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Decay in time for a one-dimensional two-component plasma

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SUMMARY

The motion of a collisionless plasma is described by the Vlasov–Poisson (VP) system, or in the presence of large velocities, the relativistic VP system. Both systems are considered in one space and one momentum dimension, with two species of oppositely charged particles. A new identity is derived for both systems and is used to study the behavior of solutions for large times. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: partial differential equations; kinetic theory; Vlasov equations

1. INTRODUCTION

Consider the Vlasov–Poisson system (VP)

$$\begin{cases} \partial_t f + v\partial_x f + E(t,x)\partial_v f = 0 \\ \partial_t g + \dfrac{v}{m}\partial_x g - E(t,x)\partial_v g = 0 \\ \rho(t,x) = \displaystyle\int (f(t,x,v) - g(t,x,v))\,\mathrm{d}v \\ E(t,x) = \dfrac{1}{2}\left(\displaystyle\int_{-\infty}^{x} \rho(t,y)\,\mathrm{d}y - \int_{x}^{\infty} \rho(t,y)\,\mathrm{d}y\right) \end{cases} \tag{1}$$

where $t \geqslant 0$ is time, $x \in \mathbb{R}$ is position, $v \in \mathbb{R}$ is momentum, f is the number density in phase space of particles with mass one and positive unit charge, and g is the number density of particles with

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mass $m>0$ and negative unit charge. The effect of collisions is neglected. The initial conditions

$$f(0,x,v)=f_0(x,v)\geqslant 0$$

and

$$g(0,x,v)=g_0(x,v)\geqslant 0$$

for $(x,v)\in\mathbb{R}^2$ are given where it is assumed that $f_0, g_0\in C^1(\mathbb{R}^2)$ are nonnegative, compactly supported, and satisfy the neutrality condition

$$\iint f_0\,\mathrm{d}v\,\mathrm{d}x=\iint g_0\,\mathrm{d}v\,\mathrm{d}x \tag{2}$$

Using the notation

$$\hat{v}_m=\frac{v}{\sqrt{m^2+v^2}}$$

the relativistic Vlasov–Poisson system (RVP) is

$$\begin{cases}\partial_t f+\hat{v}_1\partial_x f+E\partial_v f=0\\ \partial_t g+\hat{v}_m\partial_x g-E\partial_v g=0\\ \rho(t,x)=\displaystyle\int (f-g)\,\mathrm{d}v\\ E(t,x)=\dfrac{1}{2}\left(\displaystyle\int_{-\infty}^{x}\rho\,\mathrm{d}y-\int_{x}^{\infty}\rho\,\mathrm{d}y\right)\end{cases} \tag{3}$$

It is well known that solutions to (1) and (3) remain smooth for all $t\geqslant 0$ with $f(t,\cdot,\cdot)$ and $g(t,\cdot,\cdot)$ compactly supported for all $t\geqslant 0$. In fact, this is known for the three-dimensional version of (1) [1, 2], but not for the three-dimensional version of (3). The literature regarding large time behavior of solutions is quite limited. Some time decay is known for the three-dimensional analog of (1) [3–5]. Also, there are time decay results for (1) (in dimension 1) when the plasma is monocharged (set $g\equiv 0$) [6–8]. In the work that follows, two species of particles with opposite charge are considered, thus the methods used in [6–8] do not apply. References [9–11] are also mentioned as they deal with time-dependent rescalings and time decay for other kinetic equations.

In the following section an identity is derived for (1) that shows certain positive quantities are integrable in t on the interval $[0,\infty)$. The identity is modified to address (3) also, but the results are weaker. These identities seem to be linked to the one-dimensional situation and do not readily generalize to higher dimension. Additionally, it is not clear if there is an extension that allows for more than two species of particles. However, as this model allows for attractive forces between ions of differing species, it is sensible to expect that additional species of ions will only strengthen repulsive forces and cause solutions to decay faster in time. In Section 3, the L^4 integrability of both the positive and negative charge is derived and used to show time decay of the local charge. Finally, in Section 4, the main identity and L^4 integrability will be used to show decay in time of the electric field for both (1) and (3).

DOI: 10.1002/mma

2. THE IDENTITY

The basic identities for (1) and (3) will be derived in this section. The following theorem lists their main consequences:

Theorem 2.1
Assume that f_0 and g_0 are nonnegative, compactly supported, C^1, and satisfy (2). Then, for a solution to (1), there exists $C>0$ depending only on f_0, g_0, and m such that

$$\int_0^\infty \iiint f(t,x,w) f(t,x,v)(w-v)^2 \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

$$\int_0^\infty \iiint g(t,x,w) g(t,x,v)(w-v)^2 \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

and

$$\int_0^\infty \int E^2 \int (f+g)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

For a solution to (3) there is $C>0$ depending only on f_0, g_0, and m such that

$$\int_0^\infty \iiint f(t,x,w) f(t,x,v)(w-v)(\hat{w}_1-\hat{v}_1)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

$$\int_0^\infty \iiint g(t,x,w) g(t,x,v)(w-v)(\hat{w}_m-\hat{v}_m)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

and

$$\int_0^\infty \int E^2 \int (f+g)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

Moreover, $(w-v)(\hat{w}_m-\hat{v}_m)\geqslant 0$ for all $w, v\in\mathbb{R}$, $m>0$.

Proof
Suppose

$$\partial_t a+\omega(v)\partial_x a+B(t,x)\partial_v a=0 \tag{4}$$

where $a(t,x,v), \omega(v), B(t,x)$ are C^1, and $a(t,\cdot,\cdot)$ is compactly supported for each $t\geqslant 0$. Let

$$A(t,x)=\int a\,\mathrm{d}v$$

and

$$\mathscr{A}(t,x)=\int_{-\infty}^{x} A(t,y)\,\mathrm{d}y$$

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Note that $\partial_x \mathscr{A} = A$ and

$$\partial_t \mathscr{A} = -\int_{-\infty}^{x} \int (\omega(v)\partial_y a(t,y,v) + B(t,y)\partial_v a(t,y,v))\,\mathrm{d}y\,\mathrm{d}v$$
$$= -\int \omega(v) a(t,x,v)\,\mathrm{d}v$$

By (4) it follows that

$$0 = \mathscr{A}(t,x) \int v(\partial_t a + \omega(v)\partial_x a + B(t,x)\partial_v a)\,\mathrm{d}v$$
$$=: I + II + III \tag{5}$$

Then,

$$I = \partial_t \left(\mathscr{A} \int a v\,\mathrm{d}v \right) - (\partial_t \mathscr{A}) \int a v\,\mathrm{d}v$$
$$= \partial_t \left(\mathscr{A} \int a v\,\mathrm{d}v \right) + \left(\int a v\,\mathrm{d}v \right) \left(\int a \omega(v)\,\mathrm{d}v \right) \tag{6}$$

$$II = \partial_x \left(\mathscr{A} \int a v \omega(v)\,\mathrm{d}v \right) - A \int a v \omega(v)\,\mathrm{d}v \tag{7}$$

and

$$III = -\mathscr{A} B \int a\,\mathrm{d}v$$
$$= -\mathscr{A} B A$$
$$= -B \partial_x \left(\frac{1}{2} \mathscr{A}^2 \right)$$
$$= -\partial_x \left(\frac{1}{2} \mathscr{A}^2 B \right) + \frac{1}{2} \mathscr{A}^2 \partial_x B \tag{8}$$

Using (6), (7), and (8) in (5) we obtain

$$0 = \partial_t \left(\mathscr{A} \int a v\,\mathrm{d}v \right) + \partial_x \left(\mathscr{A} \int a v \omega(v)\,\mathrm{d}v - \frac{1}{2} \mathscr{A}^2 B \right)$$
$$+ \left(\int a v\,\mathrm{d}v \right) \left(\int a \omega(v)\,\mathrm{d}v \right) - A \int a v \omega(v)\,\mathrm{d}v + \frac{1}{2} \mathscr{A}^2 \partial_x B \tag{9}$$

Next consider (1) and let

$$F(t,x) := \int f\,\mathrm{d}v, \quad G(t,x) := \int g\,\mathrm{d}v$$

DOI: 10.1002/mma

and

$$\mathscr{F}(t,x) := \int_{-\infty}^{x} F(t,y)\,\mathrm{d}y, \quad \mathscr{G} := \int_{-\infty}^{x} G(t,y)\,\mathrm{d}y$$

Applying (9) twice, once with $a=f$, $\omega(v)=v$, and $B=E$, and once with $a=g$, $\omega(v)=v/m$, and $B=-E$, and adding the results we find

$$\begin{aligned} 0 &= \partial_t\left(\mathscr{F}\int fv\,\mathrm{d}v + \mathscr{G}\int gv\,\mathrm{d}v\right) + \partial_x\left(\mathscr{F}\int fv^2\,\mathrm{d}v + m^{-1}\mathscr{G}\int gv^2\,\mathrm{d}v\right) \\ &\quad -\partial_x\left(\frac{1}{2}\mathscr{F}^2 E - \frac{1}{2}\mathscr{G}^2 E\right) + \left(\int fv\,\mathrm{d}v\right)^2 + m^{-1}\left(\int gv\,\mathrm{d}v\right)^2 \\ &\quad -F\int fv^2\,\mathrm{d}v - m^{-1}G\int gv^2\,\mathrm{d}v + \frac{1}{2}\mathscr{F}^2\rho - \frac{1}{2}\mathscr{G}^2\rho \end{aligned} \tag{10}$$

It follows directly from (1) and (2) that

$$\int \rho(t,x)\,\mathrm{d}x = \int \rho(0,x)\,\mathrm{d}x = 0$$

and hence that $E \to 0$ as $|x| \to \infty$. In addition,

$$E = \mathscr{F} - \mathscr{G}$$

Hence

$$\begin{aligned} \int (\mathscr{F}^2 - \mathscr{G}^2)\rho\,\mathrm{d}x &= \int (\mathscr{F}+\mathscr{G})E\,\partial_x E\,\mathrm{d}x \\ &= -\frac{1}{2}\int \partial_x(\mathscr{F}+\mathscr{G})E^2\,\mathrm{d}x \\ &= -\frac{1}{2}\int (F+G)E^2\,\mathrm{d}x \end{aligned}$$

Integration of (10) in x yields

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathscr{F}\int fv\,\mathrm{d}v\,\mathrm{d}x + \int \mathscr{G}\int gv\,\mathrm{d}v\,\mathrm{d}x\right) \\ &\quad + \int\left[\left(\int fv\,\mathrm{d}v\right)^2 - F\int fv^2\,\mathrm{d}v + m^{-1}\left(\left(\int gv\,\mathrm{d}v\right)^2 - G\int gv^2\,\mathrm{d}v\right)\right]\mathrm{d}x \\ &\quad -\frac{1}{4}\int (F+G)E^2\,\mathrm{d}x \end{aligned} \tag{11}$$

DOI: 10.1002/mma

Note that exchanging w and v we can express

$$\begin{aligned}
-\left(\int f v\,\mathrm{d}v\right)^2 + F\int f v^2\,\mathrm{d}v &= \left(\int f(t,x,w)\,\mathrm{d}w\right)\left(\int f(t,x,v)v^2\,\mathrm{d}v\right)\\
&\quad -\left(\int f(t,x,w)w\,\mathrm{d}w\right)\left(\int f(t,x,v)v\,\mathrm{d}v\right)\\
&= \iint f(t,x,w)f(t,x,v)\left(\frac{1}{2}w^2+\frac{1}{2}v^2-wv\right)\mathrm{d}w\,\mathrm{d}v\\
&= \frac{1}{2}\iint f(t,x,w)f(t,x,v)(w-v)^2\,\mathrm{d}w\,\mathrm{d}v
\end{aligned}$$

and similarly for g. Thus, (11) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathscr{F}\int f v\,\mathrm{d}v\,\mathrm{d}x + \int \mathscr{G}\int g v\,\mathrm{d}v\,\mathrm{d}x\right) &= \frac{1}{2}\iiint f(t,x,w)f(t,x,v)(w-v)^2\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\\
&\quad +\frac{1}{2}m^{-1}\iiint g(t,x,w)g(t,x,v)(w-v)^2\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\\
&\quad +\frac{1}{4}\int (F+G)E^2\,\mathrm{d}x\\
&\geqslant 0
\end{aligned} \tag{12}$$

Consider the energy

$$\iint (f+m^{-1}g)v^2\,\mathrm{d}v\,\mathrm{d}x + \int E^2\,\mathrm{d}x$$

Note that due to (2), $E(t,\cdot)$ is compactly supported and $\int E^2\,\mathrm{d}x$ is finite (this would fail without (2)). It is standard to show that the energy is constant in t. Similarly, $\iint f\,\mathrm{d}v\,\mathrm{d}x = \iint g\,\mathrm{d}v\,\mathrm{d}x$ is constant and $f, g \geqslant 0$. Hence,

$$\begin{aligned}
\left|\int \mathscr{F}\int f v\,\mathrm{d}v\,\mathrm{d}x\right| &\leqslant C\iint f|v|\,\mathrm{d}v\,\mathrm{d}x\\
&\leqslant C\left(\iint f\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\iint f v^2\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\\
&\leqslant C
\end{aligned}$$

and similarly for g. Now it follows from (12) that

$$\int_0^\infty \iiint (f(t,x,w)f(t,x,v)+g(t,x,w)g(t,x,v))(w-v)^2\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C \tag{13}$$

and

$$\int_0^\infty \int (F+G)E^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C \tag{14}$$

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Next consider (3). Applying (9) twice, once with $a=f$, $\omega(v)=\hat{v}_1$, $B=E$ and once with $a=g$, $\omega(v)=\hat{v}_m$, $B=-E$, and adding the results we obtain

$$\begin{aligned}
0 &= \partial_t\left(\mathscr{F}\int f v\,\mathrm{d}v+\mathscr{G}\int g v\,\mathrm{d}v\right)+\partial_x\left(\mathscr{F}\int f v\hat{v}_1\,\mathrm{d}v+\mathscr{G}\int g v\hat{v}_m\,\mathrm{d}v\right)\\
&\quad -\partial_x\left(\frac{1}{2}\mathscr{F}^2E-\frac{1}{2}\mathscr{G}^2E\right)+\left(\int f v\,\mathrm{d}v\right)\left(\int f\hat{v}_1\,\mathrm{d}v\right)+\left(\int g v\,\mathrm{d}v\right)\left(\int g\hat{v}_m\,\mathrm{d}v\right)\\
&\quad -F\int f v\hat{v}_1\,\mathrm{d}v-G\int g v\hat{v}_m\,\mathrm{d}v+\frac{1}{2}\mathscr{F}^2\rho-\frac{1}{2}\mathscr{G}^2\rho
\end{aligned}$$

Proceeding as before we obtain the result

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int\mathscr{F}\int f v\,\mathrm{d}v\,\mathrm{d}x+\int\mathscr{G}\int g v\,\mathrm{d}v\,\mathrm{d}x\right)\\
&\quad +\int\left[\left(\int f v\,\mathrm{d}v\right)\left(\int f\hat{v}_1\,\mathrm{d}v\right)-F\int f v\hat{v}_1\,\mathrm{d}v\right.\\
&\quad \left.+\left(\int g v\,\mathrm{d}v\right)\left(\int g\hat{v}_m\,\mathrm{d}v\right)-G\int g v\hat{v}_m\,\mathrm{d}v\right]\mathrm{d}x\\
&\quad -\frac{1}{4}\int (F+G)E^2\,\mathrm{d}x
\end{aligned}\tag{15}$$

Note that

$$\begin{aligned}
&-\left(\int f v\,\mathrm{d}v\right)\left(\int f\hat{v}_1\,\mathrm{d}v\right)+F\int f v\hat{v}_1\,\mathrm{d}v\\
&=\left(\int f(t,x,w)\,\mathrm{d}w\right)\left(\int f(t,x,v)v\hat{v}_1\,\mathrm{d}v\right)-\left(\int f(t,x,w)w\,\mathrm{d}w\right)\left(\int f(t,x,v)\hat{v}_1\,\mathrm{d}v\right)\\
&=\frac{1}{2}\iint f(t,x,w)f(t,x,v)(v\hat{v}_1+w\hat{w}_1-w\hat{v}_1-v\hat{w}_1)\,\mathrm{d}w\,\mathrm{d}v\\
&=\frac{1}{2}\iint f(t,x,w)f(t,x,v)(w-v)(\hat{w}_1-\hat{v}_1)\,\mathrm{d}w\,\mathrm{d}v
\end{aligned}$$

By the mean value theorem for any w and v, there is ξ between them such that

$$\hat{w}_1-\hat{v}_1=(1+\xi^2)^{-3/2}(w-v)$$

and hence

$$(w-v)(\hat{w}_1-\hat{v}_1)=(1+\xi^2)^{-3/2}(w-v)^2\geqslant 0 \tag{16}$$

Similar results hold for g. For solutions to (3)

$$\iint \left(f\sqrt{1+|v|^2}+g\sqrt{m^2+|v|^2}\right)\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\int E^2\,\mathrm{d}x=\text{const}$$

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and mass is conserved so

$$\left|\int \mathscr{F}\int f v\,\mathrm{d}v\,\mathrm{d}x+\int \mathscr{G}\int g v\,\mathrm{d}v\,\mathrm{d}x\right| \leqslant C\iint f|v|\,\mathrm{d}v\,\mathrm{d}x+C\iint g|v|\,\mathrm{d}v\,\mathrm{d}x \leqslant C$$

Hence, it follows by integrating (15) in t that

$$\int_0^\infty \iiint f(t,x,w)f(t,x,v)(w-v)(\hat{w}_1-\hat{v}_1)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$
$$\int_0^\infty \iiint g(t,x,w)g(t,x,v)(w-v)(\hat{w}_m-\hat{v}_m)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

and

$$\int_0^\infty \int (F+G)E^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C$$

Theorem 2.1 now follows. □

3. DECAY ESTIMATES

In this section we will derive some consequences of the identity from the previous section. We begin by taking $m=1$ and defining $\hat{v}:=\hat{v}_m=\hat{v}_1$. Consider solutions to either system (1) or system (3), and define as above

$$F(t,x)=\int f(t,x,v)\,\mathrm{d}v, \quad G(t,x)=\int g(t,x,v)\,\mathrm{d}v$$

Theorem 3.1
Let f,g satisfy the VP system (1). Assume that the data functions f_0,g_0 satisfy the hypotheses of Theorem 2.1. Then

$$\int_0^\infty \int F^4(t,x)\,\mathrm{d}x\,\mathrm{d}t<\infty$$

and

$$\int_0^\infty \int G^4(t,x)\,\mathrm{d}x\,\mathrm{d}t<\infty$$

When f,g satisfy the RVP system (3) and the data functions f_0,g_0 satisfy the hypotheses of Theorem 2.1 we have

$$\int_0^\infty \left(\int F(t,x)^{7/4}\,\mathrm{d}x\right)^4 \mathrm{d}t<\infty$$

with the same result valid for G.

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Proof
Consider the classical case (1). By Theorem 2.1 we know that

$$k(t,x):=\iint (w-v)^2 f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w$$

is integrable over all x,t. Next we partition the set of integration

$$F(t,x)^2=\iint f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w=\int_{|v-w|<R}+\int_{|v-w|>R}=:I_1+I_2$$

Clearly, we have $I_2\leqslant R^{-2}k(t,x)$. In the integral for I_1 we express

$$\int_{|v-w|<R} f(t,x,w)\,\mathrm{d}w=\int_{v-R}^{v+R} f(t,x,w)\,\mathrm{d}w\leqslant 2\|f_0\|_\infty R$$

Thus

$$I_1\leqslant c\cdot R\cdot F$$

Set $RF=R^{-2}k$ or $R=k^{1/3}F^{-1/3}$. Then $F^4(t,x)\leqslant ck(t,x)$; hence, $F^4(t,x)$ is integrable over all x,t. The result for G is exactly the same.

Now we will find by a similar process the corresponding estimate for solutions to the relativistic version (3). To derive it we will use the estimate from (16), which implies that for $1+|v|+|w|\leqslant S$, there is a constant $c>0$ such that

$$(v-w)(\hat{v}-\hat{w})\geqslant cS^{-3}|v-w|^2$$

From Theorem 2.1 with $m=1$ we know that

$$k_r(t,x):=\iint (v-w)(\hat{v}-\hat{w})f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w$$

is integrable over all x,t. Now express

$$F(t,x)^2=\iint f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w=\int_{(v-w)(\hat{v}-\hat{w})<R}+\int_{(v-w)(\hat{v}-\hat{w})>R}=:I_1+I_2$$

Clearly, $I_2\leqslant R^{-1}k_r(t,x)$. To estimate I_1 we partition it as

$$I_1=\iint_{\substack{(v-w)(\hat{v}-\hat{w})<R\\ 1+|v|+|w|<S}} f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w+\iint_{\substack{(v-w)(\hat{v}-\hat{w})<R\\ 1+|v|+|w|>S}} f(t,x,v)f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w=:I_1'+I_1''$$

On I_1' we have by the above estimate

$$R\geqslant(v-w)(\hat{v}-\hat{w})\geqslant c|v-w|^2S^{-3}$$

Therefore, on I_1' we have $|v-w|<cR^{1/2}S^{3/2}$ so that

$$I_1'\leqslant c\int f(t,x,v)\int_{v-cR^{1/2}S^{3/2}}^{v+cR^{1/2}S^{3/2}} f(t,x,w)\,\mathrm{d}w\,\mathrm{d}v\leqslant c\cdot F(t,x)\cdot R^{1/2}S^{3/2}$$

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I_1'' is more troublesome. By the energy and mass bounds,

$$I_1'' \leqslant S^{-1} \iint (1+|v|+|w|) f(t,x,v) f(t,x,w)\,\mathrm{d}v\,\mathrm{d}w \leqslant cS^{-1} e(t,x) F(t,x)$$

where $e(t,x) = \int \sqrt{1+v^2} f(t,x,v)\,\mathrm{d}v$. Find S first by setting

$$F(t,x)\cdot R^{1/2} S^{3/2} = S^{-1} e(t,x) F(t,x)$$

that is,

$$S = e(t,x)^{2/5} R^{-1/5}$$

Thus, we obtain for I_1 the bound

$$I_1 \leqslant cS^{-1} e(t,x) F(t,x) = cF(t,x) R^{1/5} e(t,x)^{3/5}$$

Above we had $I_2 \leqslant R^{-1} k_r(t,x)$. Therefore, now set

$$F(t,x) R^{1/5} e(t,x)^{3/5} = R^{-1} k_r(t,x)$$

to find R. The result is

$$R = k_r(t,x)^{5/6} F(t,x)^{-5/6} e(t,x)^{-1/2}$$

Finally, then

$$F(t,x)^2 \leqslant cR^{-1} k_r(t,x) = ck_r(t,x)^{1/6} F(t,x)^{5/6} e(t,x)^{1/2}$$

which is the same as $F(t,x)^7/e(t,x)^3 \leqslant ck_r(t,x)$. At this point we may integrate in time to produce the result

$$\int_0^\infty \int \frac{(\int f(t,x,v)\,\mathrm{d}v)^7}{(\int \sqrt{1+v^2} f(t,x,v)\,\mathrm{d}v)^3}\,\mathrm{d}x\,\mathrm{d}t < \infty \tag{17}$$

Alternatively, we can isolate $F(t,x)^7$ on the left-hand side to find $F(t,x)^7 \leqslant ck_r(t,x) e(t,x)^3$. Then raise both sides to the $\frac{1}{4}$th power, integrate in x and use Hölder's inequality. Hence, we obtain the bound

$$\int F(t,x)^{7/4}\,\mathrm{d}x \leqslant \left(\int k_r(t,x)\,\mathrm{d}x\right)^{1/4} \left(\int e(t,x)\,\mathrm{d}x\right)^{3/4}$$

We use the time-independent bound on $\int e(t,x)\,\mathrm{d}x$ from conservation of energy to obtain the estimate

$$\int F(t,x)^{7/4}\,\mathrm{d}x \leqslant C\left(\int k_r(t,x)\,\mathrm{d}x\right)^{1/4}$$

Finally, we raise both sides to the fourth power and integrate in time to produce the result

$$\int_0^\infty \left(\int F(t,x)^{7/4}\,\mathrm{d}x\right)^4 \mathrm{d}t < \infty \tag{18}$$

This is the corresponding estimate for solutions to (3). □

DOI: 10.1002/mma

Now we will use these estimates to show that the local charges tend to 0 as $t\to\infty$ for solutions to both sets of equations.

Theorem 3.2
Let f, g be solutions to either the classical VP system (1) or to the RVP system (3) for which the assumptions of Theorem 3.1 hold. Then for any fixed $R>0$ the local charges satisfy

$$\lim_{t\to\infty}\int_{|x|<R} F(t,x)\,\mathrm{d}x=\lim_{t\to\infty}\int_{|x|<R} G(t,x)\,\mathrm{d}x=0$$

Proof
We begin with solutions to the classical equation (1). From above we know that

$$\int_0^\infty\int F^4(t,x)\,\mathrm{d}x\,\mathrm{d}t<\infty$$

By the Hölder inequality

$$\int_{|x|<R} F(t,x)\,\mathrm{d}x\leqslant\left(\int F^4(t,x)\,\mathrm{d}x\right)^{1/4}(2R)^{3/4}$$

and therefore

$$\int_0^\infty\left[\int_{|x|<R} F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}t<\infty \tag{19}$$

Now by the Vlasov equation for f

$$F_t=-\int(vf_x+Ef_v)\,\mathrm{d}v=-\partial_x\int vf\,\mathrm{d}v$$

Integrate this formula in x over $|x|<R$

$$\partial_t\int_{|x|<R} F(t,x)\,\mathrm{d}x=-\int_{|x|<R}\partial_x\int vf\,\mathrm{d}v\,\mathrm{d}x=-\int vf(t,R,v)\,\mathrm{d}v+\int vf(t,-R,v)\,\mathrm{d}v$$

Let

$$j_f(t,x)=\int vf(t,x,v)\,\mathrm{d}v$$

Then $j_f(t,x)$ is boundedly integrable over all x by the energy. Next we compute

$$\begin{aligned}\partial_t\left[\int_{|x|<R} F(t,x)\,\mathrm{d}x\right]^4&=4\left[\int_{|x|<R} F(t,x)\,\mathrm{d}x\right]^3\int_{|x|<R} F_t(t,x)\,\mathrm{d}x\\&=4\left[\int_{|x|<R} F(t,x)\,\mathrm{d}x\right]^3[-j_f(t,R)+j_f(t,-R)]\end{aligned}$$

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For $0<R_1<R_2$ integrate this in R over $[R_1,R_2]$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}R &= 4\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^3[-j_f(t,R)+j_f(t,-R)]\,\mathrm{d}R\\
&\leqslant 4\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3\int_{R_1}^{R_2}|-j_f(t,R)+j_f(t,-R)|\,\mathrm{d}R\\
&\leqslant c\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3
\end{aligned}
$$

for some constant c depending only on the data. For $t_2>t_1>1$ multiply this by $t-t_1$ and integrate in t over $[t_1,t_2]$

$$
\int_{t_1}^{t_2}(t-t_1)\partial_t\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}R\,\mathrm{d}t\leqslant c\int_{t_1}^{t_2}(t-t_1)\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3\mathrm{d}t
$$

Integrating the left-hand side by parts we obtain

$$
(t_2-t_1)\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t_2,x)\,\mathrm{d}x\right]^4\mathrm{d}R-\int_{t_1}^{t_2}\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}R\,\mathrm{d}t
$$

Now take $t_2=t$, $t_1=t-1$. Then we have

$$
\begin{aligned}
&\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}R\\
&\quad\leqslant\int_{t-1}^{t}\int_{R_1}^{R_2}\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}R\,\mathrm{d}t+c\int_{t-1}^{t}(t-t_1)\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3\mathrm{d}t\\
&\quad\leqslant\int_{t-1}^{t}(R_2-R_1)\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}t+c\int_{t-1}^{t}\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3\mathrm{d}t \qquad (20)
\end{aligned}
$$

Now take $R_2=2R_1=2R$ say. Then the left-hand side of (20) is bounded below by

$$
(R_2-R_1)\left[\int_{|x|<R_1}F(t,x)\,\mathrm{d}x\right]^4=R\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^4
$$

and we claim that the right-hand side tends to 0 as $t\to\infty$. This is clear for the first term on the right-hand side of (20) by using (19). The second term goes to 0, as well, as

$$
\int_{t-1}^{t}\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^3\mathrm{d}t\leqslant\left(\int_{t-1}^{t}\left[\int_{|x|<R_2}F(t,x)\,\mathrm{d}x\right]^4\mathrm{d}t\right)^{3/4}\cdot\left(\int_{t-1}^{t}\mathrm{d}t\right)^{1/4}
$$

The same computation establishes the estimate for G, and the result now follows in the classical case.

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The proof for the relativistic case is similar. From above we know that

$$\int_0^\infty \left(\int F(t,x)^{7/4}\,\mathrm{d}x\right)^4 \mathrm{d}t<\infty$$

By the Hölder inequality

$$\int_{|x|<R} F(t,x)\,\mathrm{d}x\leqslant c_R\left(\int F^{7/4}(t,x)\,\mathrm{d}x\right)^{4/7}$$

and therefore

$$\int_0^\infty \left[\int_{|x|<R} F(t,x)\,\mathrm{d}x\right]^7 \mathrm{d}t<\infty$$

Using the Vlasov equation (3) for f we have

$$F_t=-\int(\hat{v}f_x+Ef_v)\,\mathrm{d}v=-\partial_x\int\hat{v}f\,\mathrm{d}v$$

Integrate this formula in x over $|x|<R$

$$\partial_t\int_{|x|<R}F(t,x)\,\mathrm{d}x=-\int_{|x|<R}\partial_x\int\hat{v}f\,\mathrm{d}v\,\mathrm{d}x=-\int\hat{v}f(t,R,v)\,\mathrm{d}v+\int\hat{v}f(t,-R,v)\,\mathrm{d}v$$

Let

$$j_f^r(t,x)=\int\hat{v}f(t,x,v)\,\mathrm{d}v$$

Then $j_f^r(t,x)$ is boundedly integrable over all x by the mass bound. Next we compute

$$\begin{aligned}\partial_t\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^7&=7\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^6\int_{|x|<R}F_t(t,x)\,\mathrm{d}x\\&=7\left[\int_{|x|<R}F(t,x)\,\mathrm{d}x\right]^6[-j_f^r(t,R)+j_f^r(t,-R)]\end{aligned}$$

The proof now concludes exactly as in the classical case. □

4. TIME DECAY OF ELECTRIC FIELD

We conclude this paper with results concerning the time integrability and decay of the electric field for both the classical and relativistic systems, (1) and (3).

Theorem 4.1
Let the assumptions of Theorem 3.1 hold and consider solutions f, g to either (1) or (3). Then

$$\int_0^\infty \|E(t)\|_\infty^3\,\mathrm{d}t<\infty$$

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Proof
This will follow immediately from the result in Theorem 2.1 that

$$Q(t):=\int_{-\infty}^{\infty} E^2(t,x)\,[F(t,x)+G(t,x)]\,\mathrm{d}x$$

is integrable in time. Indeed by the equation $E_x=\rho=\int(f-g)\,\mathrm{d}v=F-G$, we have

$$\frac{\partial}{\partial x}E^3=3E^2\rho=3E^2(F-G)$$

Integrate in x to obtain

$$E^3(t,x)=\int_{-\infty}^{x} 3E^2(F-G)\,\mathrm{d}x$$

so that

$$|E(t,x)|^3\leqslant\int_{-\infty}^{\infty} 3E^2(F+G)\,\mathrm{d}x=3Q(t) \tag{21}$$

and the result follows as claimed. □

Our final results will show that for solutions to the classical VP system (1) and RVP system (3), the electric field E tends to 0 in the maximum norm.

Theorem 4.2
Let the assumptions of Theorem 3.1 hold and consider solutions f, g to the classical VP system (1). Then

$$\lim_{t\to\infty}\|E(t)\|_\infty=0$$

Proof
We will show that

$$\lim_{t\to\infty}Q(t)=0$$

The conclusion will then follow from (21). As $Q(t)$ is integrable over $[0,\infty)$, $\liminf Q(t)=0$. Therefore, there is a sequence t_n tending to infinity such that $Q(t_n)\to 0$ as $n\to\infty$. As above, we denote

$$F(t,x)=\int f(t,x,v)\,\mathrm{d}v,\quad G(t,x)=\int g(t,x,v)\,\mathrm{d}v$$

Using $E_x=\rho=F-G$ and $E_t=-j=-\int v(f-g)\,\mathrm{d}v$ we first compute

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t}&=-2\int jE(F+G)\,\mathrm{d}x+\int E^2\partial_t(F+G)\,\mathrm{d}x\\&=-2\int jE(F+G)\,\mathrm{d}x-\int E^2\partial_x\int v(f+g)\,\mathrm{d}v\,\mathrm{d}x\\&=-2\int jE(F+G)\,\mathrm{d}x+2\int \rho E\int v(f+g)\,\mathrm{d}v\,\mathrm{d}x\end{aligned}$$

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Now, E is uniformly bounded because by definition in (1),

$$|E(t,x)| \leqslant \int_{-\infty}^{x} (F+G)(t,x)\,\mathrm{d}x \leqslant \int_{-\infty}^{\infty} (F+G)(t,x)\,\mathrm{d}x \leqslant \text{const}$$

where the last inequality follows by conservation of mass. Therefore

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right| \leqslant c \int (F+G) \int |v|(f+g)\,\mathrm{d}v\,\mathrm{d}x$$

Define e to be the kinetic energy density

$$e(t,x) := \int v^2 (f+g)\,\mathrm{d}v$$

Then in the usual manner we obtain

$$\begin{aligned}\int |v|(f+g)\,\mathrm{d}v &= \int_{|v|<R} |v|(f+g)\,\mathrm{d}v + \int_{|v|>R} |v|(f+g)\,\mathrm{d}v\\ &\leqslant \|f+g\|_\infty \cdot R^2 + R^{-1}e\\ &\leqslant c(R^2 + R^{-1}e)\end{aligned}$$

Choosing $R^3 = e$ we find that

$$\int |v|(f+g)\,\mathrm{d}v \leqslant c e^{2/3}(t,x)$$

and therefore

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right| \leqslant c \int (F+G) e^{2/3}\,\mathrm{d}x \leqslant c \left(\int (F+G)^3\,\mathrm{d}x\right)^{1/3} \tag{22}$$

by the Hölder inequality and the bound on kinetic energy from Section 2. By interpolation, for suitable functions w,

$$\|w\|_3 \leqslant \|w\|_1^\theta \cdot \|w\|_4^{1-\theta}$$

where

$$\frac{1}{3} = \frac{\theta}{1} + \frac{1-\theta}{4}$$

Therefore, $\theta = \frac{1}{9}$. Apply this to $w = F+G$ and use the boundedness of $F+G$ in L^1 to obtain

$$\|F+G\|_3 \leqslant c\|F+G\|_4^{8/9}$$

Using this above we conclude that

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right| \leqslant c\|F+G\|_4^{8/9}$$

DOI: 10.1002/mma

From Theorem 3.1 we know that $\int (F^4+G^4)\,\mathrm{d}x$ is integrable in time. Thus, $|\mathrm{d}Q/\mathrm{d}t|^{9/2}$ is integrable in time. Now, for any $0<R_1<R_2$ express

$$Q(R_2)^{16/9}-Q(R_1)^{16/9}=\frac{16}{9}\int_{R_1}^{R_2} Q(t)^{7/9}\dot{Q}(t)\,\mathrm{d}t$$

By the Hölder inequality again, with $p=\frac{9}{7}$ and $q=\frac{9}{2}$,

$$|Q(R_2)^{16/9}-Q(R_1)^{16/9}|\leqslant c\left(\int_{R_1}^{R_2} Q(t)\,\mathrm{d}t\right)^{7/9}\cdot\left(\int_{R_1}^{R_2} |\dot{Q}(t)|^{9/2}\,\mathrm{d}t\right)^{2/9}\to 0$$

as $R_1, R_2\to\infty$. Therefore, the limit

$$\lim_{R\to\infty} Q(R)^{16/9}$$

exists and equals ω, say. By taking $R=t_n$ and letting $n\to\infty$ we obtain that $\omega=0$. This concludes the proof. □

Theorem 4.3

Let the assumptions of Theorem 2.1 hold and consider solutions f, g to the RVP system (3). Then, also in this case

$$\lim_{t\to\infty}\|E(t)\|_\infty=0$$

Proof

As is to be expected, the proof is similar to that of Theorem 4.2. From Theorem 2.1 we have again that $Q(t)$ is integrable in time, where exactly as in the non-relativistic case

$$Q(t)=\int_{-\infty}^{\infty} E^2(t,x)[F(t,x)+G(t,x)]\,\mathrm{d}x$$

In this situation, we have $\rho=\int (f-g)\,\mathrm{d}v$ and (with $m=1$) $j=\int \hat{v}(f-g)\,\mathrm{d}v$ where $\hat{v}=v/\sqrt{1+v^2}$ so that $|\hat{v}|<1$. The computation of the derivative in time is now

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t}&=-2\int jE(F+G)\,\mathrm{d}x+\int E^2\partial_t(F+G)\,\mathrm{d}x\\&=-2\int jE(F+G)\,\mathrm{d}x-\int E^2\partial_x\int \hat{v}(f+g)\,\mathrm{d}v\,\mathrm{d}x\\&=-2\int jE(F+G)\,\mathrm{d}x+2\int \rho E\int \hat{v}(f+g)\,\mathrm{d}v\,\mathrm{d}x\end{aligned}$$

It follows that

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right|\leqslant c\int |E|(F+G)^2\,\mathrm{d}x\leqslant c\int (F+G)^2\,\mathrm{d}x$$

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because E is uniformly bounded. Let e be the relativistic kinetic energy density,

$$e(t,x)=\int\sqrt{1+v^2}(f+g)\,\mathrm{d}v$$

Then as above

$$\begin{aligned}F+G&=\int(f+g)\,\mathrm{d}v\\&=\int_{|v|<R}(f+g)\,\mathrm{d}v+\int_{|v|>R}(f+g)\,\mathrm{d}v\\&\leqslant\|f+g\|_\infty\cdot 2R+R^{-1}e\\&\leqslant c(R+R^{-1}e)\end{aligned}$$

Hence, with $R^2=e$ we find that $F+G\leqslant ce^{1/2}$. Thus, we see that

$$\left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right|\leqslant c\int e\,\mathrm{d}x\leqslant c$$

In view of Remark 1 then, $Q(t)\to 0$ as $t\to\infty$, which implies the result for E as in the classical case. □

Remarks

1. Once $Q(t)$ is integrable in time, the uniform boundedness of $|\mathrm{d}Q/\mathrm{d}t|$ also implies that $Q(t)\to 0$ as $t\to\infty$. Estimate (22) provides the desired bound in the classical case because $(F+G)^3$ is dominated by the energy integral in this situation.
2. For solutions to (1) or (3), using interpolation with Theorem 4.2 or Theorem 4.3, and the bound on $\|E(t)\|_2$ from energy conservation we find that

$$\lim_{t\to\infty}\|E(t)\|_p=0$$

   for any $p>2$.
3. We have been unable to find a rate of decay for the maximum norm of E. For solutions to the classical VP system in three-space dimensions such a rate follows from differentiating in time an expression essentially of the form

$$\iint|x-tv|^2(f+g)\,\mathrm{d}v\,\mathrm{d}x$$

   (cf. [4, 5]). This estimate fails to imply time decay in the current one-dimensional case.
4. An identity similar to that in the proof of Theorem 2.1 holds for solutions to the 'one- and one-half-dimensional' Vlasov–Maxwell system. However, we have been unable to show that certain terms arising from the linear parts of the differential operators have the proper sign.
5. As stated in the Introduction, such decay theorems should be true for several species under the hypothesis of neutrality. However, we have been unable to achieve this generalization for more than two species.
6. After suitable approximation, these results can be seen to be valid for weak solutions as well.

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