

# HYPERSINGULAR INTEGRALS: HOW SMOOTH MUST THE DENSITY BE?\*

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## Abstract

Hypersingular integrals are guaranteed to exist at a point  $x$  only if the density function  $f$  in the integrand satisfies certain conditions in a neighbourhood of  $x$ . It is well known that a sufficient condition is that  $f$  has a Hölder-continuous first derivative. This is a stringent condition, especially when it is incorporated into boundary-element methods for solving hypersingular integral equations. This paper is concerned with finding weaker conditions for the existence of one-dimensional Hadamard finite-part integrals: it is shown that it is sufficient for the even part of  $f$  (with respect to  $x$ ) to have a Hölder-continuous first derivative – the odd part is allowed to be discontinuous. A similar condition is obtained for Cauchy principal-value integrals. These simple results have non-trivial consequences. They are applied to the calculation of the tangential derivative of a single-layer potential and to the normal derivative of a double-layer potential. Particular attention is paid to discontinuous densities and to discontinuous boundary conditions. Also, despite the weaker sufficient conditions, it is reaffirmed that, for hypersingular integral equations, collocation at a point  $x$  at the junction between two standard conforming boundary elements is not permissible, theoretically. Various modifications to the definition of finite-part integral are explored.

## 1 INTRODUCTION

There are many papers in the literature on singular and hypersingular integral equations. By definition, such equations have kernels that are not integrable in the ordinary (improper) sense. Specifically, for one-dimensional equations, the basic integrals are of the form

$$I_n(x) = \int_A^B \frac{F(t)}{(t-x)^n} dt, \quad n = 1, 2,$$

where  $A < x < B$  and  $F(t)$  is called the *density function*. When the integrals are properly defined (various definitions are considered below),  $I_1$  is a Cauchy principal-value (CPV) integral and  $I_2$  is a Hadamard finite-part (HFP) integral;  $I_1$  is also called a *singular* integral and  $I_2$  is also called a *hypersingular* integral.<sup>1-4</sup>

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To be sure that  $I_n(x)$  exists,  $F(t)$  must have certain smoothness or continuity properties. Classically, these are usually expressed in terms of Hölder continuity and the function spaces  $C^{m,\alpha}$  (functions in  $C^{0,\alpha}$  are called Hölder continuous; a function  $f$  is in  $C^{1,\alpha}$  if  $f'$  is in  $C^{0,\alpha}$ ; see section 2 for more details). Thus, it is well known that<sup>3, 4</sup>

$$\text{if } F \in C^{m-1,\alpha}, \text{ then } I_n \text{ exists.} \quad (1)$$

This condition is *local*; if it holds in a neighbourhood of  $x$ , then  $I_n(x)$  exists. Note that condition (1) is a *sufficient* condition for  $I_n$  to exist; thus, there are functions  $F$  that do not satisfy (1) and yet  $I_n$  exists. It is this situation that we explore here, along with the possibility and/or the desirability of actually changing the definition of ‘integral’.

Conditions similar to (1) also arise when considering the limiting values of layer potentials when the field point approaches the boundary. For example, the tangential derivative of a single-layer potential is closely related to  $I_1$ , whereas the normal derivative of a double-layer potential is closely related to  $I_2$ . We examine these limiting values below (sections 4 and 6).

The primary motivation for this work is a widespread desire to weaken continuity requirements. This is especially true for hypersingular integral equations, as the condition (1) for  $n = 2$  is stringent. For example, a standard boundary-element method proceeds by approximating  $F$  by quadratic isoparametric conforming elements: this gives inter-element continuity but not differentiability. Condition (1) suggests that, for a hypersingular integral equation, we cannot then collocate at a point  $x$  at the junction between two elements because  $I_2(x)$  does not exist, in general (see section 8). Although this fact is recognised widely in the literature,<sup>3–8</sup> there still exist recent claims<sup>9</sup> that ordinary conforming elements suffice. This claim is demonstrably false, unless one collocates away from the element edges or changes the definition of the HFP integral; a third possibility is to use a Galerkin method,<sup>10</sup> but we do not pursue this option here.

In this paper, we show that the condition (1) *can* be weakened, using simple arguments. We start by separating the density into its even and odd parts, with respect to the singular point  $x$ :

$$\begin{aligned} F(t) &= F_{\text{even}}(t; x) + F_{\text{odd}}(t; x), \\ F_{\text{even}}(t; x) &= \frac{1}{2}\{F(t) + F(2x - t)\}, \\ F_{\text{odd}}(t; x) &= \frac{1}{2}\{F(t) - F(2x - t)\}. \end{aligned} \quad (2)$$

Assume that  $F$  is merely continuous. Then, we show that the CPV integral  $I_1(x)$  exists if  $F_{\text{odd}}$  is Hölder continuous, and the HFP integral  $I_2(x)$  exists if  $F_{\text{even}}$  is in  $C^{1,\alpha}$ . In fact, for the HFP integral, we can even allow  $F$  to be discontinuous at  $x$  if we make a minor change to the definition of the finite-part integral.

These results are simple but they seem to be new (Gray<sup>11</sup> has used the vanishing of certain integrals of odd integrands over symmetric intervals). They have several non-trivial applications. For example, they give weaker (sufficient) conditions for the existence of the limiting values of derivatives of layer potentials. Specifically, if one solves a boundary-value problem with discontinuous data, one usually obtains density functions that do not satisfy (1); however, they do satisfy our weaker conditions, and so we can assert that the appropriate limits exist (section 7).

For a second example, we look again at the use of conforming elements for hypersingular boundary integral equations (section 8); for definiteness, we consider quadratic elements although our conclusions would remain unchanged if we were to use higher-order elements. Although we have weaker sufficient conditions for the existence of the normal derivative of a double-layer potential on the boundary, these conditions do not cover the use of collocation at the junction between two elements. Indeed, the non-existence of the limit can be explicitly demonstrated; this reinforces

our earlier findings. One possible way out of this difficulty is to change the hypersingular integral equation itself by changing the definition of finite-part integral; we shall argue (section 8.1) that this is undesirable, although it may be worthy of further study.

Throughout the paper, we make use of examples to illustrate the results. A particularly useful example is the simple discontinuous function,

$$f(t) = \begin{cases} f_L, & t < 0, \\ f_R, & t > 0, \end{cases} \quad (4)$$

where  $f_L$  and  $f_R$  are constants. We also consider two ‘smoothed’ versions of this function (defined by (20) and (39) below). Another useful example is  $f(t) = |t|$ , because it is not differentiable at  $t = 0$ . Several other examples are constructed so as to show that the sufficient conditions obtained cannot be weakened much further.

## 2 CLASSES OF FUNCTIONS

We are interested in functions with various smoothness properties. It is convenient to list these here. Thus, assume that  $f$  is a given *bounded* function, defined on an interval  $a < t < b$ , except perhaps for finitely many isolated points (these are irrelevant, as we will be interested in various integrals of  $f$ ).

- $f$  is piecewise continuous,  $f \in PC$ . Such functions are continuous, except for finite discontinuities. Thus, for a discontinuity at  $t = x$ , the left-hand limit,  $f(x-)$ , and the right-hand limit,  $f(x+)$ , both exist, with  $f(x-) \neq f(x+)$ ;  $f(x)$  need not be defined. For example, the discontinuous function (4) has  $f(0-) = f_L$  and  $f(0+) = f_R$ , and is not defined at  $t = 0$ .
- $f$  is continuous,  $f \in C$ . Note that continuous functions are defined for all  $t$  with  $a < t < b$ ; in particular, we have  $f(x-) = f(x+) = f(x)$ .
- $f$  is Hölder continuous,  $f \in C^{0,\alpha}$ . This means that we can find positive constants  $A$  and  $\alpha$  so that

$$|f(t_1) - f(t_2)| < A|t_1 - t_2|^\alpha \quad \text{with } 0 < \alpha \leq 1$$

for all  $t_1$  and  $t_2$  in  $a < t < b$ . In particular, when we say that  $f$  is Hölder continuous at a point such as  $t = 0$ , then we have

$$|f(t) - f(0)| < A|t|^\alpha \quad \text{with } 0 < \alpha \leq 1 \quad (5)$$

for all  $t$  in some interval containing  $t = 0$ . Roughly speaking, functions in  $C^{0,\alpha}$  are smoother than merely continuous functions, but they need not be differentiable (even if  $\alpha = 1$ ). For example,  $f(t) = |t|$  is in  $C^{0,1}$  but it is not differentiable at  $t = 0$ .

- $f$  has a Hölder-continuous first derivative,  $f' \in C^{0,\alpha}$  or  $f \in C^{1,\alpha}$ . Roughly speaking, functions in  $C^{1,\alpha}$  are smoother than merely differentiable functions, but they need not have two continuous derivatives.

## 3 CAUCHY PRINCIPAL-VALUE INTEGRALS

The Cauchy principal-value (CPV) integral of a function  $F$  is defined by

$$\int_A^B \frac{F(t)}{t-x} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{t-x} dt + \int_{x+\varepsilon}^B \frac{F(t)}{t-x} dt \right\}, \quad (6)$$

where  $A < x < B$ . Choose  $a > 0$  so that  $A < x - a < x + a < B$ . Then

$$\int_A^B \frac{F(t)}{t-x} dt = I + \int_A^{x-a} \frac{F(t)}{t-x} dt + \int_{x+a}^B \frac{F(t)}{t-x} dt,$$

where the last two integrals are nonsingular (we ignore them henceforth),

$$I = \int_{x-a}^{x+a} \frac{F(t)}{t-x} dt = \int_{-a}^a f(t) \frac{dt}{t}$$

and  $f(t) = F(t+x)$ . Thus, without loss of generality, we can take  $x = 0$  and a symmetric interval of integration.

It is well known<sup>12</sup> that a sufficient condition for the existence of  $I$  is that  $f$  be Hölder continuous,  $f \in C^{0,\alpha}$ ; we shall refine this condition.

We start by splitting  $f$  into its even and odd parts:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t) \tag{7}$$

where

$$f_{\text{even}}(t) = \frac{1}{2}[f(t) + f(-t)] \quad \text{and} \quad f_{\text{odd}}(t) = \frac{1}{2}[f(t) - f(-t)].$$

In particular,  $f_{\text{even}}(0+) = f_{\text{even}}(0-) = \frac{1}{2}[f(0+) + f(0-)]$ . By taking this value as the definition of  $f_{\text{even}}(0)$ , we ensure that  $f_{\text{even}}$  is continuous at  $t = 0$ , even when  $f$  is not.

Substituting for  $f$  into  $I$  gives

$$I = \int_{-a}^a f(t) \frac{dt}{t} = \int_{-a}^a f_{\text{odd}}(t) \frac{dt}{t} = 2 \lim_{\varepsilon \rightarrow 0} L(\varepsilon) \tag{8}$$

where

$$L(\varepsilon) = \int_{\varepsilon}^a f_{\text{odd}}(t) \frac{dt}{t}.$$

In order to examine the limit  $\varepsilon \rightarrow 0$ , we write

$$\begin{aligned} f_{\text{odd}}(t) &= f_{\text{odd}}(0+) + f_1(t), \\ f_1(t) &= f_{\text{odd}}(t) - f_{\text{odd}}(0+), \end{aligned} \tag{9}$$

where  $f_1(t) \rightarrow 0$  as  $t \rightarrow 0+$ . Then

$$\begin{aligned} L(\varepsilon) &= f_{\text{odd}}(0+) \log(a/\varepsilon) + L_{\text{add}}(\varepsilon), \\ L_{\text{add}}(\varepsilon) &= \int_{\varepsilon}^a f_1(t) \frac{dt}{t}. \end{aligned} \tag{10}$$

Now, we examine the limit  $\varepsilon \rightarrow 0$ , given various conditions on  $f$ . Note that, in order to obtain a finite limit, both terms on the right-hand side of (10) must have finite limits.

- Suppose that  $f(t)$  is discontinuous at  $t = 0$ ,  $f \in PC$ . Then,  $f_{\text{odd}}(0+) \neq 0$  and so the CPV integral does not exist: it diverges as  $-2f_{\text{odd}}(0+) \log \varepsilon$  (assuming that  $L_{\text{add}}$  has a finite limit). For a simple example, take (4).
- Suppose that  $f_{\text{odd}}(t)$  is Hölder continuous. This implies that  $f_{\text{odd}}(0+) = 0$ , whence the logarithmic term is absent in (10). Moreover, the Hölder continuity of  $f_{\text{odd}}$  is sufficient to ensure that the ‘added back’ integral,  $L_{\text{add}}(\varepsilon)$ , has a finite limit as  $\varepsilon \rightarrow 0$ . Note that these conclusions follow even if  $f_{\text{even}}$  (and hence  $f$ ) is merely continuous. For example,

$$f(t) = \begin{cases} [\log |t|]^{-1}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

is even and continuous, but it is not Hölder continuous; its CPV integral exists (it is zero).

- Suppose that  $f_{\text{odd}}(t)$  is merely continuous. Again, this implies that  $f_{\text{odd}}(0+) = 0$ , whence the logarithmic term is absent in (10). However, continuity is *not* sufficient to guarantee that  $L_{\text{add}}(\varepsilon)$  has a finite limit. For example, take  $a < 1$  and

$$f(t) = \begin{cases} [\log t]^{-1}, & t > 0, \\ 0, & t = 0, \\ -[\log(-t)]^{-1}, & t < 0, \end{cases} \quad (11)$$

which is continuous at  $t = 0$ ; we have  $f \equiv f_{\text{odd}}$ , whence

$$L(\varepsilon) = \log |\log a| - \log |\log \varepsilon|,$$

which does not have a finite limit as  $\varepsilon \rightarrow 0$ .

**THEOREM 1** *Suppose that  $F(t)$  is continuous at  $t = x$ . Suppose that  $F_{\text{odd}}(t; x)$ , defined by (3), is Hölder continuous at  $t = x$ . Then, the Cauchy principal-value integral (6) exists.*

This theorem gives a sufficient condition for the existence of the CPV integral. We have also shown that the continuity of  $F_{\text{odd}}$  is necessary, but not sufficient. Similar remarks can be made concerning all the subsequent theorems.

### 3.1 Change the definition

As we have seen, the CPV integral (6) does not exist if  $F(t)$  is discontinuous at  $t = x$ . One way to obtain a finite limit is to take a finite part! In other words, we change the definition of ‘integral’; define two *one-sided Cauchy principal-value integrals* of  $F$  by (Schwartz,<sup>13</sup> p. 104)

$$\begin{aligned} \oint_A^x \frac{F(t)}{t-x} dt &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{t-x} dt - F(x-) \log \varepsilon \right\}, \\ \oint_x^B \frac{F(t)}{t-x} dt &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{x+\varepsilon}^B \frac{F(t)}{t-x} dt + F(x+) \log \varepsilon \right\}, \end{aligned}$$

where  $A < x < B$ . These definitions supply finite values even if  $F$  is discontinuous at  $t = x$ . Moreover, if  $F$  is actually smooth enough for its CPV integral to exist, then the sum of its two one-sided CPV integrals will give the same value; this suggests generalizing the CPV integral to

$$\oint_A^B \frac{F(t)}{t-x} dt = \oint_A^x \frac{F(t)}{t-x} dt + \oint_x^B \frac{F(t)}{t-x} dt, \quad (12)$$

this being the definition of the integral on the left-hand side. For example, take the simple discontinuous function  $f$ , defined by (4), with  $A = -a$ ,  $B = a$  and  $x = 0$ . Then

$$\oint_{-a}^a f(t) \frac{dt}{t} = (f_R - f_L) \log a. \quad (13)$$

An equivalent definition to (12) is

$$\oint_A^B \frac{F(t)}{t-x} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{t-x} dt + \int_{x+\varepsilon}^B \frac{F(t)}{t-x} dt + [F(x+) - F(x-)] \log \varepsilon \right\}, \quad (14)$$

equivalent in the sense that the right-hand side of (12) exists if and only if the right-hand side of (14) exists. However, care is needed with all of these definitions, as the presence of the logarithmic terms means that the integrals are not independent of simple changes of variable; an example of this unpleasant property will be given at the end of section 5.2.

## 4 TANGENTIAL DERIVATIVE OF A SINGLE-LAYER POTENTIAL

Consider the single-layer potential

$$S(x, y) = \int_{-a}^a f(t) G(x, y; t, 0) dt$$

where  $G$  is the fundamental solution defined by

$$G(x, y; t, \eta) = \log \sqrt{(t-x)^2 + (y-\eta)^2}. \quad (15)$$

The problem is to compute the tangential derivative of  $S$  on  $y = 0$ . Thus, we seek the limit of

$$T(x, y) = -\frac{\partial S}{\partial x} = \int_{-a}^a \frac{(t-x)f(t)}{(t-x)^2 + y^2} dt \quad (16)$$

as  $y \rightarrow 0$ . For simplicity, let us take  $x = 0$  and define  $T(y) \equiv T(0, y)$ . Then, we have

$$\begin{aligned} T(y) &= \int_{-a}^a \frac{t f(t)}{t^2 + y^2} dt = 2 \int_0^a \frac{t f_{\text{odd}}(t)}{t^2 + y^2} dt \\ &= f_{\text{odd}}(0+) \log \frac{a^2 + y^2}{y^2} + T_{\text{add}}(y), \end{aligned} \quad (17)$$

$$T_{\text{add}}(y) = 2 \int_0^a \frac{t f_1(t)}{t^2 + y^2} dt,$$

where  $f_1$  is defined by (9).

- Suppose that  $f(t)$  is discontinuous at  $t = 0$ . Then,  $f_{\text{odd}}(0+) \neq 0$  and so  $T(y)$  diverges logarithmically as  $y \rightarrow 0$ . We discuss this case further in section 4.1.
- Suppose that  $f_{\text{odd}}(t)$  is Hölder continuous. In this case,

$$T(0) = T_{\text{add}}(0) = 2 \int_0^a f_1(t) \frac{dt}{t}$$

exists. Also, using (5),

$$\begin{aligned} |T(y) - T(0)| &= 2 \left| \int_0^a \frac{-y^2}{t(t^2 + y^2)} f_1(t) dt \right| \leq 2Ay^2 \int_0^a \frac{t^{\alpha-1}}{t^2 + y^2} dt \\ &< 2Ay^\alpha \int_0^\infty \frac{s^{\alpha-1}}{s^2 + 1} ds, \end{aligned}$$

and the last integral is finite, since  $0 < \alpha \leq 1$ ; hence, as  $\alpha > 0$ ,  $T(y) \rightarrow T(0)$  as  $y \rightarrow 0$ . Comparing with (8) and (10), we see that the limiting value is

$$T(0) = \int_{-a}^a f(t) \frac{dt}{t}.$$

- Suppose that  $f_{\text{odd}}(t)$  is merely continuous. Again, we obtain  $T(y) = T_{\text{add}}(y)$ , but now we cannot guarantee that  $T(0)$  exists. For example,  $T(0)$  diverges for the choice (11); nevertheless, is it possible that  $T(y)$  has a finite limit as  $y \rightarrow 0$ ? It does not: to see this, choose  $a < 1$ , whence (11) and (17) give

$$\begin{aligned} |T(y)| &= -2 \int_0^a \frac{t dt}{(t^2 + y^2) \log t} > -2 \int_y^a \frac{t dt}{(t^2 + y^2) \log t} \\ &> -\int_y^a \frac{dt}{t \log t} = \log |\log y| - \log |\log a|, \end{aligned}$$

which is unbounded as  $y \rightarrow 0$ .

THEOREM 2 Suppose that  $f(t)$  is continuous at  $t = x$ . Suppose that

$$f_{\text{odd}}(t; x) = \frac{1}{2}\{f(t) - f(2x - t)\}$$

is Hölder continuous at  $t = x$ . Then,

$$\lim_{y \rightarrow 0} T(x, y) = \oint_{-a}^a \frac{f(t)}{t - x} dt,$$

where  $T(x, y)$  is defined by (16).

#### 4.1 More on discontinuous densities

Here, we discuss the simple example (4) in more detail. From (17), we obtain

$$\begin{aligned} T(y) &= -(f_R - f_L) \log y + T_0(y), \\ T_0(y) &= (f_R - f_L) \log \sqrt{a^2 + y^2}, \end{aligned}$$

so that  $T(y)$  is unbounded as  $y \rightarrow 0$ ; note that, from (13),

$$T_0(0) = \oint_{-a}^a f(t) \frac{dt}{t} = (f_R - f_L) \log a. \quad (18)$$

Thus, we can use the finite-part generalization of the CPV integral to express the ‘finite part’ of the limit. However, the limit  $y \rightarrow 0$  is genuinely divergent, not an artifice of any definition of integral. Despite this fact, attempts have been made to extract limiting values that are intended to be physically meaningful; some of these attempts are described next.

#### Evaluate the limit near the discontinuity

Return to  $T(x, y)$ , defined by (16), and evaluate the limit at  $x = \pm\delta$  on either side of the discontinuity (assume  $\delta > 0$ ). The density is smooth at these points, so there is no difficulty in calculating the limiting values; they are given by

$$\begin{aligned} T(\pm\delta, 0) &= -(f_R - f_L) \log \delta + T_1(\pm\delta), \\ T_1(\nu) &= f_R \log(a - \nu) - f_L \log(a + \nu). \end{aligned}$$

We note that  $T(\delta, 0)$  and  $T(-\delta, 0)$  both diverge logarithmically as  $\delta \rightarrow 0$ . However, we also note that

$$T(\delta, 0) - T(-\delta, 0) = (f_R + f_L) \log \frac{a - \delta}{a + \delta} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (19)$$

Thus  $T(\delta, 0)$  and  $T(-\delta, 0)$  both approach the same value as  $\delta \rightarrow 0$  (there is no discontinuity across the discontinuity), *but that value is infinite!* Note also that  $T_1(\pm\delta) \rightarrow T_0(0)$  as  $\delta \rightarrow 0$ , so that we obtain the same ‘finite part’ as before, (18).

#### Smooth the discontinuity

Huang and Cruse<sup>9</sup> have replaced the discontinuous function (4) by a piecewise-linear approximation (see their Figure 2). Thus, let

$$f(t) = \begin{cases} f_L, & t \leq -\delta, \\ \frac{1}{2}(f_R - f_L)(t/\delta) + \frac{1}{2}(f_R + f_L), & -\delta < t < \delta, \\ f_R, & t \geq \delta, \end{cases} \quad (20)$$

where we again think of  $\delta$  as being small and positive. Since  $f(t)$  is Hölder continuous at  $t = 0$ , we readily obtain

$$\begin{aligned} T(0) &= \int_{-a}^a f(t) \frac{dt}{t} = -(f_R - f_L) \log \delta + T_2, \\ T_2 &= (f_R - f_L)(1 + \log a). \end{aligned}$$

Note that  $T(0)$  diverges as  $\delta \rightarrow 0$  exactly as before. However, the ‘finite part’,  $T_2$ , is independent of  $\delta$  and differs from  $T_0(0) = T_1(0)$ .

### *Combination of the two*

Suppose that, following Huang and Cruse,<sup>9</sup> we take the piecewise-linear approximation (20), and then evaluate  $T(x, 0)$  at  $x = \pm\delta$ . We find that

$$\begin{aligned} T(\pm\delta, 0) &= -(f_R - f_L) \log \delta + T_3(\pm\delta) \\ T_3(\nu) &= f_R \log(a - \nu) - f_L \log(a + \nu) + (f_R - f_L)(1 - \log 2). \end{aligned}$$

Again, we note that  $T(\delta, 0)$  and  $T(-\delta, 0)$  both diverge logarithmically as  $\delta \rightarrow 0$ , and that their difference tends to zero as  $\delta \rightarrow 0$ . However,  $T_3(0) = (f_R - f_L)(1 + \log \frac{1}{2}a)$ , which is yet another value for the ‘finite part’!

Huang and Cruse<sup>9</sup> examined the difference  $T(\delta, 0) - T(-\delta, 0)$  (actually, they studied the corresponding problem in plane elastostatics, where the aim is to calculate boundary stresses). We have seen that this difference vanishes as  $\delta \rightarrow 0$ ; in fact, it is given exactly by (19), so that it is not affected by the replacement of (4) by (20). This seems to disagree with the results of Huang and Cruse,<sup>9</sup> although we note that their argument on p. 2050 is incomplete: having expanded all the various kernels in powers of  $\varepsilon/r$ , where  $\varepsilon$  (our  $\delta$ ) is small, one cannot then let  $r \rightarrow 0$  without further justification.

## **Summary**

These differing treatments for dealing with the unbounded limit at a simple discontinuity yield various answers for the ‘correction’; such behaviour is not unexpected and shows that these devices are inappropriate.

## **5 HADAMARD FINITE-PART INTEGRALS**

The Hadamard finite-part (HFP) integral of a function  $F$  is defined by

$$\int_A^B \frac{F(t)}{(t-x)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2} dt + \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2} dt - \frac{2F(x)}{\varepsilon} \right\}, \quad (21)$$

where  $A < x < B$ . This definition assumes implicitly that  $F$  is continuous at  $x$ . Choose  $a > 0$  so that  $A < x - a < x + a < B$ . Then

$$\int_A^B \frac{F(t)}{(t-x)^2} dt = J + \int_A^{x-a} \frac{F(t)}{(t-x)^2} dt + \int_{x+a}^B \frac{F(t)}{(t-x)^2} dt,$$

where the last two integrals are nonsingular (we ignore them, as before),

$$J = \int_{x-a}^{x+a} \frac{F(t)}{(t-x)^2} dt = \int_{-a}^a f(t) \frac{dt}{t^2}$$



and  $f(t) = F(t + x)$ . As in section 3, we take  $x = 0$ .

It is well known<sup>3</sup> that a sufficient condition for the existence of  $J$  is that  $f'$  be Hölder continuous,  $f \in C^{1,\alpha}$ ; we shall refine this condition by splitting  $f$  into its even and odd parts, (7). This gives

$$J = \int_{-a}^a f(t) \frac{dt}{t^2} = \int_{-a}^a f_{\text{even}}(t) \frac{dt}{t^2} = 2 \lim_{\varepsilon \rightarrow 0} M(\varepsilon) \quad (22)$$

where

$$M(\varepsilon) = \int_{\varepsilon}^a f_{\text{even}}(t) \frac{dt}{t^2} - \frac{f_{\text{even}}(0)}{\varepsilon}. \quad (23)$$

Note that, as  $f$  is continuous at  $t = 0$ , we must have  $f_{\text{odd}}(0) = 0$ .

Now, we examine the limit  $\varepsilon \rightarrow 0$ , given various conditions on  $f$ .

- Suppose that  $f(t)$  is discontinuous at  $t = 0$ ,  $f \in PC$ . Then, the HFP integral does not exist as we cannot even use its definition (it contains the undefined term  $-2f(0)/\varepsilon$ ). However, see section 5.1 for a simple way of overcoming this difficulty.
- Suppose that  $f_{\text{even}}(t)$  is Hölder continuous,  $f_{\text{even}} \in C^{0,\alpha}$ . Then, in general, the HFP integral does not exist. For example, take  $f_{\text{even}}(t) = |t| \in C^{0,1}$ . Then,  $M(\varepsilon) = \log(a/\varepsilon)$ , which is unbounded as  $\varepsilon \rightarrow 0$ .
- Suppose that  $f_{\text{even}} \in C^{1,\alpha}$ . Then, we can integrate by parts in (23) to give

$$M(\varepsilon) = \frac{1}{\varepsilon}[f_{\text{even}}(\varepsilon) - f_{\text{even}}(0)] - \frac{1}{a}f_{\text{even}}(a) + \int_{\varepsilon}^a f'_{\text{even}}(t) \frac{dt}{t}. \quad (24)$$

The first term tends to zero as  $\varepsilon \rightarrow 0$ , because  $f'_{\text{even}}(0) = 0$  (as  $f_{\text{even}}$  is even and differentiable) implying that  $f_{\text{even}}(\varepsilon) - f_{\text{even}}(0)$  is  $o(\varepsilon)$  as  $\varepsilon \rightarrow 0$ . The limit of the integral in (24) exists because of the assumed Hölder-continuity of  $f'_{\text{even}}$ , and so the HFP integral exists. In fact, combining (22) and (24) gives

$$\int_{-a}^a f(t) \frac{dt}{t^2} = \frac{-2}{a}f_{\text{even}}(a) + \int_{-a}^a f'_{\text{even}}(t) \frac{dt}{t}, \quad (25)$$

which can be viewed as a regularization of the HFP integral. Note that these conclusions follow even if  $f_{\text{odd}}$  (and hence  $f$ ) is merely continuous.

- Suppose that  $f_{\text{even}}(t)$  is differentiable,  $f_{\text{even}} \in C^1$ . Again, the first term in (24) tends to zero as  $\varepsilon \rightarrow 0$ , but mere differentiability is not sufficient to guarantee that the integral in (24) has a finite limit. For example, take  $a < 1$  and

$$f(t) = \begin{cases} |t| [\log |t|]^{-1}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

which is continuously differentiable at  $t = 0$ ; we have  $f \equiv f_{\text{even}}$ , whence

$$M(\varepsilon) = \log |\log a| - \log |\log \varepsilon|,$$

which does not have a finite limit as  $\varepsilon \rightarrow 0$ .

**THEOREM 3** *Suppose that  $F(t)$  is continuous at  $t = x$ . Suppose that  $F_{\text{even}}(t; x)$ , defined by (2), has a Hölder-continuous first derivative at  $t = x$ . Then, the Hadamard finite-part integral (21) exists.*

### 5.1 A useful generalization of the Hadamard finite-part definition

The definition of HFP integral, (21), cannot be used if  $F(t)$  is discontinuous at  $t = x$ . Instead, we generalise the definition slightly as follows:

$$\rlap{-}\int_A^B \frac{F(t)}{(t-x)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2} dt + \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2} dt - \frac{1}{\varepsilon} [F(x+) + F(x-)] \right\}. \quad (26)$$

This definition only differs from (21) if  $F$  is discontinuous. If we take  $A = -a$ ,  $B = a$  and  $x = 0$ , we obtain (cf. (22))

$$\rlap{-}\int_{-a}^a f(t) \frac{dt}{t^2} = \rlap{-}\int_{-a}^a f_{\text{even}}(t) \frac{dt}{t^2} = 2 \lim_{\varepsilon \rightarrow 0} M(\varepsilon)$$

where  $M(\varepsilon)$  is defined *exactly* as before by (23). It follows that the properties of (26) are very similar to those of (21), the exception being that discontinuous functions can now be treated.

**THEOREM 4** *Suppose that  $F(t)$  is bounded at  $t = x$ . Suppose that  $F_{\text{even}}(t; x)$ , defined by (2), has a Hölder-continuous first derivative at  $t = x$ . Then, the generalized Hadamard finite-part integral (26) exists.*

As an example, consider our standard discontinuous function (4); we obtain

$$\rlap{-}\int_{-a}^a f(t) \frac{dt}{t^2} = -\frac{1}{a} (f_R + f_L). \quad (27)$$

Moreover, it turns out that the definition (26) is preferable to (21) as it is more closely related to the normal derivative of a double-layer potential, whose properties will be described below in section 6. With this in mind, we note that (25) generalises to

$$\rlap{-}\int_{-a}^a f(t) \frac{dt}{t^2} = \frac{-2}{a} f_{\text{even}}(a) + \rlap{-}\int_{-a}^a f'_{\text{even}}(t) \frac{dt}{t}, \quad (28)$$

this formula being valid if  $f_{\text{even}} \in C^{1,\alpha}$ ; a finite discontinuity in  $f(t)$  at  $t = 0$  is allowed.

### 5.2 Change the definition

As for the CPV integral, we can also introduce one-sided finite-part integrals; define two *one-sided Hadamard finite-part integrals* of a function  $F$  by (Schwartz,<sup>13</sup> p. 104)

$$\rlap{-}\int_A^x \frac{F(t)}{(t-x)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2} dt - F'(x-) \log \varepsilon - \frac{F(x-)}{\varepsilon} \right\}, \quad (29)$$

$$\rlap{-}\int_x^B \frac{F(t)}{(t-x)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2} dt + F'(x+) \log \varepsilon - \frac{F(x+)}{\varepsilon} \right\}, \quad (30)$$

where  $A < x < B$ . These definitions supply finite values even if  $F$  or its derivative are discontinuous at  $t = x$ . Moreover, if  $F$  is actually smooth enough for its HFP integral to exist, then the sum of its two one-sided HFP integrals will give the same value; this suggests another generalization of the HFP integral to

$$\rlap{-}\int_A^B \frac{F(t)}{(t-x)^2} dt = \rlap{-}\int_A^x \frac{F(t)}{(t-x)^2} dt + \rlap{-}\int_x^B \frac{F(t)}{(t-x)^2} dt, \quad (31)$$

this being the definition of the integral on the left-hand side. For a simple example, evaluation of (31) for the discontinuous function (4) gives the same value as (27).

An equivalent definition to (31) is<sup>14</sup>

$$\begin{aligned} \rlap{-}\int_A^B \frac{F(t)}{(t-x)^2} dt &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_A^{x-\varepsilon} \frac{F(t)}{(t-x)^2} dt + \int_{x+\varepsilon}^B \frac{F(t)}{(t-x)^2} dt \right. \\ &\quad \left. + [F'(x+) - F'(x-)] \log \varepsilon - \frac{1}{\varepsilon} [F(x+) + F(x-)] \right\}. \end{aligned} \quad (32)$$

This definition generalises (26) to functions that do not satisfy  $F'(x+) = F'(x-)$ . We will discuss the utility of this generalization in section 8.1. Here, we note that care is needed because of the logarithmic terms in (29), (30) and (32). For example, take  $f(t) = |t|$ . Then, taking  $x = 0$  in (32), we obtain

$$\rlap{-}\int_{-1}^1 f(t) \frac{dt}{t^2} = 0 \quad \text{but} \quad \rlap{-}\int_{-1/2}^{1/2} f(2s) \frac{2 ds}{(2s)^2} = -2 \log 2,$$

which shows that the value of the integral is not independent of simple changes of variable.

## 6 NORMAL DERIVATIVE OF A DOUBLE-LAYER POTENTIAL

Consider the double-layer potential

$$D(x, y) = \int_{-a}^a f(t) \left( \frac{\partial}{\partial \eta} G(x, y; t, \eta) \right) \Big|_{\eta=0} dt = \int_{-a}^a \frac{-y f(t)}{(t-x)^2 + y^2} dt,$$

where  $G$  is defined by (15). The problem is to compute the normal derivative of  $D$  on  $y = 0$ . Thus, we seek the limit of

$$N(x, y) = -\frac{\partial D}{\partial y} = \int_{-a}^a \frac{(t-x)^2 - y^2}{[(t-x)^2 + y^2]^2} f(t) dt \quad (33)$$

as  $y \rightarrow 0$ . For simplicity, let us take  $x = 0$  and define  $N(y) \equiv N(0, y)$ . Then, we have

$$N(y) = \int_{-a}^a \frac{t^2 - y^2}{(t^2 + y^2)^2} f(t) dt = 2 \int_0^a \frac{t^2 - y^2}{(t^2 + y^2)^2} f_{\text{even}}(t) dt. \quad (34)$$

- Suppose that  $f_{\text{even}}(t)$  is Hölder continuous at  $t = 0$ . Then, in general,  $N(y)$  does not have a finite limit. For example, if  $f_{\text{even}}(t) = |t|$ , then

$$N(y) = 2 \int_0^a \frac{t(t^2 - y^2)}{(t^2 + y^2)^2} dt = \log \frac{a^2 + y^2}{y^2} - \frac{2a^2}{a^2 + y^2},$$

which is unbounded as  $y \rightarrow 0$  (this is Lyapunov's famous example; see p. 71 of Günter's book<sup>15</sup>).

- Suppose that  $f'_{\text{even}}(t)$  is Hölder continuous at  $t = 0$ . Then, we can integrate by parts in (34) to give

$$N(y) = \frac{-2a}{a^2 + y^2} f_{\text{even}}(a) + 2 \int_0^a \frac{t f'_{\text{even}}(t)}{t^2 + y^2} dt. \quad (35)$$

The first term is well behaved as  $y \rightarrow 0$ . The second term is of the same form as  $T(y)$ , given by (17); its limit exists as  $y \rightarrow 0$ . Hence, we see that

$$\lim_{y \rightarrow 0} N(y) = N(0) = \frac{-2}{a} f_{\text{even}}(a) + \int_{-a}^a f'_{\text{even}}(t) \frac{dt}{t},$$

which we recognise as the finite-part integral (28). Thus, we can simply set  $y = 0$  in the formula for  $N(y)$ , (34), provided we interpret the integral correctly.

**THEOREM 5** *Suppose that  $f(t)$  is bounded at  $t = x$ . Suppose that*

$$f_{\text{even}}(t; x) = \frac{1}{2}\{f(t) + f(2x - t)\}$$

*has a Hölder-continuous first derivative at  $t = x$ . Then,*

$$\lim_{y \rightarrow 0} N(x, y) = \mathcal{F}_{-a}^a \frac{f(t)}{(t-x)^2} dt, \quad (36)$$

where  $N(x, y)$  is defined by (33) and the integral is defined by (26).

Ervin *et al.*<sup>14</sup> claim that Theorem 5 can be generalized to

$$\lim_{y \rightarrow 0} N(x, y) = \mathcal{H}_{-a}^a \frac{f(t)}{(t-x)^2} dt, \quad (37)$$

whenever  $f$  is piecewise-smooth (the finite-part integral on the right-hand side is defined by (32)). This is false. To see this, take  $f(t) = |t|$  and  $x = 0$ : the right-hand side of (37) is  $2 \log a$ , but the left-hand side does not exist.

### 6.1 More on discontinuous densities

Consider the discontinuous function (4), for which  $f'(0+) = f'(0-)$  and  $f_{\text{even}}(t) = \frac{1}{2}(f_R + f_L)$ , a constant. Then, (35) gives

$$N(y) = -2a(a^2 + y^2)^{-1} f_{\text{even}}(a) \rightarrow -(f_R + f_L)/a \quad \text{as } y \rightarrow 0, \quad (38)$$

which is precisely the result (27).

However, if we calculate  $N(x, y)$  as  $y \rightarrow 0$  for the same  $f$  but other non-zero values of  $x$ , we find that the limiting function  $N(x, 0)$  is not continuous at  $x = 0$ :

$$N(x, 0) = -f_L/(a+x) - f_R/(a-x) - (f_R - f_L)/x.$$

This result can be found either by calculating  $N(x, y)$  directly and then letting  $y \rightarrow 0$ , or by evaluating the finite-part integral on the right-hand side of (36).

#### *The construction of Huang and Cruse*

In order to deal with discontinuous densities, we could introduce the piecewise-linear approximation (20), as used by Huang and Cruse<sup>9</sup> for a different problem. Generalizing their approach, we calculate  $N(x, 0)$  at  $x = \pm\delta$ , and then examine  $N(\delta, 0) \pm N(-\delta, 0)$ . First, we note that  $N(x, 0)$  is not defined at  $x = \pm\delta$ , because of the discontinuity in slope of  $f$  at these points (this can be shown by explicit calculation; see section 8 below). However, let us overcome this difficulty by using a piecewise-quadratic approximation,

$$f(t) = \begin{cases} f_L, & t \leq -\delta, \\ \frac{1}{2}(f_R - f_L)[2(t/\delta) + (t/\delta)^2] + \frac{1}{2}(f_R + f_L), & -\delta < t \leq 0, \\ \frac{1}{2}(f_R - f_L)[2(t/\delta) - (t/\delta)^2] + \frac{1}{2}(f_R + f_L), & 0 < t < \delta, \\ f_R, & t \geq \delta. \end{cases} \quad (39)$$

$f(t)$  is in  $C^{1,\alpha}$ ; in particular,  $f'(\pm\delta) = 0$ . For this function, we can evaluate  $N(\pm\delta, y)$  as  $y \rightarrow 0$ , by direct calculation from (33); the result is

$$\begin{aligned} N(\pm\delta, 0) &= -(f_R + f_L)a(a^2 - \delta^2)^{-1} + N_0(\pm\delta), \\ N_0(\nu) &= -(f_R - f_L)\left\{\nu(a^2 - \nu^2)^{-1} + \nu^{-1} \log 4\right\}. \end{aligned}$$

Hence,

$$\frac{1}{2} \{N(\delta, 0) + N(-\delta, 0)\} = -(f_R + f_L)a(a^2 - \delta^2)^{-1}, \quad (40)$$

$$\frac{1}{2} \{N(\delta, 0) - N(-\delta, 0)\} = -(f_R - f_L)\left\{\delta(a^2 - \delta^2)^{-1} + \delta^{-1} \log 4\right\}. \quad (41)$$

Thus, the average value, given by (40), has a finite limit as  $\delta \rightarrow 0$ , namely  $-(f_R + f_L)/a$ , in agreement with (38); thus, as in section 4.1, this quantity is not affected by smoothing the discontinuity in  $f$ . On the other hand, the discontinuity in  $N$ , given by (41), is proportional to the discontinuity in  $f$  but the limit is unbounded as  $\delta \rightarrow 0$ .

## 7 DISCONTINUOUS BOUNDARY CONDITIONS

In applications of the boundary-element method, one often meets discontinuous boundary conditions. These induce density functions that are not smooth at the points of discontinuity. For a model problem with this property, we choose a boundary-value problem for Laplace's equation  $\nabla^2 u = 0$  in the half-plane  $y > 0$  with boundary condition

$$\frac{\partial u}{\partial y} = \begin{cases} 2\pi, & |x| < a, \\ 0, & |x| > a, \end{cases} \quad \text{on } y = 0. \quad (42)$$

In addition, we require that  $u = 4a \log r + o(1)$  as  $r \rightarrow \infty$  for  $0 < \theta < \pi$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ .

We can write down the solution of this problem. It is  $u = u_0$ , where

$$\begin{aligned} u_0(x, y) &= (a - x) \log [(a - x)^2 + y^2] + (a + x) \log [(a + x)^2 + y^2] \\ &\quad + 2y \tan^{-1} [(a - x)/y] + 2y \tan^{-1} [(a + x)/y] - 4a, \end{aligned} \quad (43)$$

as can be easily verified. We note two features of this exact solution. First, if we evaluate  $\partial u / \partial y$  on  $y = 0$  at  $x = a$ , say, we find that

$$\lim_{y \rightarrow 0} \frac{\partial u(a, y)}{\partial y} = \lim_{y \rightarrow 0} \left( 2 \tan^{-1} \frac{2a}{y} \right) = \pi, \quad (44)$$

which should be compared with its (prescribed) values for  $x < a$  and  $x > a$ , given by (42). Second, we find that

$$u(x, 0) = (a - x) \log [(a - x)^2] + (a + x) \log [(a + x)^2] - 4a,$$

which shows that  $u(x, 0)$  has weak singularities at  $x = \pm a$ ; in fact,  $u(x, 0)$  is not differentiable at  $x = \pm a$ , but it is Hölder continuous there.

Now, suppose we wanted to solve the above model problem using a boundary-element method. First, we need an integral representation; because of the growth at infinity, it is convenient to subtract the logarithmic term by defining a new function  $v$  by

$$u(x, y) = 4a \log r + v(x, y),$$

so that  $v = o(1)$  as  $r \rightarrow \infty$ ,

$$\frac{\partial v}{\partial y} = \begin{cases} 2\pi, & 0 < |x| < a, \\ 0, & |x| > a, \end{cases} \quad \text{on } y = 0,$$

and  $v \sim -4a \log r$  as  $r \rightarrow 0$ . Then, a careful application of Green's theorem gives

$$v(x, y) + 2a \log r = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left\{ v(\xi, 0) \frac{\partial G}{\partial n} - G \frac{\partial v}{\partial n} \right\} d\xi,$$

where  $\partial/\partial n = \partial/\partial y$  on  $y = 0$  and  $G = G(x, y; \xi, \eta)$ . Direct calculation shows that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} G \frac{\partial v}{\partial n} d\xi = \frac{1}{2} \int_{-a}^a \log \{(x - \xi)^2 + y^2\} d\xi = \frac{1}{2} u_0(x, y),$$

where  $u_0$  is defined by (43). Hence

$$v(x, y) = -2a \log r + \frac{1}{2} u_0(x, y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y v(\xi, 0)}{[(\xi - x)^2 + y^2]^2} d\xi. \quad (45)$$

This representation for  $v(x, y)$  is valid for all  $y > 0$ .

We can obtain a boundary integral equation for  $v(x, 0)$  by letting  $y \rightarrow 0$  in (45) in the usual way. Having done this, we could then solve for  $v(x, 0)$  and then construct  $v(x, y)$  from (45). For this particular problem, we know the solution for  $v(x, 0)$ : it is

$$v(x, 0) = u_0(x, 0) - 4a \log |x|.$$

Now, consider the representation (45). The integral is a double-layer potential with density  $v(\xi, 0)$ ; near  $\xi = \pm a$ ,  $v$  is Hölder continuous. So, if we wanted to calculate (cf. (44))

$$\lim_{y \rightarrow 0} \frac{\partial v(a, y)}{\partial y}, \quad (46)$$

say, the classical theorems on the existence of the normal derivative of a double-layer potential cannot be used (they would require  $v \in C^{1,\alpha}$ ). However, near  $\xi = a$ , we have

$$v(\xi, 0) = (a - \xi) \log [(a - \xi)^2] + \text{a smooth term.}$$

In particular, we note that the first term is an *odd* function of  $(a - \xi)$ , and so it has no bearing on the existence of the limit (46), which is assured by Theorem 5.

## 8 BOUNDARY-ELEMENT APPROXIMATIONS

Assuming that the geometry is smooth, a common approximation for the unknown density uses piecewise quadratics:

$$f(t) = \begin{cases} a_0^L + a_1^L t + a_2^L t^2, & t < 0, \\ a_0^R + a_1^R t + a_2^R t^2, & t > 0. \end{cases}$$

Thus, splitting as in (7), we obtain

$$\begin{aligned} f_{\text{even}}(t) &= \frac{1}{2}(a_0^R + a_0^L) + \frac{1}{2}(a_1^R - a_1^L) |t| + \frac{1}{2}(a_2^R + a_2^L) t^2 \\ f_{\text{odd}}(t) &= \frac{1}{2}(a_0^R - a_0^L) \operatorname{sgn} t + \frac{1}{2}(a_1^R + a_1^L) t + \frac{1}{2}(a_2^R - a_2^L) t^2 \operatorname{sgn} t \end{aligned}$$

where  $\operatorname{sgn} t = 1$  if  $t > 0$  and  $\operatorname{sgn} t = -1$  if  $t < 0$ . This approximation is called *conforming* if  $a_0^R = a_0^L$ ; this ensures continuity at the junction  $t = 0$ . In fact, one can see that a conforming approximation actually gives Hölder continuity, with  $f \in C^{0,1}$ . We have the following results.

- For the tangential derivative of a single-layer potential to exist on the boundary  $y = 0$  at  $x = 0$ , Theorem 2 requires that  $f$  be continuous and  $f_{\text{odd}} \in C^{0,\alpha}$ . These conditions are clearly satisfied for conforming approximations,  $a_0^R = a_0^L$ .
- For the normal derivative of a double-layer potential to exist on the boundary  $y = 0$  at  $x = 0$ , Theorem 5 requires that  $f$  be bounded and  $f_{\text{even}} \in C^{1,\alpha}$ . For these conditions to be met, we do not need a conforming approximation, *but we must have*  $a_1^R = a_1^L$ , or  $f'(0+) = f'(0-)$ . Without this latter condition, there would be a term proportional to  $|t|$  in  $f_{\text{even}}(t)$ , and it is known (section 6) that the limit does not exist for such a density.

Thus, we have found weaker sufficient conditions for the existence of various integrals and limits. However, in the context of hypersingular boundary integral equations, these conditions explicitly forbid the use of collocation at junctions between conforming quadratic elements, in general. Moreover, the same conclusion obtains with conforming higher-order elements.

### 8.1 Change the integral equation

Another point of view is as follows. First, derive a hypersingular integral equation in a standard way, assuming that all densities are sufficiently smooth; here, the connection between the original boundary-value problem and the boundary integral equation is made, using results such as Theorem 5. Now, change the definition of integral so as to allow densities with less smoothness, such as conforming piecewise quadratics. Finally, solve the new integral equation. This has been done for a model problem by Ervin *et al.*,<sup>14</sup> using various elements: in particular, convergence was observed when using conforming quadratic elements, with collocation at the mid-point and end-points of each element. It remains to ask how the solution of the new integral equation is related to the (boundary values of the) solution of the original boundary-value problem. This does not seem to be a trivial question, for we do not know how the matrix entries obtained by collocating at the element end-points are related to the exact values. As a simple example, suppose we approximate the quadratic function

$$f_1(t) = \frac{1}{4}(t^2 - a^2)/a$$

over  $-a \leq t \leq a$  by the piecewise-linear function

$$f_2(t) = \frac{1}{4}(|t| - a),$$

so that  $f_1(t) = f_2(t)$  at  $t = 0$  and at  $t = \pm a$ . Then,

$$\int_{-a}^a f_1(t) \frac{dt}{t^2} = 1$$

(independent of  $a$ ), but

$$\int_{-a}^a f_2(t) \frac{dt}{t^2} = \frac{1}{2}(1 + \log a),$$

which can take any value, depending on the choice of  $a$ .

## 9 CLOSING REMARKS

The boundary element community regularly deals with representation integrals, over the boundary of a domain, for field variables and their derivatives. Limits of these integrals as the field point approaches the boundary are of primary interest, as such limits give rise to the boundary integral

equations so important for computations. It is customary and usually convenient (although not necessary) to express these limits-to-the-boundary (LTBs) as CPV or HFP integrals, which themselves involve limits, albeit with at least some formal differences (for example, different parameters go to zero). In practice, the integrals of most interest are often tangential derivatives of single-layer potentials or normal derivatives of double-layer potentials. Hence the detailed attention to all of these matters in this paper and throughout the community.

It is important to note, in any case, that although we express our LTBs in sections 4 and 6 as CPV and/or HFP integrals, no additional smoothness of the density functions is required for existence of the LTB by so doing. Even though prior regularization would allow us to avoid the CPV and HFP concepts, the smoothness requirements are exactly the same. Indeed, without proper smoothness, unbounded terms will arise in the LTB, whether or not the CPV and/or HFP concepts are employed.

Nevertheless, some limiting processes involving the density and kernel functions at issue here do seem to require less smoothness, and still not produce unbounded terms; see, for example, equations (1) and (2) in Guiggiani's paper.<sup>16</sup> However, these limits, although correct, do not give rise to boundary integral equations as required for computations; this is acknowledged by Guiggiani.<sup>16</sup> Moreover, they are not LTBs as considered in sections 4 and 6.

On the basis of our earlier findings and work in this paper, we can summarise as follows.

- Largely-accepted sufficient conditions for the existence of certain LTBs, and CPV and HFP integrals, can be weakened, but only somewhat, based on the character of the even and odd parts of the density, as shown above (extensions of these ideas to three-dimensional problems are expected);
- however, collocation at the junction between two standard conforming boundary elements, with hypersingular integral equations cannot be theoretically justified.
- Whether or not a LTB exists, for certain integrals, depends on the smoothness of the density. This smoothness in turn is related to the order of singularity of the kernel. The smoothness demand may not be relaxed, for a given kernel singularity, just because a regularization process designed to lower the singularity of the integrand containing that kernel has been used. It also may not be relaxed if the integral in question happens to be over a closed curve (or surface) rather than an open one.
- Whether or not a LTB exists cannot be a function of the type of regularization procedure used.
- Whether or not a LTB exists cannot be a function of whether regularization procedures are applied globally (before discretization) or locally (to a particular boundary element or group of elements containing the singularity).
- Smoothness conditions required for existence of LTBs cannot be avoided or altered by first deriving an integral equation based on sufficient smoothness requirements, and subsequently relaxing these requirements (without further justification).

In brief, for a given smoothness-of-density in the vicinity of a (collocation) point on the boundary, either the LTB exists or it does not. Conditions for the existence of a LTB are not negotiable, notwithstanding the variety of expressions, special integral definitions and/or regularization procedures one may use (or avoid) to determine the value of that limit.

As noted above, we agree with Huang and Cruse,<sup>9</sup> that the terminology 'CPV', 'Hypersingular', and 'HFP', and the special definitions of integrals and some ideas associated with these terms may



be avoided, if certain regularization procedures are done before the LTB is taken, but we do not agree that such terms are ‘unnatural’ or ‘artifices’. Indeed, we believe that it is important to retain these concepts to help call attention to the rich and useful role played by Cauchy-singular and hypersingular equations. We believe this regardless of how integrands, containing Cauchy-singular or hypersingular kernels, can be or may have been weakened through regularization.

Finally, if smoothness requirements are ignored or believed to be unimportant, and by some argument or tactic an infinity does not appear in an integral equation, where theory says that one should, and one then computes with the (finite) terms which do appear, it is difficult to predict what will happen. The fact that in many numerical experiments, including some of the authors’ own, ostensibly good data can be obtained by simply ignoring infinities, the authors regard as seductive — but dangerously so.

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