforth, we write $\iint$ when the integration limits are as in (3). Thus

$$f(\boldsymbol{x}')=\frac{1}{4\pi}\iint|\boldsymbol{\xi}|^{-1}U(\boldsymbol{\xi})\exp(-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}')\,\mathrm{d}\boldsymbol{\xi}\tag{4}$$

and

$$I=\frac{1}{2}\iint|\boldsymbol{\xi}|^{-1}\,|U(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi},\tag{5}$$

where

$$U(\boldsymbol{\xi})=\frac{1}{2\pi}\int_\Omega\sigma(\boldsymbol{x})\,\exp(\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\tag{6}$$

For an elliptical plate, we write the Fourier-transform variable $\boldsymbol{\xi}$ as

$$\xi=(\lambda/a)\cos\psi\quad\text{and}\quad\eta=(\lambda/b)\sin\psi.$$

Then, using (2), $\boldsymbol{\xi}\cdot\boldsymbol{x}=\lambda\rho\cos(\phi-\psi)$. Hence,

$$\exp(\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x})=\sum_{n=0}^{\infty}\epsilon_n\,\mathrm{i}^n J_n(\lambda\rho)\cos n(\phi-\psi),$$

where J_n is a Bessel function, $\epsilon_0=1$ and $\epsilon_n=2$ for $n\geq1$.

In order to evaluate $U(\boldsymbol{\xi})$, defined by (6), we suppose that σ has a Fourier expansion,

$$\sigma(\boldsymbol{x})=\sum_{m=0}^{\infty}\sigma_m(\rho)\cos m\phi+\sum_{m=1}^{\infty}\tilde{\sigma}_m(\rho)\sin m\phi.\tag{7}$$

Then, using $\mathrm{d}\boldsymbol{x}=ab\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$ and defining

$$\mathcal{S}_n[g_n;\lambda]=\int_0^1 g_n(\rho)J_n(\lambda\rho)\,\rho\,\mathrm{d}\rho,\tag{8}$$

we obtain

$$U(\boldsymbol{\xi})=ab\sum_{n=0}^{\infty}\mathrm{i}^n\mathcal{S}_n[\sigma_n;\lambda]\cos n\psi+ab\sum_{n=1}^{\infty}\mathrm{i}^n\mathcal{S}_n[\tilde{\sigma}_n;\lambda]\sin n\psi.$$

We have $\mathrm{d}\boldsymbol{\xi}=(ab)^{-1}\lambda\,\mathrm{d}\lambda\,\mathrm{d}\psi$ and $|\boldsymbol{\xi}|=(\lambda/b)(1-k^2\cos^2\psi)^{1/2}$, where $k^2=1-(b/a)^2$; k is the eccentricity of the ellipse.

From (4), we obtain

$$f(\boldsymbol{x})=f_0(\rho)+2\sum_{n=1}^{\infty}\left\{f_n(\rho)\cos n\phi+\tilde{f}_n(\rho)\sin n\phi\right\}$$

where

$$f_n(\rho)=\frac{b}{2\pi}\sum_{m=0}^{\infty}I^c_{mn}(k)\int_0^\infty J_n(\lambda\rho)\,\mathcal{S}_m[\sigma_m;\lambda]\,\mathrm{d}\lambda,\tag{9}$$

$$\tilde{f}_n(\rho)=\frac{b}{2\pi}\sum_{m=1}^{\infty}I^s_{mn}(k)\int_0^\infty J_n(\lambda\rho)\,\mathcal{S}_m[\tilde{\sigma}_m;\lambda]\,\mathrm{d}\lambda,\tag{10}$$

$$I^c_{mn}(k) = \mathrm{i}^m(-\mathrm{i})^n \int_0^\pi \frac{\cos m\psi \cos n\psi}{\sqrt{1-k^2\cos^2\psi}}\,\mathrm{d}\psi, \tag{11}$$

$$I^s_{mn}(k) = \mathrm{i}^m(-\mathrm{i})^n \int_0^\pi \frac{\sin m\psi \sin n\psi}{\sqrt{1-k^2\cos^2\psi}}\,\mathrm{d}\psi \tag{12}$$

and we have noticed that $|\boldsymbol{\xi}|$ is an even function of ψ. The integrals I^c_{mn} and I^s_{mn} can be reduced to combinations of complete elliptic integrals, $K(k)$ and $E(k)$. They are zero unless m and n are both even or both odd. (See [5, p. 276] for a discussion of similar integrals.) Explicit formulas for a few of these integrals will be given later.

For the energy, I, (5) gives

$$\begin{aligned} I &= \frac{1}{2a}\int_0^\infty \int_{-\pi}^{\pi} |U(\boldsymbol{\xi})|^2 \frac{\mathrm{d}\psi\,\mathrm{d}\lambda}{\sqrt{1-k^2\cos^2\psi}} \\ &= ab^2 \sum_{m=0}^\infty \sum_{n=0}^\infty I^c_{mn}(k) \int_0^\infty \mathcal{S}_m[\sigma_m;\lambda]\,\overline{\mathcal{S}_n[\sigma_n;\lambda]}\,\mathrm{d}\lambda \\ &\quad + ab^2 \sum_{m=1}^\infty \sum_{n=1}^\infty I^s_{mn}(k) \int_0^\infty \mathcal{S}_m[\tilde\sigma_m;\lambda]\,\overline{\mathcal{S}_n[\tilde\sigma_n;\lambda]}\,\mathrm{d}\lambda. \end{aligned} \tag{13}$$

4. Polynomial expansions

To make further progress, we must be able to evaluate $\mathcal{S}_n[g_n;\lambda]$, defined by (8). We do this by expanding $g_n(\rho)$ using the functions

$$G_j^{(n,\nu)}(\rho) = \rho^n(1-\rho^2)^\nu P_j^{(n,\nu)}(1-2\rho^2),$$

where $P_j^{(n,\nu)}$ is a Jacobi polynomial. The parameter ν controls the behaviour near the edge of the ellipse, $\rho = 1$. Thus, when $\nu = 0$, $G_j^{(n,0)}(\rho)$ is a polynomial; this covers the case discussed in [1]. Setting $\nu = -\frac{1}{2}$ gives appropriate expansion functions when the goal is to solve (1) for σ. We note that Boyd [6, §18.5.1] has advocated using the polynomials $G_j^{(n,0)}(r)$ as radial basis functions in spectral methods for problems posed on a disc, $0 \le r < 1$.

The functions $G_j^{(n,\nu)}$ are orthogonal. To see this, note that Jacobi polynomials satisfy

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(x)\,\mathrm{d}x = h_i(\alpha,\beta)\delta_{ij},$$

where h_i is known and δ_{ij} is the Kronecker delta; see [7, §18.3]. Hence, the substitution $x = 1-2\rho^2$ gives

$$\int_0^1 G_i^{(n,\nu)}(\rho) G_j^{(n,\nu)}(\rho)\,\frac{\rho\,\mathrm{d}\rho}{(1-\rho^2)^\nu} = 2^{-n-\nu-2} h_i(n,\nu)\delta_{ij}. \tag{14}$$

Next, we use *Tranter's integral* [8, 9] to evaluate $\mathcal{S}_n[G_j^{(n,\nu)};\lambda]$:

$$\int_0^1 J_n(\lambda\rho) G_j^{(n,\nu)}(\rho)\,\rho\,\mathrm{d}\rho = \frac{2^\nu}{\lambda^{\nu+1}\,j!}\Gamma(\nu+j+1) J_{2j+n+\nu+1}(\lambda).$$

Thus, if we write

$$\sigma_n(\rho) = \sum_{j=0} \frac{j!\,s_j^n}{2^\nu\Gamma(\nu+j+1)}\, G_j^{(n,\nu)}(\rho), \tag{15}$$

where s_j^n are coefficients, we find that

$$\mathcal{S}_n[\sigma_n;\lambda] = \sum_{j=0} \frac{s_j^n}{\lambda^{\nu+1}}\, J_{2j+n+\nu+1}(\lambda). \tag{16}$$

We also expand $\tilde{\sigma}_n(\rho)$ as (15) but with coefficients $\tilde{s}_j^n$.

If we substitute (16) in (9), we encounter Weber–Schafheitlin integrals; these can be evaluated. We give a simple example later.

If we substitute (16) in (13), we encounter integrals of the type

$$\int_0^\infty \lambda^{-2\mu} J_{p+\mu}(\lambda) J_{q+\mu}(\lambda)\,\mathrm{d}\lambda \tag{17}$$

where $\mu = \nu + 1$, and p and q are non-negative integers. The integral (17) is known as the critical case of the Weber–Schafheitlin integral; its value is [7, eqn 10.22.57]

$$\frac{\Gamma(\frac{1}{2}[p+q+1])\,\Gamma(2\mu)}{2^{2\mu}\,\Gamma(\frac{1}{2}[2\mu+p-q+1])\,\Gamma(\frac{1}{2}[2\mu+q-p+1])\,\Gamma(\frac{1}{2}[4\mu+p+q+1])}. \tag{18}$$

5. Three examples

We discuss three examples. In the first, we examine the dependence on the parameter ν but, for simplicity, we ignore any dependence on the angle ϕ. In the second example, we compare with some results of Roy and Sabina [2] for $\nu = -\frac{1}{2}$. In the third example, we assume that $\sigma(x, y)$ is a general quadratic function of x and y (so that $\nu = 0$); this extends the calculations in [1], where σ was taken as a linear function.

5.1. Example: dependence on ν [1]

For a very simple example, suppose that $\sigma(\boldsymbol{x}) = (1-\rho^2)^\nu$ for some $\nu > -1$. Thus, as $P_0^{(n,\nu)} = 1$, (15) gives $s_0^0 = 2^\nu\Gamma(\nu+1)$. All other coefficients s_j^n and $\tilde{s}_j^n$ are zero. Then, from (16), $\mathcal{S}_0[\sigma_0;\lambda] = s_0^0\lambda^{-\nu-1}J_{\nu+1}(\lambda)$. Hence

$$f(\boldsymbol{x}) = f_0(\rho) = \frac{bs_0^0}{2\pi} I_{00}^c(k) \int_0^\infty \lambda^{-\nu-1} J_0(\lambda\rho) J_{\nu+1}(\lambda)\,\mathrm{d}\lambda, \quad 0 \le \rho < 1. \tag{19}$$

From (11), we obtain

$$I_{00}^c = 2\int_0^{\pi/2} \frac{\mathrm{d}x}{\Delta} = 2K(k), \tag{20}$$

where $\Delta = (1 - k^2\sin^2 x)^{1/2}$. From [7, eqn 10.22.56], the integral in (19) evaluates to

$$\frac{\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} F(\tfrac{1}{2}, -\nu-\tfrac{1}{2}; 1; \rho^2),$$

where F is the Gauss hypergeometric function. Hence

$$f(\boldsymbol{x}) = \frac{b}{2\pi} K(k) \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})} F(\tfrac{1}{2}, -\nu-\tfrac{1}{2}; 1; \rho^2), \quad 0 \le \rho < 1.$$

When $\nu = -\frac{1}{2}$, $F(\frac{1}{2}, 0; 1; \rho^2) = 1$ and $f(\boldsymbol{x}) = \frac{1}{2}bK(k)$, a constant, in accord with Galin's theorem.

When $\nu = 0$, we obtain $f(\boldsymbol{x}) = (2b/\pi^2)K(k)E(\rho)$ for $0 \le \rho < 1$, using [7, eqn 19.5.2]. Thus, for this particular f, the solution of the integral equation (1) is $\sigma = 1$. Although this solution is bounded, we see that adding a small constant to f adds a constant multiple of $(1-\rho^2)^{-1/2}$ to σ. In other words, the integral equation (1) has bounded solutions for some f, but these solutions are not typical: singular behaviour around the edge of Ω should be expected.

[1] There are errors in the published version of this Example; see Addendum

5.2. Example: comparison with Roy and Sabina

Roy and Sabina [2] consider $\sigma(\boldsymbol{x}) = (1-\rho^2)^{-1/2} g(x,y)$ when $g(x,y)$ is a quadratic in x and y. As a particular example, let us take $g(x,y) = 4\pi x = 4\pi a\rho\cos\phi$. Thus, $n=1$, $\nu = -\frac{1}{2}$ and $j=0$ in (15), giving $s_0^1 = 4\pi a\sqrt{\pi/2}$; all other coefficients s_j^n are zero. Then, from (16), $\mathcal{S}_1[\sigma_1;\lambda] = s_0^1 \lambda^{-1/2} J_{3/2}(\lambda)$. Hence

$$f(\boldsymbol{x}) = 2f_1(\rho)\cos\phi = \frac{bs_0^1}{\pi} I_{11}^c(k)\cos\phi \int_0^\infty J_1(\lambda\rho) J_{3/2}(\lambda)\,\frac{\mathrm{d}\lambda}{\sqrt{\lambda}}, \quad 0 \le \rho < 1. \tag{21}$$

It is shown in section 5.3 that $I_{11}^c(k) = 2(K-E)/k^2$. From [7, eqn 10.22.56], the integral in (21) evaluates to $\frac{1}{2}\rho\sqrt{\pi/2}$. Hence $f(\boldsymbol{x}) = \pi b x I_{11}^c$, in agreement with [2, eqn (14b)].

5.3. Example: quadratic σ

Suppose that

$$\begin{aligned}\sigma(\boldsymbol{x}) &= \alpha_0 + \alpha_1(x/a) + \alpha_2(y/b) + 2\alpha_3(x/a)^2 + 2\alpha_4(xy)/(ab) + 2\alpha_5(y/b)^2 \\ &= \{\alpha_0 + \rho^2(\alpha_3+\alpha_5)\} + \alpha_1\rho\cos\phi + \alpha_2\rho\sin\phi + (\alpha_3-\alpha_5)\rho^2\cos 2\phi + \alpha_4\rho^2\sin 2\phi,\end{aligned}$$

with constants α_j; Laurens and Tordeux [1] have $\alpha_3 = \alpha_4 = \alpha_5 = 0$. Then (7) gives

$$\sigma_0(\rho) = \alpha_0 + (\alpha_3+\alpha_5)\rho^2, \tag{22}$$

$\sigma_1 = \alpha_1\rho$, $\tilde{\sigma}_1 = \alpha_2\rho$, $\sigma_2 = (\alpha_3-\alpha_5)\rho^2$ and $\tilde{\sigma}_2 = \alpha_4\rho^2$. All other terms in (7) are absent.

Next, we use $P_0^{(n,\nu)} = 1$ and $\nu = 0$. These give $s_0^1 = \alpha_1$, $\tilde{s}_0^1 = \alpha_2$, $s_0^2 = \alpha_3 - \alpha_5$ and $\tilde{s}_0^2 = \alpha_4$. For s_j^0, we use $P_1^{(0,0)}(x) = P_1(x) = x$, giving

$$\sigma_0(\rho) = s_0^0 G_0^{(0,0)} + s_1^0 G_1^{(0,0)} = s_0^0 + s_1^0(1-2\rho^2).$$

Comparison with (22) gives $\alpha_0 = s_0^0 + s_1^0$ and $\alpha_3 + \alpha_5 = -2s_1^0$; these determine s_0^0 and s_1^0. Apart from the six mentioned, all other coefficients s_j^n and $\tilde{s}_j^n$ are zero.

Then, from (16), we obtain

$$\begin{aligned}&\lambda\mathcal{S}_0[\sigma_0;\lambda] = s_0^0 J_1(\lambda) + s_1^0 J_3(\lambda),\\ &\lambda\mathcal{S}_1[\sigma_1;\lambda] = s_0^1 J_2(\lambda), \quad \lambda\mathcal{S}_1[\tilde{\sigma}_1;\lambda] = \tilde{s}_0^1 J_2(\lambda),\\ &\lambda\mathcal{S}_2[\sigma_2;\lambda] = s_0^2 J_3(\lambda), \quad \lambda\mathcal{S}_2[\tilde{\sigma}_2;\lambda] = \tilde{s}_0^2 J_3(\lambda).\end{aligned}$$

We use these to compute the energy, I, given by (13). We will need the integrals (see (18))

$$\begin{aligned}\mathcal{J}_{pq} &= \int_0^\infty \frac{1}{\lambda^2} J_{p+1}(\lambda) J_{q+1}(\lambda)\,\mathrm{d}\lambda \\ &= \frac{\Gamma(\frac{1}{2}[p+q+1])}{4\,\Gamma(\frac{1}{2}[3+p-q])\,\Gamma(\frac{1}{2}[3+q-p])\,\Gamma(\frac{1}{2}[5+p+q])}.\end{aligned} \tag{23}$$

Thus

$$\begin{aligned}\frac{I}{ab^2} &= I_{00}^c \int_0^\infty |\mathcal{S}_0[\sigma_0;\lambda]|^2\,\mathrm{d}\lambda + I_{11}^c \int_0^\infty |\mathcal{S}_1[\sigma_1;\lambda]|^2\,\mathrm{d}\lambda \\ &\quad + I_{22}^c \int_0^\infty |\mathcal{S}_2[\sigma_2;\lambda]|^2\,\mathrm{d}\lambda + 2I_{02}^c\,\mathrm{Re}\int_0^\infty \mathcal{S}_0[\sigma_0;\lambda]\,\overline{\mathcal{S}_2[\sigma_2;\lambda]}\,\mathrm{d}\lambda \\ &\quad + I_{11}^s \int_0^\infty |\mathcal{S}_1[\tilde{\sigma}_1;\lambda]|^2\,\mathrm{d}\lambda + I_{22}^s \int_0^\infty |\mathcal{S}_2[\tilde{\sigma}_2;\lambda]|^2\,\mathrm{d}\lambda \\ &= I_{00}^c \left\{ \left|s_0^0\right|^2 \mathcal{J}_{00} + 2\,\mathrm{Re}\Big(s_0^0\,\overline{s_1^0}\Big)\,\mathcal{J}_{02} + \left|s_1^0\right|^2 \mathcal{J}_{22} \right\} + I_{11}^c \left|s_0^1\right|^2 \mathcal{J}_{11} \\ &\quad + I_{22}^c \left|s_0^2\right|^2 \mathcal{J}_{22} + 2I_{02}^c\,\mathrm{Re}\Big(s_0^0\overline{s_0^2}\mathcal{J}_{02} + s_1^0\overline{s_0^2}\mathcal{J}_{22}\Big) \\ &\quad + I_{11}^s \left|\tilde{s}_0^1\right|^2 \mathcal{J}_{11} + I_{22}^s \left|\tilde{s}_0^2\right|^2 \mathcal{J}_{22}.\end{aligned} \tag{24}$$

From (23), we obtain

$$\mathcal{J}_{00} = \frac{4}{3\pi}, \quad \mathcal{J}_{11} = \frac{4}{15\pi}, \quad \mathcal{J}_{22} = \frac{4}{35\pi}, \quad \mathcal{J}_{02} = \frac{4}{45\pi}.$$

For I^c_{mn} and I^s_{mn}, we have $I^c_{00} = 2K(k)$ (see (20)), $I^c_{mm} + I^s_{mm} = I^c_{00}$,

$$I^s_{11} - I^c_{11} = I^c_{02} = 2\int_0^{\pi/2} \frac{\cos 2x}{\Delta}\,\mathrm{d}x = \frac{2}{k^2}(k^2-2)K(k) + \frac{4}{k^2}E(k),$$

$$I^c_{22} - I^s_{22} = 2\int_0^{\pi/2} \frac{\cos 4x}{\Delta}\,\mathrm{d}x = \frac{32k'^2}{3k^4}K + 2K + \frac{16}{3k^4}(k^2-2)E,$$

where $k'^2 = 1 - k^2 = (b/a)^2$. Thus

$$I^c_{11} = 2(K-E)/k^2, \quad I^s_{11} = 2(E - k'^2K)/k^2,$$
$$I^c_{22} = 2\{(3k^4 + 8k'^2)K + 4(k^2-2)E\}/(3k^4),$$
$$I^s_{22} = 8\{(2-k^2)E - 2k'^2K\}/(3k^4).$$

One can check that these all have the correct limiting values as $k \to 0$.

This completes the computation of all the quantities required in (24). In the special case considered by Laurens and Tordeux [1], we have $s^0_0 = \alpha_0$, $s^1_0 = \alpha_1$, $\tilde{s}^1_0 = \alpha_2$ and $s^0_1 = s^2_0 = \tilde{s}^2_0 = 0$, whence

$$\begin{aligned} I/(ab^2) &= |\alpha_0|^2 I^c_{00}\mathcal{J}_{00} + |\alpha_1|^2 I^c_{11}\mathcal{J}_{11} + |\alpha_2|^2 I^s_{11}\mathcal{J}_{11} \\ &= \frac{8}{15\pi}\left\{5|\alpha_0|^2K + |\alpha_1|^2\frac{K-E}{k^2} + |\alpha_2|^2\frac{E-k'^2K}{k^2},\right\} \end{aligned}$$

in agreement with [1, Theorem 1.1].

6. Discussion

The (weakly singular) integral equation (1) arises when Laplace's equation holds in the three-dimensional region exterior to a thin flat plate Ω with Dirichlet boundary conditions on both sides of Ω. There are analogous (hypersingular) integral equations when a Neumann boundary condition is imposed. Explicit formulas for σ in terms of f are known when Ω is circular; for a review, see [10].

Expansion methods of the kind used above for problems involving elliptical plates, screens or cracks have a long history. The author's 1986 paper [5] gives references for Neumann problems, in the context of crack problems. For Dirichlet problems, see [2, 3, 4]. Similar expansion methods have been used recently for the problem of internal wave generation in a continuously stratified fluid by an oscillating elliptical plate [11].

References

[1] S. Laurens, S. Tordeux, Explicit computation of the electrostatic energy for an elliptical charged disc, Appl. Math. Lett. 26 (2013) 301–305.

[2] A. Roy, F.J. Sabina, Low-frequency acoustic diffraction by a soft elliptic disk, J. Acoust. Soc. Amer. 73 (1983) 1494–1498.

[3] A.K. Gautesen, F.J. Sabina, On the problem of the electrified elliptic disk, Quart. J. Mech. Appl. Math. 43 (1990) 363–372.

[4] J. Boersma, E. Danicki, On the solution of an integral equation arising in potential problems for circular and elliptic disks, SIAM J. Appl. Math. 53 (1993) 931–941.

[5] P.A. Martin, Orthogonal polynomial solutions for pressurized elliptical cracks, Quart. J. Mech. Appl. Math. 39 (1986) 269–287.

[6] J.P. Boyd, Chebyshev and Fourier Spectral Methods, 2nd edition, Dover, New York, 2001.

[7] NIST Digital Library of Mathematical Functions, `http://dlmf.nist.gov/`, Release 1.0.5 of 2012-10-01.

[8] C.J. Tranter, Integral Transforms in Mathematical Physics, 3rd edition, Methuen, London, 1966.

[9] S. Krenk, Some integral relations of Hankel transform type and applications to elasticity theory, Integral Eqns and Operator Theory 5 (1982) 548–561.

[10] P.A. Martin, Exact solution of some integral equations over a circular disc, J. Integral Eqns and Applications 18 (2006) 39–58.

[11] P.A. Martin, S.G. Llewellyn Smith, Generation of internal gravity waves by an oscillating horizontal elliptical plate, SIAM J. Appl. Math. 72 (2012) 725–739.

Addendum: corrections to Example 5.1

The formula for $f_0(\rho)$, (19), is correct but other Fourier components of $f(\boldsymbol{x})$ are also non-zero, in general. Thus, it is easy to see that f_{2m+1} and $\tilde{f}_n$ are all zero, leaving

$$f(\boldsymbol{x}) = f_0(\rho) + 2\sum_{m=1}^{\infty} f_{2m}(\rho)\cos 2m\phi$$

with f_{2m} given by (9),

$$f_{2m}(\rho) = \frac{bs_0^0}{2\pi} I_{0,2m}^c(k) \int_0^\infty \lambda^{-\nu-1} J_{2m}(\lambda\rho) J_{\nu+1}(\lambda)\,\mathrm{d}\lambda, \quad 0 \le \rho < 1. \tag{25}$$

From [7, eqn 10.22.56], the integral in (25) evaluates to

$$\frac{\rho^{2m}\Gamma(m+\frac{1}{2})}{2^{\nu+1}(2m)!\,\Gamma(\nu-m+\frac{3}{2})} F(m+\tfrac{1}{2},\, m-\nu-\tfrac{1}{2};\, 2m+1;\, \rho^2) = \mathcal{I}_m^\nu(\rho), \tag{26}$$

say. This gives the stated result when $m = 0$.

When $\nu = -\frac{1}{2}$, $\mathcal{I}_m^{-1/2}(\rho) = 0$ for $m = 1, 2, 3, \ldots$ (because of the Γ function in the denominator). Then, $f(\boldsymbol{x}) = f_0(\rho) = \frac{1}{2}bK(k)$, a constant, in accord with Galin's theorem.

When $\nu = 0$, $\mathcal{I}_0^0(\rho) = (2/\pi)E(\rho)$ for $0 \le \rho < 1$, using [7, eqn 19.5.2]. For $m \ge 1$, $\mathcal{I}_m^0(\rho)$ is given by (26) but the hypergeometric function does not seem to simplify. However, we find that

$$\lim_{\rho\to 1-} \mathcal{I}_m^0(\rho) = (2/\pi)(-1)^m/(1-4m^2),$$

implying that $f(\boldsymbol{x})$ is bounded around the edge of Ω. Having constructed f is this way, the last three sentences of Example 5.1 remain valid.

29 May 2013

ellipse-aml3.tex