PARTITIONING, BOUNDARY INTEGRAL EQUATIONS, AND EXACT GREEN'S FUNCTIONS*

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Abstract

Discretization of boundary integral equations leads to large full systems of algebraic equations, in practice. Partitioning is a method for solving such systems by breaking them down into smaller systems. It may be viewed merely as a technique from linear algebra. However, it is profitable to view it as arising directly from partitions of the boundary; these partitions could be natural (such as two separate boundaries) but they need not be. We investigate partitioning in the context of multiple scattering of acoustic waves by two sound-hard obstacles (the ideas extend to other physical situations). Specifically, we make a connection between partitioning and the use of the exact Green's function for a single obstacle in isolation. This suggests computing the action of this Green's function once-and-for-all, storing it (perhaps on a compact disc), and then using it to solve other problems in which the second obstacle is altered. One example of this approach is computing the stress distribution around a cavity of a standard-but-complicated shape inside a structure whose shape is varied. The theoretical foundation for these ideas is given, as well as a connection with the use of generalized Born series for multiple-scattering problems. Important distinctions between the partitioning/Green's function idea in this paper and seemingly similar ideas such as substructuring, multi-zoning, and domain decomposition are made.

INTRODUCTION

Integral representations for the solutions of second-order elliptic boundary-value problems usually involve two distinct boundary quantities. For example, the Helmholtz formula in acoustics gives the pressure outside a body in terms of the boundary values of the pressure and the normal velocity; and the Somigliana formula in elastostatics gives the displacement at a point in a body in terms of the surface displacements and the surface tractions. Typically, one boundary quantity is known (so that the normal velocity or the traction, say, is specified through a boundary condition) and the other is not (it may even be the desired quantity).

It is now standard practice to compute the unknown boundary values by solving an appropriate boundary integral equation, perhaps using a boundary-element method. The basic ingredient here is a fundamental solution of the governing partial differential equation; usually, the simplest fundamental solution is used. However, if one looks in older books, such as those of Kellogg (Chpt. 9),[1] Webster (§66)[2] or Garabedian (Chpt. 7),[3] one finds detailed discussions of (exact) Green's functions. These are particular fundamental solutions that satisfy a suitable homogeneous boundary condition on the boundary of a particular domain; their use eliminates the unknown boundary quantity, giving an integral representation involving the prescribed data only.

Exact Green's functions are clearly useful for addressing analytical questions, but what is their relevance to the modern (computational) use of boundary integral equations? It turns out, perhaps not unexpectedly, that if you can derive and solve a conventional boundary integral equation (BIE), using any fundamental

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solution, then you have in fact constructed a numerical approximation to the exact Green's function, and vice versa. This observation has important pedagogical value and potential strategic value for solving large, difficult problems in modern computational environments. However, the observation seems to have escaped notice by the BIE community.

Exact Green's functions can also be used in problems involving two or more bodies, such as acoustic scattering by two rigid targets. The idea here is to use the exact Green's function for one body in isolation as a fundamental solution for the two-body problem; the result is an integral equation over the boundary of the other body.

When formulating the two-body problem via direct boundary integral equations, we can use a process which we call *partitioning*. We begin with a conventional boundary integral equation, containing source points and field points on both body surfaces. In particular, we can identify the contribution arising from both points being on the first body, say, as being the same as when that body is in isolation. If we take the view that we already know how to solve for that contribution (through an inverse operator or a matrix approximation thereof), we can derive a new integral equation over the boundary of the other body. It turns out that this equation, and the one obtained using the exact Green's function, are identical. Our purpose here is to show this connection explicitly and to suggest some new points of view and possible new solution strategies based upon it.

These ideas generalise. Many physical problems can be broken down into subproblems. Good examples are cracks in a solid body or engine nacelles on a wing. It may be attractive to store a matrix corresponding to the inverse operator for a crack in an unbounded solid (the operator would act on any given tractions on the crack to give the desired crack-opening displacements); this could then be used in an exact treatment of a particular cracked body — one would only need to solve an integral equation over the surface of the body. Similarly, one might want to experiment with different engines on the same wing, or vice versa. In fact, the idea of partitioning is applicable to *single* bodies too: partition the surface into two parts. It gives a way of solving large problems in smaller steps.

In any case, we have the option of breaking down a problem into subproblems, including a particular component such as a penny-shaped crack or a cavity of a standard-but-complicated shape embedded in an otherwise unbounded medium; call this the 'standard subproblem'. We assume that we can compute and store a (possibly large) matrix (a matrix inverse or LU decomposition) corresponding to having pre-solved the standard subproblem to high accuracy. With today's technology, it is possible to think of storing such matrices on compact disks or mass-storage devices and making them widely available for reuse perhaps on international networks. This would constitute an electronic library of such matrices which could be called 'discretized Green's functions'.[4, 5]

The computational efficacy of the partitioning process, and how it might fare in terms of operation counts,[6] CPU time, and accuracy of a given run in particular, versus doing a given large computation all at once is not what this paper is mainly about; although our own limited experience, and what might be gleaned from the literature on solving large linear systems of equations in pieces, suggest that partitioning can fare well indeed by these measures. More generally, the idea of having and using a library of Green's functions for a given class of problems, is a larger issue than just consideration of the measures just mentioned. Indeed, matters of shared or distributed modelling effort, cutting down on duplication of effort, necessary expertise to do a computation, required hardware and software for various strategies, matters of speed, expense and more are all involved in the wisdom and feasibility of such a library. These issues and others are taken up elsewhere.[5]

Again, the main intent of this paper is to make some observations about exact Green's functions and BIE's which seem to have escaped notice. In passing, we speculate about some possible practical consequences of these observations in modern environments.

The partitioning process described above is exact, in principle. Analytical approximations can be attempted. One example is the *generalized Born series* for scattering by two obstacles.[7, 8] This approximation can be derived formally from our exact formulation: for a rigorous justification of such approximations, one needs to prove that a certain operator B, say, has norm less than one, so that the inverse $(I - B)^{-1}$ can be replaced by the geometric series.

Finally, we note that, for a single body, partitioning has a resemblance to multi-zoning and domain decomposition, as practiced, for example, with the finite element method (FEM). However, there is an essential difference. We partition the boundary of the body, not the body itself. Thus, partitioning is readily

applicable to exterior problems. Most importantly though, the partitioning-of-the-boundary-only process, as described here, gives rise to 'pieces of the problem', which are interpretable in terms of exact Green's functions, whether viewed in discrete or integral-operator form. There is no counterpart to this with finite elements. While precomputing may be done for pieces of a problem (cf. the DMAP formalism in NASTRAN), these pieces are not Green's functions.

ONE OBSTACLE, TWO APPROACHES

Consider the scattering of time-harmonic acoustic waves by a three-dimensional bounded obstacle with a smooth surface S. Thus, the problem is to solve the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0 \tag{1}$$

in D, the unbounded region exterior to S, subject to a radiation condition at infinity and a boundary condition on S; we take the latter to be

$$\frac{\partial u}{\partial n} = f \qquad \text{on } S, \tag{2}$$

where f is a given function and $\partial/\partial n$ denotes normal differentiation from S into D.

We remark that we could consider other boundary conditions on S, or other field equations: our intention is to describe rather general ideas in the context of a specific concrete example.

Standard boundary integral equation

We start with a fundamental solution for (1). The simplest is the free-space Green's function,

$$G(P,Q) = G(Q,P) = -\mathrm{e}^{\mathrm{i}kR}/(2\pi R),$$

where R is the distance between the two points P and Q. G satisfies (1) everywhere, with respect to P or Q, except when $P = Q$; it also satisfies the radiation condition (for a time-dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$). Then, an application of Green's theorem to $u(Q)$ and $G(Q,P)$, for fixed P, gives

$$2u(P) = \int_S \left(\frac{\partial u(q)}{\partial n} G(q,P) - u(q) \frac{\partial}{\partial n_q} G(q,P) \right) ds_q. \tag{3}$$

This is an integral representation for $u(P)$, with $P \in D$, in terms of the boundary values of u and $\partial u/\partial n$. For our problem, the latter are known from the boundary condition (2), whence

$$2u(P) = \int_S \left(f(q)\, G(q,P) - u(q) \frac{\partial}{\partial n_q} G(q,P) \right) ds_q. \tag{4}$$

It remains to determine $u(q)$ for $q \in S$. Letting P go to p on S gives the familiar integral equation,

$$u(p) + \int_S u(q) \frac{\partial}{\partial n_q} G(q,p)\, ds_q = \int_S f(q)\, G(q,p)\, ds_q.$$

This is a Fredholm integral equation of the second kind with a weakly-singular kernel; we write it concisely as

$$Au \equiv (I + K)u = S_0 f, \tag{5}$$

where K and S_0 are integral operators. It is known that (5) is uniquely solvable, except at certain irregular values of k^2.[9] We disregard these here (several methods for their elimination, leading to different integral equations are available[9]). Thus, formally, we obtain

$$u = A^{-1} S_0 f \tag{6}$$

for the solution of (5).

In practice, we cannot find A^{-1} analytically. However, we can solve (5) numerically using a boundary-element method. This gives a discrete approximation to A^{-1}, which we can think of as an $N \times N$ matrix; increasing N gives a better approximation to A^{-1}.

The exact Green's function

Let us introduce a different fundamental solution G^{E}, defined as follows. Fix the point P. Then, write

$$G^{\mathrm{E}}(Q;P) = G(Q,P) + w(Q;P),$$

and choose w so that it (i) satisfies (1) for all $Q \in D$, (ii) satisfies the radiation condition, and (iii) is such that G^{E} satisfies

$$\frac{\partial}{\partial n_q} G^{\mathrm{E}}(q;P) = 0 \qquad \text{for } q \in S. \tag{7}$$

We call G^{E} the *exact Green's function*; Bergman and Schiffer[10] call it the *Neumann function*. Like G, G^{E} is *symmetric* (see Appendix A):

$$G^{\mathrm{E}}(P;Q) = G^{\mathrm{E}}(Q;P) \qquad \text{for all } P \text{ and } Q \text{ in } D \cup S,\ P \neq Q. \tag{8}$$

Since G^{E} is a fundamental solution, we can use it to derive the integral representation (4), with G replaced by G^{E}. However, the representation simplifies because of (7), giving

$$2u(P) = \int_S f(q)\, G^{\mathrm{E}}(q;P)\, ds_q. \tag{9}$$

This is an explicit formula for the solution of the one-obstacle problem. In particular, for $p \in S$, it gives

$$u(p) = \tfrac{1}{2} \int_S f(q)\, G^{\mathrm{E}}(q;p)\, ds_q = \tfrac{1}{2} S_0^{\mathrm{E}} f,$$

say. Comparison of this formula with (6) gives

$$A^{-1} S_0 f = \tfrac{1}{2} S_0^{\mathrm{E}} f. \tag{10}$$

As this holds for every f, we deduce that

$$A^{-1} S_0 = \tfrac{1}{2} S_0^{\mathrm{E}}. \tag{11}$$

Moreover, (10) gives $A S_0^{\mathrm{E}} f = 2 S_0 f$, which implies that G^{E} solves the integral equation

$$G^{\mathrm{E}}(q;p) + \int_S G^{\mathrm{E}}(l;p) \frac{\partial}{\partial n_l} G(l,q)\, ds_l = 2G(q,p). \tag{12}$$

In this equation, the point p occurs as a parameter; indeed, the same equation holds when p is replaced by $P \in D$. See Appendix A for a direct derivation of (12), and other similar equations. The idea of constructing G^{E} by solving a boundary integral equation can be found in a paper by Boley.[11]

In summary, if we want to find G^{E} for a particular geometry, we typically have to solve a boundary integral equation such as (12): we have shown above that this is equivalent to calculating A^{-1}.

MULTIPLE SCATTERING BY TWO OBSTACLES

In this section, we consider the same scattering problem as in the preceding section but with two bounded obstacles. For the moment, we assume that their surfaces, S_1 and S_2, are disjoint; other configurations will be discussed later. Thus, the problem is to solve (1) in D, the unbounded region exterior to S_1 and S_2, subject to a radiation condition and the boundary conditions

$$\frac{\partial u}{\partial n} = f_j \qquad \text{on } S_j,\ j = 1,2, \tag{13}$$

where f_1 and f_2 are given functions.

We describe two methods for solving this problem. First, we derive a pair of coupled boundary integral equations using G, in a standard way; these equations are weighted equally between S_1 and S_2. In practice, we may already have information on how to scatter by one of the obstacles (S_2, say) in isolation, such as A^{-1}, G^{E} or one of their discrete approximations. How can we use this information? One way is to 'partition' the pair of integral equations; another is to replace G by G^{E}. We prove that these two approaches lead to exactly the same equations. A third approach is to assume that we have two exact Green's functions, one for each scatterer; this leads naturally to the generalized Born series, discussed later.

Partitioning

The method used above to derive the boundary integral equation (5) works equally well for two obstacles: simply replace S by $S_1 \cup S_2$. The resulting equation can be written in the following form:

$$A_{11}u_1 + A_{12}u_2 = S_{11}f_1 + S_{12}f_2, \tag{14}$$
$$A_{21}u_1 + A_{22}u_2 = S_{21}f_1 + S_{22}f_2. \tag{15}$$

Here, $u_j = u(p_j)$ where $p_j \in S_j$ for $j = 1, 2$,

$$A_{jk}u_k = \delta_{jk}u(p_k) + \int_{S_k} u(q_k)\frac{\partial}{\partial n_q}G(q_k, p_j)\, ds_q \tag{16}$$

$$S_{jk}f_k = \int_{S_k} f_k(q_k)\, G(q_k, p_j)\, ds_q \tag{17}$$

and δ_{ij} is the Kronecker delta. We note that A_{jj} is simply the operator A for S_j (both the field point p_j and the source point (integration point) q_j are on S_j), whereas A_{12} and A_{21} give the interactions between S_1 and S_2 (these integral operators have smooth kernels, as the field and source points are on different surfaces).

It is known that the pair of integral equations (14) and (15) suffers from irregular frequencies. Again, we disregard these here; for a discussion on methods for eliminating irregular frequencies from such equations, see Martin.[12]

Now, suppose we already have (a discrete approximation to) A_{22}^{-1}; for example, we may have solved (5) using an accurate boundary-element method. Then, (15) gives

$$u_2 = A_{22}^{-1}\{S_{21}f_1 + S_{22}f_2 - A_{21}u_1\}. \tag{18}$$

Eliminating u_2 from (14), we obtain

$$\mathcal{A}_{11}u_1 = \mathcal{S}_{11}f_1 + \mathcal{S}_{12}f_2, \tag{19}$$

where

$$\mathcal{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{20}$$
$$\mathcal{S}_{1j} = S_{1j} - A_{12}A_{22}^{-1}S_{2j}, \quad j = 1, 2. \tag{21}$$

Equation (19) is an integral equation to solve for u on S_1. We shall return to it later.

We could view partitioning as merely a method for solving systems of linear algebraic equations. However, we shall argue later that it is profitable to view partitioning as arising directly from partitions of the boundary.

Use of the exact Green's function

Suppose that G^{E} is the exact Green's function for S_2 (in isolation). Proceeding as for the one-obstacle problem, using G^{E} for our chosen fundamental solution, we obtain the following integral representation for the two-obstacle problem:

$$2u(P) = \int_{S_1}\left(f_1(q_1)\, G^{\mathrm{E}}(q_1; P) - u(q_1)\frac{\partial}{\partial n_q}G^{\mathrm{E}}(q_1; P)\right) ds_q + \int_{S_2} f_2(q_2)\, G^{\mathrm{E}}(q_2; P)\, ds_q. \tag{22}$$

This representation does not involve the unknown boundary values of u on S_2. Moreover, in some applications, $f_2 \equiv 0$, whence the second integral in (22) is absent.

To find u on S_1, we let $P \to p_1 \in S_1$, as usual; the result can be written as

$$A_{11}^{\mathrm{E}}u_1 = S_{11}^{\mathrm{E}}f_1 + S_{12}^{\mathrm{E}}f_2, \tag{23}$$

where

$$A_{11}^{\mathrm{E}}u_1 = u(p_1) + \int_{S_1} u(q_1)\frac{\partial}{\partial n_q}G^{\mathrm{E}}(q_1, p_1)\, ds_q \tag{24}$$

$$S_{1j}^{\mathrm{E}}f_j = \int_{S_j} f_j(q_j)\, G^{\mathrm{E}}(q_j, p_1)\, ds_q, \quad j = 1, 2. \tag{25}$$

If we subsequently want u on S_2, we can calculate it by simply setting $P = p_2$ in (22); there is no jump. Again, we note the obvious simplification if $f_2 \equiv 0$.

Having solved (23) for u_1, we can calculate $u(P)$ for P in D from (22). To do this, we need $G^{\rm E}(q_2;P)$, $G^{\rm E}(q_1;P)$ and $(\partial/\partial n_q)G^{\rm E}(q_1;P)$. We can obtain $G^{\rm E}(q_2;P)$ by solving (A.4), and then $G^{\rm E}(q_1;P)$ is given by (A.1). Similarly, $(\partial/\partial n_q)G^{\rm E}(q_1;P)$ can be found by first solving (A.12) and then using (A.13).

Comparison

We have two boundary integral equations for u on S_1, namely (19) and (23). It turns out that these equations are identical: in Appendix B, we give direct proofs that

$$\mathcal{A}_{11} = A^{\rm E}_{11}, \quad \mathcal{S}_{11} = S^{\rm E}_{11}, \quad \text{and} \quad \mathcal{S}_{12} = S^{\rm E}_{12}. \tag{26}$$

DISCUSSION AND CONCLUSIONS

The derivation of the partitioning integral equation (19) does not require two disjoint surfaces: S_2 could be the surface of a defect buried inside a body with outer surface S_1 (such as a crack in an elastic body); or $S_1 \cup S_2$ could be a partition of a single surface, S (the partition could be artificial or physical, such as an edge-crack in a body). This latter interpretation suggests that partitioning is related to other methods; it can also be related to certain iterative methods for solving multiple-scattering problems. These relations are discussed next.

Generalized Born series

The standard integral equations, (14) and (15), can be written as

$$\begin{aligned} A_{11}u_1 &= F_1 - A_{12}u_2, \\ A_{22}u_2 &= F_2 - A_{21}u_1, \end{aligned}$$

where $F_j = S_{j1}f_1 + S_{j2}f_2$. The generalized Born series[7, 8] is a method for solving this pair iteratively in the context of multiple scattering by two obstacles (actually, it can be recognised as the block Jacobi method for linear algebraic equations): assuming that A_{11} and A_{22} are non-singular, construct $u_j^{(m)}$ according to

$$\begin{aligned} u_1^{(m+1)} &= A_{11}^{-1}\left\{F_1 - A_{12}u_2^{(m)}\right\}, \\ u_2^{(m+1)} &= A_{22}^{-1}\left\{F_2 - A_{21}u_1^{(m)}\right\}, \end{aligned}$$

with $u_1^{(0)} = u_2^{(0)} = 0$. Eliminating $u_2^{(m)}$, we obtain

$$u_1^{(m+1)} = g_1 + B_{11}u_1^{(m-1)} \tag{27}$$

for $m = 1, 2, \ldots$, where

$$g_1 = A_{11}^{-1}\left\{F_1 - A_{12}A_{22}^{-1}F_2\right\} \qquad \text{and} \qquad B_{11} = A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}.$$

Hence,

$$u_1^{(2M)} = \sum_{m=0}^{M-1} B_{11}^m g_1 \qquad \text{for } M = 1, 2, \ldots. \tag{28}$$

Also, since $u_1^{(1)} = A_{11}^{-1}F_1$, (27) gives

$$u_1^{(2M+1)} = u_1^{(2M)} + B_{11}^M A_{11}^{-1}F_1 \qquad \text{for } M = 1, 2, \ldots. \tag{29}$$

The (geometric) series in (28) and (29) converges if

$$\|B_{11}\| < 1 \tag{30}$$

(Kress,[13] p. 16), with any reasonable norm. Under this condition (which will be satisfied if the two scatterers are sufficiently far apart), the last term in (29) tends to zero and both sequences ($\{u_1^{(2M)}\}$ and $\{u_1^{(2M+1)}\}$) converge to $u_1^{(\infty)}$, say, where

$$\begin{aligned} u_1^{(\infty)} &= (I - B_{11})^{-1} g_1 = \mathcal{A}_{11}^{-1} A_{11} g_1 \\ &= \mathcal{A}_{11}^{-1} \left\{ F_1 - A_{12} A_{22}^{-1} F_2 \right\} \\ &= \mathcal{A}_{11}^{-1} \left\{ \mathcal{S}_{11} f_1 + \mathcal{S}_{12} f_2 \right\}, \end{aligned}$$

which is the solution of the partitioning equation (19). This latter equation is not subject to the condition (30).

Note that the generalized Born series requires a knowledge of both A_{11}^{-1} and A_{22}^{-1}; this is equivalent to knowing two exact Green's functions, one for each obstacle. Rudgers[14] has used the sum of these two exact Green's functions as a fundamental solution, followed by an iterative method.

Comparison with other methods

Partitioning is simple and superficially similar to other methods, such as *domain decomposition* and *multi-zoning*. In these methods, the original problem is broken down into smaller problems; each problem is solved and then the solutions are patched together. For the partitioning method, however, the original problem itself is reduced to a boundary integral equation; this equation is then manipulated further by partitioning the boundary.

To expose these differences further, consider the following model problem: solve $\nabla^2 u = 0$ inside a bounded domain Ω, subject to $u = f$ on the boundary of Ω, S (f is given). Partition S into two pieces, $S = S_1 \cup S_2$. Then the usual boundary integral equation over S can be written in the form

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or $\mathbf{Bv} = \mathbf{f}$; here, $v_j = \partial u / \partial n$ on S_j. For the partitioning method, we eliminate v_1 or v_2: if we eliminate v_1, we obtain

$$\mathcal{B}_{22} v_2 = b_2 - B_{21} B_{11}^{-1} b_1$$

where

$$\mathcal{B}_{22} = B_{22} - B_{21} B_{11}^{-1} B_{12}.$$

In discretized form (where B_{ij} are matrices), this process is sometimes called *condensation* and the matrix $\mathcal{B}_{22}$ is called the *Schur complement of* B_{11} *in* $\mathbf{B}$; see, for example, Golub and Van Loan (p. 103)[15] or Fiedler (p. 19).[16]

Now, cut Ω into two subdomains, Ω_1 and Ω_2, using an interface Γ_0, so that $\Gamma_j \equiv S_j \cup \Gamma_0$ is the closed boundary of Ω_j ($j = 1, 2$). If we knew the values of u or $\partial u / \partial n$ on Γ_0 (we do not), we could solve $\nabla^2 u = 0$ inside each subdomain. One way of proceeding is to find these values. Write down an integral equation over Γ_1 and another one over Γ_2. Then, enforce continuity of u and $\partial u / \partial n$ across Γ_0. This leads to a larger system of equations (because of the additional unknowns on Γ_0), of the form

$$\begin{pmatrix} B_{11} & A_{10} & B_{10} & 0 \\ B_{01} & A_{00} & B_{00} & 0 \\ 0 & -A_{00} & -B_{00} & B_{02} \\ 0 & -A_{20} & -B_{20} & B_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ u_0 \\ v_0 \\ v_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_{01} \\ b_{02} \\ b_2 \end{pmatrix},$$

where $v_j = \partial u / \partial n$ on Γ_j ($j = 0, 1, 2$) and u_0 is u on Γ_0. The occurrence of zero blocks can be exploited; for a sophisticated recent treatment of such multi-zone analyses, see Kane *et al.*[17] Furthermore, the system can be condensed, eliminating some unknowns in favour of others; see, for example, Kane *et al.*[17] or §3.6 of Banerjee's book.[18]

An alternative to multi-zone analysis is *domain decomposition*. The basic idea here is: guess u_0 (the boundary values of u on the interface Γ_0), solve the (Dirichlet) subproblems in Ω_1 and Ω_2, and then adjust

the guess until the normal derivatives of the two solutions are continuous across Γ_0. This idea is exploited in Chapter 8 of Przemieniecki's book,[19] where it is called *substructure analysis*. The subproblems can be solved, in parallel, using boundary elements,[20, 21] and this leads naturally to the Dirichlet-to-Neumann (DtN) mappings, one for each subdomain, mapping Dirichlet data on Γ_0 into the corresponding Neumann data. A symmetric form of the DtN mapping is called the *Poincaré-Steklov operator*; for a single interior domain with boundary S, this operator is given by

$$2\frac{\partial u}{\partial n} = \left\{N - (I - K')S_0^{-1}(I - K)\right\} u,$$

where K and S_0 are given by (5) (with $k = 0$), K' is the adjoint of K (replace $\partial/\partial n_q$ by $\partial/\partial n_p$) and N is the usual hypersingular operator (normal derivative of a double-layer potential). Note that the Poincaré-Steklov operator has the structure of a Schur complement.

We emphasise that, in both multi-zoning and domain decomposition, the interface Γ_0 is artificial. Of course, there are problems where Γ_0 is an actual interface separating different media in Ω_1 and Ω_2 (transmission problem), or where part of Γ_0 is occupied by a crack.[22] Such problems can be solved by multi-zoning. However, the extension to exterior problems introduces further complications: one can surround Ω by a closed surface S_∞ (such as a sphere) and then solve the problem exterior to S_∞ explicitly (find the DtN mapping for the region exterior to S_∞);[23] or one can decompose the exterior of Ω into infinite subdomains, using radial partitions.[24] These difficulties do not arise with (boundary) partitioning; its inherent simplicity makes it worthy of further study.

Appendix A. Integral equations for G^{E}

Fix points P and Q in D, and then apply Green's theorem to $G^{\mathrm{E}}(L;P)$ and $G(L,Q)$ with respect to the point L; as both of these are singular solutions, the result is

$$2G^{\mathrm{E}}(Q;P) - 2G(P,Q) = -\int_S G^{\mathrm{E}}(l;P)\frac{\partial}{\partial n_l}G(l,Q)\,ds_l. \tag{A.1}$$

If we apply the same argument, but with $G(L,Q)$ replaced by $G^{\mathrm{E}}(L;Q)$, we deduce that

$$G^{\mathrm{E}}(P;Q) = G^{\mathrm{E}}(Q;P) \qquad \text{for all } P \text{ and } Q \text{ in } D,\ P \neq Q. \tag{A.2}$$

Then, combining this symmetry property with (A.1) gives

$$2G^{\mathrm{E}}(Q;P) - 2G(P,Q) = -\int_S G^{\mathrm{E}}(l;Q)\frac{\partial}{\partial n_l}G(l,P)\,ds_l. \tag{A.3}$$

We can now obtain integral equations from the two different integral representations for G^{E}. First, let $Q \to q \in S$ in (A.1), giving

$$G^{\mathrm{E}}(q;P) + \int_S G^{\mathrm{E}}(l;P)\frac{\partial}{\partial n_l}G(l,q)\,ds_l = 2G(q,P), \tag{A.4}$$

which is a boundary integral equation for $G^{\mathrm{E}}(q;P)$; note that, for fixed P, the operator on the left-hand side is precisely A, defined by (5). In particular, letting $P \to p \in S$, we obtain

$$G^{\mathrm{E}}(q;p) + \int_S G^{\mathrm{E}}(l;p)\frac{\partial}{\partial n_l}G(l,q)\,ds_l = 2G(q,p). \tag{A.5}$$

Second, interchange P and Q in (A.3), and then let $Q \to q \in S$; the result is

$$G^{\mathrm{E}}(P;q) + \int_S G^{\mathrm{E}}(l;P)\frac{\partial}{\partial n_l}G(l,q)\,ds_l = 2G(q,P). \tag{A.6}$$

Subtracting this equation from (A.4) shows that

$$G^{\mathrm{E}}(q;P) = G^{\mathrm{E}}(P;q) \qquad \text{for all } P \in D \text{ and } q \in S. \tag{A.7}$$

Third, let $Q \to q \in S$ in (A.3), giving

$$2G^{\mathrm{E}}(q;P) = 2G(q,P) - \int_S G^{\mathrm{E}}(l;q)\frac{\partial}{\partial n_l}G(l,P)\,ds_l \tag{A.8}$$

(there is no jump as $Q \to q$), which is not an integral equation. However, if we let $P \to p \in S$, we obtain

$$G^{\mathrm{E}}(q;p) + \int_S G^{\mathrm{E}}(l;q)\frac{\partial}{\partial n_l}G(l,p)\,ds_l = 2G(q,p). \tag{A.9}$$

If we interchange p and q in this equation and then subtract the result from (A.5), we deduce that

$$G^{\mathrm{E}}(q;p) = G^{\mathrm{E}}(p;q) \qquad \text{for all } p \text{ and } q \text{ on } S, \tag{A.10}$$

whence (A.9) becomes

$$G^{\mathrm{E}}(q;p) + \int_S G^{\mathrm{E}}(q;l)\frac{\partial}{\partial n_l}G(l,p)\,ds_l = 2G(q,p). \tag{A.11}$$

Multiplying this equation by $f(q)$ and then integrating over $q \in S$ gives $AS_0^{\mathrm{E}}f = 2S_0f$, in agreement with (11).

Combining (A.2), (A.7) and (A.10), we see that G^{E} is symmetric for all locations of the source and field points, (8).

Finally, we note that we can derive integral equations giving the gradient of G^{E} at any point P not on S, $\mathrm{grad}_P G(q;P)$. Thus, taking the gradient of (A.4) gives

$$\mathrm{grad}_P G^{\mathrm{E}}(q;P) + \int_S \mathrm{grad}_P G^{\mathrm{E}}(l;P)\frac{\partial}{\partial n_l}G(l,q)\,ds_l = 2\,\mathrm{grad}_P G(q,P), \tag{A.12}$$

which is an integral equation for $\mathrm{grad}_P G^{\mathrm{E}}(q;P)$. Then, (A.1) gives

$$2\,\mathrm{grad}_P G^{\mathrm{E}}(Q;P) = 2\,\mathrm{grad}_P G(P,Q) - \int_S \mathrm{grad}_P G^{\mathrm{E}}(l;P)\frac{\partial}{\partial n_l}G(l,Q)\,ds_l \tag{A.13}$$

for any point Q not on S.

Appendix B. Proof of (26)

We make use of various formulae from Appendix A, but with S replaced by S_2.

Proof that $A_{11}^{\mathrm{E}} = \mathcal{A}_{11}$

From (A.3), we can let $P \to p_1 \in S_1$ to give

$$2G^{\mathrm{E}}(Q;p_1) = 2G(Q,p_1) - \int_{S_2} G^{\mathrm{E}}(l_2;Q)\frac{\partial}{\partial n_l}G(l_2,p_1)\,ds_l. \tag{B.1}$$

Computing the normal derivative at $q_1 \in S_1$ yields

$$\frac{\partial}{\partial n_q}G^{\mathrm{E}}(q_1;p_1) = \frac{\partial}{\partial n_q}G(q_1,p_1) - \tfrac{1}{2}\int_{S_2}\frac{\partial}{\partial n_q}G^{\mathrm{E}}(l_2;q_1)\frac{\partial}{\partial n_l}G(l_2,p_1)\,ds_l. \tag{B.2}$$

Multiply this formula by $u(q_1)$ and integrate over S_1. Adding $u(p_1)$ to the result, making use of the definitions (16) and (24), gives

$$A_{11}^{\mathrm{E}}u_1 = A_{11}u_1 - \tfrac{1}{2}A_{12}U_2^{\mathrm{E}} \tag{B.3}$$

where

$$U_2^{\mathrm{E}}(p_2) = \int_{S_1}\frac{\partial}{\partial n_q}G^{\mathrm{E}}(q_1;p_2)u(q_1)\,ds_q$$

for $p_2 \in S_2$. $U_2^{\rm E}$ satisfies a boundary integral equation over S_2: from (A.4), we have

$$G^{\rm E}(Q;p_2) + \int_{S_2} G^{\rm E}(Q;l_2) \frac{\partial}{\partial n_l} G(l_2,p_2)\, ds_l = 2G(Q,p_2); \tag{B.4}$$

multiply the normal derivative of this equation at q_1 by $u(q_1)$, and then integrate over S_1 to give $A_{22}U_2^{\rm E} = 2A_{21}u_1$. Inverting this equation and then eliminating $U_2^{\rm E}$ from (B.3), we obtain the desired result:

$$A_{11}^{\rm E} = A_{11} - A_{12}A_{22}^{-1}A_{21}.$$

Proof that $S_{11}^{\rm E} = \mathcal{S}_{11}$

Evaluate (A.3) at $P = p_1$ and $Q = q_1$. Multiply the result by $f_1(q_1)$ and then integrate over S_1 to give

$$S_{11}^{\rm E} f_1 = S_{11} f_1 - \tfrac{1}{2} A_{12} F_2^{\rm E} \tag{B.5}$$

where

$$F_2^{\rm E}(p_2) = \int_{S_1} G^{\rm E}(q_1;p_2) f_1(q_1)\, ds_q$$

for $p_2 \in S_2$. $F_2^{\rm E}$ satisfies a boundary integral equation over S_2: evaluate (B.4) at $Q = q_1$, multiply the result by $f_1(q_1)$ and integrate over S_1 to give $A_{22}F_2^{\rm E} = 2S_{21}f_1$. Eliminating $F_2^{\rm E}$ from (B.5) gives the final result:

$$S_{11}^{\rm E} = S_{11} - A_{12}A_{22}^{-1}S_{21}.$$

Proof that $S_{12}^{\rm E} = \mathcal{S}_{12}$

Evaluate (A.8) at $P = p_1$ and $q = q_2$. Multiply the result by $f_2(q_2)$ and then integrate over S_2 to give

$$S_{12}^{\rm E} f_2 = S_{12} f_2 - \tfrac{1}{2} A_{12} \mathcal{F}_2^{\rm E} \tag{B.6}$$

where

$$\mathcal{F}_2^{\rm E}(p_2) = \int_{S_2} G^{\rm E}(q_2;p_2) f_2(q_2)\, ds_q.$$

$\mathcal{F}_2^{\rm E}$ satisfies a boundary integral equation over S_2: evaluate (A.11) at $p = p_2$ and $q = q_2$, multiply the result by $f_2(q_2)$ and integrate over S_2 to give $A_{22}\mathcal{F}_2^{\rm E} = 2S_{22}f_2$. Eliminating $\mathcal{F}_2^{\rm E}$ from (B.6) gives the final result:

$$S_{12}^{\rm E} = S_{12} - A_{12}A_{22}^{-1}S_{22}.$$

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