MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2008; **31**:375–389
Published online 14 June 2007 in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.915
MOS subject classification: 35 L 60; 35 Q 99; 82 C 21; 82 C 22; 82 D 10

Explicit solutions of the one-dimensional Vlasov–Poisson system with infinite mass

Stephen Pankavich*,†

Department of Mathematics, Indiana University, Bloomington, IN 47401, U.S.A.

Communicated by R. Glassey

SUMMARY

A collisionless plasma is modelled by the Vlasov–Poisson system in one dimension. A fixed background of positive charge, dependent only upon velocity, is assumed and the situation in which the mobile negative ions balance the positive charge as $|x|\to\infty$ is considered. Thus, the total positive charge and the total negative charge are infinite. In this paper, the charge density of the system is shown to be compactly supported. More importantly, both the electric field and the number density are determined explicitly for large values of $|x|$. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: partial differential equations; plasma physics; Vlasov–Poisson; kinetic theory; infinite mass

INTRODUCTION

Consider the one dimensional Vlasov–Poisson system with a given positive background function and initial data. We take as given the functions $F:\mathbb{R}\to[0,\infty)$ and $f_0:\mathbb{R}^2\to[0,\infty)$ and seek functions $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $E:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that

$$\left.\begin{aligned} \partial_t f + v\partial_x f - E\partial_v f = 0\\ \rho(t,x) = \int (F(v) - f(t,x,v))\,\mathrm{d}v\\ E(t,x) = \frac{1}{2}\left(\int_{-\infty}^{x}\rho(t,y)\,\mathrm{d}y - \int_{x}^{\infty}\rho(t,y)\,\mathrm{d}y\right)\\ f(0,x,v) = f_0(x,v) \end{aligned}\right\} \tag{1}$$

*Correspondence to: Stephen Pankavich, Department of Mathematics, Indiana University, Bloomington, IN 47401, U.S.A.
†E-mail: sdp@indiana.edu

Received 29 August 2006

Here $t \in [0, T]$ denotes time, $x \in \mathbb{R}$ denotes one-dimensional space, and $v \in \mathbb{R}$ denotes one-dimensional momentum. In (1), F represents a number density in phase space of positive ions which form a fixed background, and f describes the number density of mobile negative ions. Notice that if $f_0 = F$, then $f = F$ is a steady solution. Thus, we consider solutions for which $f \to F$ as $|x| \to \infty$. The existence of a unique, local-in-time solution to (1) was shown in [1]. Thus, we will include assumptions which satisfy the requirements of [1] so that the existence of a solution f on $[0, T]$ is valid for some $T > 0$, and the results which follow may be applied to the local-in-time solution.

The Vlasov–Poisson system has been studied extensively in the case where $F(v) = 0$ and solutions tend to zero as $|x| \to \infty$, both for the one-dimensional problem and the more difficult, three-dimensional problem. Most of the literature involving the one-dimensional Vlasov–Poisson system focus on time asymptotics, such as [2, 3]. Much more work has been done concerning the three-dimensional problem. Smooth solutions were shown to exist globally in-time in [4] and independently in [5]. The results of [4] were later revised in [6]. Important results preliminary to the discovery of a global-in-time solution include [7, 8]. A complete discussion of the literature concerning the Vlasov–Poisson system may be found in both [9, 10].

Only over the past decade has some work begun studying solutions of the Vlasov–Poisson system with infinite mass and energy. Under differing assumptions, distributional solutions with infinite mass or infinite kinetic energy have been constructed in [11, 12]. More recently, the three-dimensional analogue of (1), which yields solutions with both infinite mass and energy, has been studied. Local existence of smooth solutions and a continuation criteria for this problem were shown in [13]. *A priori* bounds on the current density were achieved in [14]. Finally, global existence in the case of a radial electric field was shown in [15], and global existence without the assumption of radial symmetry in [16].

SECTION 1

We will make the following assumptions throughout, for any $x, v \in \mathbb{R}$:

(I) There is $R > 0$ such that for $|x| > R$,

$$f_0(x, v) = F(v) \tag{2}$$

where $F \in \mathscr{C}_c^1(\mathbb{R})$ is even and non-negative, and $f_0 \in \mathscr{C}_c^1(\mathbb{R}^2)$ is non-negative.

(II) There is $C^{(1)} > 0$ such that

$$\int_0^T \|E(\tau)\|_\infty \, d\tau \leqslant C^{(1)} \tag{3}$$

Notice that Assumption (I) satisfies the assumptions made in [1] and thus the existence of a unique local-in-time solution is guaranteed. Define the characteristics $X(s, t, x, v)$ and $V(s, t, x, v)$ by

$$\left.\begin{aligned} \frac{\partial}{\partial s} X(s, t, x, v) &= V(s, t, x, v) \\ \frac{\partial}{\partial s} V(s, t, x, v) &= -E(s, X(s, t, x, v)) \\ X(t, t, x, v) &= x \\ V(t, t, x, v) &= v \end{aligned}\right\} \tag{4}$$

DOI: 10.1002/mma

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} f(s, X(s,t,x,v), V(s,t,x,v)) &= \partial_t f(s, X(s,t,x,v), V(s,t,x,v)) \\ &\quad + V(s,t,x,v)\partial_x f(s, X(s,t,x,v), V(s,t,x,v)) \\ &\quad - E(s, X(s,t,x,v))\partial_v f(s, X(s,t,x,v), V(s,t,x,v)) \\ &= 0\end{aligned}$$

so that f is constant along characteristics, and

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)) = f_0(X(0,t,x,v), V(0,t,x,v)) \tag{5}$$

Thus, we find that f must be non-negative and $\|f\|_\infty = \|f_0\|_\infty < \infty$.

Now, put

$$g(t,x,v) = F(v) - f(t,x,v)$$

and define for every $t \in [0,T]$,

$$Q_g(t) := \sup\{|v| : \exists x \in \mathbb{R}, \tau \in [0,t] \text{such that } g(\tau,x,v) \neq 0\} \tag{6}$$

and

$$R(t) := R + tQ_g(t) + \int_0^t \int_\tau^t \|E(s)\|_\infty \,\mathrm{d}s\,\mathrm{d}\tau \tag{7}$$

Then, $V(s,t,x,v)$ is linear in v for large $|x|$, and $\rho(t,x)$ has compact support for every $t \in [0,T]$.

Theorem 1

Let $T>0$ and f, a C^1 solution of (1) on $[0,T] \times \mathbb{R}^2$, be given and assume (2) and (3) hold. Then, for $t \in [0,T]$ and $|x| > R(t)$ we have

1. $\rho(t,x) = 0$.
2. For $s \in [0,t]$ and $|v| \leqslant Q_g(t)$,

$$\frac{\partial V}{\partial v}(s,t,x,v) = 1$$

More importantly, we may explicitly determine the unique solution to (1) for large $|x|$. Define

$$\operatorname{sign}(x) := \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0 \end{cases}$$

Theorem 2

Let $T>0$ and f, a C^1 solution of (1) on $[0,T] \times \mathbb{R}^2$, be given and assume (2) and (3) hold. Then, for $t \in [0,T]$ and $|x| > R(t)$ we have

$$E(t,x) = E^0 \operatorname{sign}(x)\cos(\omega t)$$

DOI: 10.1002/mma

and

$$f(t,x,v)=F\left(v+\frac{E^0\,\mathrm{sign}(x)}{\omega}\sin(\omega t)\right)$$

where

$$E^0=\frac{1}{2}\int_{-R}^{R}\int (F(v)-f_0(y,v))\,\mathrm{d}v\,\mathrm{d}y$$

and

$$\omega=\left(\int F(v)\,\mathrm{d}v\right)^{1/2}$$

We derive the form of $f(t,x,v)$ and $E(t,x)$ from the field bound and the initial current density. Then, using Theorem 1, we show that the current density must be a multiple of the field for large $|x|$. In order to arrive at these results, we must first show that Q_g is bounded.

Lemma 1
For any $t\in[0,T]$,

$$Q_g(t)\leqslant C$$

To control spatial characteristics, we will use the following lemma.

Lemma 2
For $t\in[0,T]$, $s\in[0,t]$, $x\in\mathbb{R}$, and $|v|\leqslant Q_g(t)$, we have

$$|x|\geqslant R(t)\Rightarrow |X(s,t,x,v)|\geqslant R(s)$$

We will delay the proofs of the lemmas until Section 4. Section 2 will be dedicated to proving Theorem 1 using the above lemmas. Then, in Section 3, we will prove Theorem 2, utilizing the first theorem.

SECTION 2

In order to prove Theorem 1, we must first bound the charge density and v-derivatives of characteristics. Notice from (2) and Lemma 2,

$$|x|>R(t)\Rightarrow f_0(X(s,t,x,v),V(s,t,x,v))=F(V(s,t,x,v))$$

for every $s\in[0,t]$, $t\in[0,T]$, and $|v|\leqslant Q_g(t)$. In particular,

$$|x|>R(t)\Rightarrow f(t,x,v)=F(V(0,t,x,v)) \tag{8}$$

DOI: 10.1002/mma

Let $|x|>R(t)$ and using (3), (8), and Lemma 1, we write

$$\begin{aligned}
|\rho(t,x)| &= \left|\int (F(v) - f(t,x,v))\,\mathrm{d}v\right| \\
&= \left|\int_{|v|\leqslant Q_g(t)} (F(v) - F(V(0,t,x,v))\,\mathrm{d}v\right| \\
&\leqslant \int_{|v|\leqslant Q_g(t)} \|F'\|_\infty |V(0,t,x,v) - v|\,\mathrm{d}v \\
&\leqslant Q_g(t)\ \|F'\|_\infty \left(\int_0^t \|E(\tau)\|_\infty\,\mathrm{d}\tau\right) \\
&\leqslant C
\end{aligned}$$

So, for $|x|>R(t)$,

$$\rho(t,x)\leqslant C \tag{9}$$

To bound derivatives of characteristics, we use (1) and (4) to find

$$\frac{\partial \dot{X}}{\partial v}(s,t,x,v) = \frac{\partial V}{\partial v}(s,t,x,v)$$

and

$$\begin{aligned}
\frac{\partial \dot{V}}{\partial v}(s,t,x,v) &= -E_x(s, X(s,t,x,v))\frac{\partial X}{\partial v}(s,t,x,v) \\
&= -\rho(s, X(s,t,x,v))\frac{\partial X}{\partial v}(s,t,x,v)
\end{aligned}$$

so that

$$\frac{\partial V}{\partial v}(s,t,x,v) = 1 + \int_s^t \rho(\tau, X(\tau,t,x,v))\frac{\partial X}{\partial v}(s,t,x,v)\,\mathrm{d}\tau \tag{10}$$

Thus,

$$\left|\frac{\partial X}{\partial v}(s,t,x,v)\right| \leqslant \int_s^t \left|\frac{\partial V}{\partial v}(\tau,t,x,v)\right|\mathrm{d}\tau$$

and

$$\left|\frac{\partial V}{\partial v}(s,t,x,v)\right| \leqslant 1 + \int_s^t \left|\rho(\tau, X(\tau,t,x,v))\frac{\partial X}{\partial v}(s,t,x,v)\right|\mathrm{d}\tau$$

DOI: 10.1002/mma

Combining the two inequalities, we use Lemma 2 and (9) so that

$$\begin{aligned}\left|\frac{\partial X}{\partial v}(s,t,x,v)\right| + \left|\frac{\partial V}{\partial v}(s,t,x,v)\right| &\leqslant 1 + \int_s^t \left(\left|\frac{\partial V}{\partial v}(\tau,t,x,v)\right| + C\left|\frac{\partial X}{\partial v}(\tau,t,x,v)\right|\right) \mathrm{d}\tau \\ &\leqslant 1 + C\int_s^t \left(\left|\frac{\partial V}{\partial v}(\tau,t,x,v)\right| + \left|\frac{\partial X}{\partial v}(\tau,t,x,v)\right|\right) \mathrm{d}\tau\end{aligned}$$

for $|x|>R(t)$ and $|v|\leqslant Q_g(t)$. Then, by Gronwall's Inequality, for $t\in[0,T]$, $s\in[0,t]$, $|x|>R(t)$, and $|v|\leqslant Q_g(t)$

$$\left|\frac{\partial X}{\partial v}(s,t,x,v)\right| + \left|\frac{\partial V}{\partial v}(s,t,x,v)\right| \leqslant C \tag{11}$$

and we have bounds on v-derivatives of characteristics.

Now that $\rho(t,x)$ and $(\partial X/\partial v)(s,t,x,v)$ are bounded, define

$$\Lambda(t) := \sup_{\substack{|x|>R(s)\\ s\in[0,t]}} |\rho(s,x)|$$

and

$$\Upsilon(t) := \sup_{\substack{|x|>R(t)\\ |v|\leqslant Q_g(t)\\ s\in[0,t]}} \left|\frac{\partial X}{\partial v}(s,t,x,v)\right|$$

Then, for $|v|\leqslant Q_g(t)$ and $|x|>R(t)$,

$$\begin{aligned}\left|\int_0^t \rho(\tau, X(\tau,t,x,v))\frac{\partial X}{\partial v}(\tau,t,x,v)\,\mathrm{d}\tau\right| &\leqslant \int_0^t \Lambda(\tau)\Upsilon(t)\,\mathrm{d}\tau \\ &= \Upsilon(t)\left(\int_0^t \Lambda(\tau)\,\mathrm{d}\tau\right) \\ &=: G(t)\end{aligned}$$

Notice $G\in C[0,T]$, G increasing, $G(t)\geqslant 0$ for every $t\in[0,T]$, and $G(0)=0$. Define

$$T_0 := \sup\{\tilde{T} : G(\tilde{T})\leqslant \tfrac{1}{2}\}$$

Thus, $G(T_0)\leqslant\frac{1}{2}$ and by the above computation,

$$\left|\int_0^t \rho(\tau, X(\tau,t,x,v))\frac{\partial X}{\partial v}(\tau,t,x,v)\,\mathrm{d}\tau\right| \leqslant \frac{1}{2}$$

DOI: 10.1002/mma

for every $t \leqslant T_0$. By (10), we find for $t \in [0, T_0]$

$$\begin{aligned}
\left|\frac{\partial V}{\partial v}(0)\right| &\geqslant 1 - \left|\int_0^t \rho(\tau, X(\tau, t, x, v)) \frac{\partial X}{\partial v}(\tau, t, x, v)\,\mathrm{d}\tau\right| \\
&\geqslant 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

so that

$$\left|\frac{1}{\dfrac{\partial V}{\partial v}(0)}\right| \leqslant 2 \tag{12}$$

for $|v| \leqslant Q_g(t)$ and $|x| > R(t)$. Once we show $\rho(t, x) = 0$ for $|x| > R(t)$ with $t \in [0, T_0]$, then $\Lambda(t) = G(t) = 0$ for every $t \in [0, T_0]$, and it follows that $T_0 = T$, thus bounding $|(\partial V/\partial v)(0)|$ from below for all $t \in [0, T]$.

Let $|x| > R(t)$. Using (8), (10), and (12), we find

$$\begin{aligned}
\rho(t, x) &= \int (F(v) - F(V(0, t, x, v)))\,\mathrm{d}v \\
&= \int F(v)\,\mathrm{d}v - \int F(w) \frac{1}{\dfrac{\partial V}{\partial v}(0)}\,\mathrm{d}w \\
&= \int F(w) \left(1 - \frac{1}{\dfrac{\partial V}{\partial v}(0)}\right) \mathrm{d}w \\
&= \int F(w) \left(\frac{\dfrac{\partial V}{\partial v}(0) - 1}{\dfrac{\partial V}{\partial v}(0)}\right) \mathrm{d}w \\
&= \int F(w) \left(\int_0^t \rho(\tau, X(\tau, t, x, w)) \frac{\partial X}{\partial v}(\tau)\,\mathrm{d}\tau\right) \frac{1}{\dfrac{\partial V}{\partial v}(0)}\,\mathrm{d}w
\end{aligned}$$

DOI: 10.1002/mma

Thus, for $|x|>R(t)$, we use Assumption (I), (11), and (12) to find

$$|\rho(t,x)| \leqslant \int_{|w|\leqslant Q_g(t)} |F(w)| \left(\int_0^t |\rho(t,X(\tau))| \left|\frac{\partial X}{\partial v}(\tau)\right| \mathrm{d}\tau\right) \left|\frac{1}{\frac{\partial V}{\partial v}(0)}\right| \mathrm{d}w$$

$$\leqslant C \int_{|w|\leqslant Q_g(t)} F(w) \int_0^t |\rho(\tau, X(\tau))| \,\mathrm{d}\tau \,\mathrm{d}w$$

Now, define

$$\mathscr{P}(s) := \sup_{|x|>R(s)} |\rho(s,x)|$$

The above inequality becomes

$$\mathscr{P}(t) \leqslant C \left(\int F(w)\,\mathrm{d}w\right) \int_0^t \mathscr{P}(\tau)\,\mathrm{d}\tau$$

By Gronwall's inequality,

$$\mathscr{P}(t) \leqslant 0$$

Thus,

$$\mathscr{P}(t) = 0$$

and

$$\rho(t,x) = 0 \tag{13}$$

for $|x|>R(t)$. As previously stated, since (13) holds for $t\in[0,T_0]$, we can conclude that it does so for all $t\in[0,T]$. Then, since $\rho(t,x)$ has compact support, we use Lemma 2 and (13) to find

$$|x|>R(t) \Rightarrow \rho(\tau, X(\tau,t,x,v)) = 0$$

for any $t\in[0,T]$, $\tau\in[0,t]$, $|v|\leqslant Q_g(t)$. Using this with (10), we may conclude

$$\frac{\partial V}{\partial v}(s,t,x,v) = 1$$

for $|x|>R(t)$, $|v|\leqslant Q_g(t)$ and $s\in[0,t]$. Thus, the proof of Theorem 1 is complete.

DOI: 10.1002/mma

SECTION 3

In order to prove Theorem 2, let us first define the current density, $j:[0,T]\times\mathbb{R}\to\mathbb{R}$ by

$$j(t,x) := \int v(F(v) - f(t,x,v))\,\mathrm{d}v$$

A well-known result, known as the equation of continuity, follows from (1):

$$\partial_t\rho(t,x) + \partial_x j(t,x) = 0 \tag{14}$$

Using the second result of Theorem 1, we find for $|x|>R(t)$, $|v|\leqslant Q_g(t)$, and $s\in[0,t]$

$$V(s,t,x,v) = v + \gamma(s,t,x)$$

for some γ. In addition, (4) implies

$$V(s,t,x,v) = v + \int_s^t E(\tau, X(\tau,t,x,v))\,\mathrm{d}\tau$$

Thus, for $|x|>R(t)$,

$$E(\tau, X(\tau,t,x,v)) = E(\tau, X(\tau,t,x,0))$$

Define for $|x|>R(t)$,

$$\gamma(s,t,x) := \int_s^t E(\tau, X(\tau,t,x,0))\,\mathrm{d}\tau$$

so that

$$V(0,t,x,v) = v + \gamma(0,t,x) \tag{15}$$

Now, let $|x|>R(t)$. By Assumption (I), (5), and Lemma 2, we find

$$\begin{aligned}
j(t,x) &= \int v(F(v) - f(t,x,v))\,\mathrm{d}v\\
&= \int vF(v)\,\mathrm{d}v - \int vf_0(X(0),V(0))\,\mathrm{d}v\\
&= -\int vF(V(0))\,\mathrm{d}v\\
&= -\int vF(v+\gamma(0,t,x))\,\mathrm{d}v\\
&= -\int F(w)[w-\gamma(0,t,x)]\,\mathrm{d}w\\
&= \int F(w)\gamma(0,t,x)\,\mathrm{d}w\\
&= \left(\int F(w)\,\mathrm{d}w\right)\int_0^t E(\tau, X(\tau,t,x,0))\,\mathrm{d}\tau
\end{aligned}$$

DOI: 10.1002/mma

Thus, from this relation, we find

$$\frac{\partial}{\partial t}(j(t,x)) = \left(\int F(w)\,\mathrm{d}w\right) E(t,x) \tag{16}$$

Since $\rho(t)$ has compact support, let us write $\mathrm{supp}(\rho(t)) \subset [-L, L]$ for some $L>0$, and so

$$E(t,x) = \frac{1}{2}\left(\int_{-L}^{x} \rho(t,y)\,\mathrm{d}y - \int_{x}^{L} \rho(t,y)\,\mathrm{d}y\right)$$

Also, notice

$$E(t,L) = -E(t,-L) \tag{17}$$

Therefore, using (14)

$$\begin{aligned}
\frac{\partial}{\partial t}(E(t,x)) &= \frac{1}{2}\left(\int_{-L}^{x} \rho_t(t,y)\,\mathrm{d}y - \int_{x}^{L} \rho_t(t,y)\,\mathrm{d}y\right)\\
&= \frac{1}{2}\left(-\int_{-L}^{x} j_x(t,y)\,\mathrm{d}y + \int_{x}^{L} j_x(t,y)\,\mathrm{d}y\right)\\
&= \frac{1}{2}[j(t,-L) - j(t,x) + j(t,L) - j(t,x)]\\
&= \frac{1}{2}(j(t,-L) + j(t,L)) - j(t,x)
\end{aligned}$$

Thus, using (16) and (17), we find for $|x|>R(t)$

$$\frac{\partial^2}{\partial t^2}(E(t,x)) = \frac{1}{2}[j_t(t,-L) + j_t(t,L)] - j_t(t,x) \tag{18}$$

$$= \frac{1}{2}\left(\int F(w)\,\mathrm{d}w\right)[E(t,-L) + E(t,L)] - \left(\int F(w)\,\mathrm{d}w\right) E(t,x) \tag{19}$$

$$= -\left(\int F(w)\,\mathrm{d}w\right) E(t,x) \tag{20}$$

Now, for $|x|>R(t)$

$$\begin{aligned}
E(t,x) &= \frac{1}{2}\,\mathrm{sign}(x) \int_{-L}^{L} \rho(t,y)\,\mathrm{d}y\\
&= \mathrm{sign}(x) E(t,L)
\end{aligned}$$

For all $t \in [0, T]$, define

$$e(t) := E(t,L)$$

DOI: 10.1002/mma

and

$$\omega := \left(\int F(w)\,\mathrm{d}w\right)^{1/2}$$

Then, (20) yields

$$e''(t) = -\omega^2 e(t)$$

and so

$$e(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t) \tag{21}$$

for some $c_1, c_2 \in \mathbb{R}$. Thus, for $|x| > R(t)$

$$E(t, x) = \mathrm{sign}(x)(c_1 \sin(\omega t) + c_2 \cos(\omega t)) \tag{22}$$

Now, we use (1) and Assumption (I) to find for $|x| > R(t)$

$$\begin{aligned} E(0, x) &= \frac{1}{2}\,\mathrm{sign}(x) \int_{-R}^{R} \rho(0, y)\,\mathrm{d}y \\ &= \frac{1}{2}\,\mathrm{sign}(x) \int_{-R}^{R} \int (F(v) - f_0(y, v))\,\mathrm{d}v\,\mathrm{d}y \\ &=: E^0\,\mathrm{sign}(x) \end{aligned}$$

However, by (22)

$$E(0, x) = (c_1 \sin(0) + c_2 \cos(0))\,\mathrm{sign}(x)$$

So, $c_2 = E^0$. Also, for $|x| > R(t)$,

$$E_t(0, x) = \tfrac{1}{2}[j(0, -L) + j(0, L)] - j(0, x) = 0$$

But, $E_t(0, x) = \omega c_1$, so that for non-trivial E, $c_1 = 0$. Finally, we may write

$$E(t, x) = E^0\,\mathrm{sign}(x) \cos(\omega t) \tag{23}$$

for $|x| > R(t)$ where

$$E^0 = \frac{1}{2} \int_{-R}^{R} \int (F(v) - f_0(y, v))\,\mathrm{d}v\,\mathrm{d}y$$

and

$$\omega = \left(\int F(w)\,\mathrm{d}w\right)^{1/2}$$

So, the proof of the first part of Theorem 2 is complete.

DOI: 10.1002/mma

Now, we may use this result to show the second part of the theorem. First, by (4), we know for $s\in[0,t]$,

$$V(s)=v+\int_s^t E(\tau, X(\tau))\,\mathrm{d}\tau$$

Using (23) and Lemma 2 in this equation, for $|x|>R(t)$,

$$\begin{aligned}V(s)&=v+\int_s^t E^0\,\mathrm{sign}(x)\cos(\omega\tau)\,\mathrm{d}\tau\\&=v+E^0\,\mathrm{sign}(x)\left(\frac{1}{\omega}\right)[\sin(\omega t)-\sin(\omega s)]\end{aligned}$$

In addition, for $|x|>R(t)$

$$\begin{aligned}X(s)&=x-\int_s^t V(\tau)\,\mathrm{d}\tau\\&=x-\mathrm{sign}(x)\int_s^t\left[v+\frac{E^0}{\omega}(\sin(\omega t)-\sin(\omega\tau))\right]\mathrm{d}\tau\\&=x-\mathrm{sign}(x)\left((t-s)\left[v+\frac{E^0}{\omega}\sin(\omega t)\right]+\frac{E^0}{\omega}\int_s^t\sin(\omega\tau)\,\mathrm{d}\tau\right)\\&=x-\mathrm{sign}(x)\left((t-s)\left[v+\frac{E^0}{\omega}\sin(\omega t)\right]-\frac{E^0}{\omega}[\cos(\omega t)-\cos(\omega s)]\right)\end{aligned}$$

Thus, for $|x|>R(t)$, we can explicitly calculate the characteristics at $s=0$ as

$$X(0)=x-\mathrm{sign}(x)\left(t\left[v+\frac{E^0}{\omega}\sin(\omega t)\right]-\frac{E^0}{\omega}[\cos(\omega t)-1]\right)$$

and

$$V(0)=v+\frac{E^0\,\mathrm{sign}(x)}{\omega}\sin(\omega t)$$

Therefore, we use (5) to find

$$f(t,x,v)=f_0(X(0),V(0))$$

for $|x|>R(t)$ where $X(0)$ and $V(0)$ are as above. Finally, using (2) and Lemma 2, we conclude

$$f(t,x,v)=F(V(0,t,x,v))=F\left(v+\frac{E^0\,\mathrm{sign}(x)}{\omega}\sin(\omega t)\right)$$

for $|x|>R(t)$, and the proof of Theorem 2 is complete. The final section will be devoted to the proofs of Lemmas 1 and 2.

DOI: 10.1002/mma

SECTION 4

We complete the paper with the proofs of the lemmas.

Proof of Lemma 1
Let $T>0$ be given and f be a solution of (1) on $[0, T]$. Define for $t \in [0, T]$,

$$Q(t) := \sup\{|v| : \exists x \in \mathbb{R}, \tau \in [0, t] \text{ s.t. } f(\tau, x, v) \neq 0\}$$

We show that all momenta characteristics are bounded as functions of s. Using (3) and (4),

$$\begin{aligned}|V(0, t, x, v)| &= \left| v + \int_0^t E(s, X(s, t, x, v))\, \mathrm{d}s \right| \\ &\geqslant |v| - C^{(1)}\end{aligned}$$

By definition of $Q(t)$, if $|V(0, t, x, v)| \geqslant Q(0)$, we have for every $y \in \mathbb{R}$,

$$f_0(y, V(0, t, x, v)) = 0$$

But, by the above equation, if $|v| \geqslant Q(0) + C^{(1)}$, then

$$|V(0, t, x, v)| \geqslant |v| - C^{(1)} \geqslant Q_g(0)$$

which implies that $f_0(y, V(0, t, x, v)) = 0$. So, if $f_0(y, V(0, t, x, v)) \neq 0$, we must have

$$|v| \leqslant Q(0) + C^{(1)} \tag{24}$$

Since we wish to consider only non-trivial f, (5) implies that $f(X(0, t, x, v), V(0, t, x, v)) \neq 0$ for some $t \in [0, T]$, $x, v \in \mathbb{R}$, and thus (24) must hold. Taking the supremum over v of both sides in (24), we find

$$Q(t) \leqslant Q(0) + C^{(1)} \leqslant C$$

for every $t \in [0, T]$. Since F is compactly supported, it follows that $Q_g(t) \leqslant C$ for all $t \in [0, T]$, as well. In addition, for $t \in [0, T]$ and $|v| > Q_g(t)$, we have

$$f(t, x, v) = f_0(X(0), V(0)) = 0 \qquad \square$$

Proof of Lemma 2
Let $T>0$ be given and f be a solution of (1) on $[0, T]$. Define $Q_g(t)$ for $t \in [0, T]$ as in (6). Consider $|v| \leqslant Q_g(t)$ and use (4) to obtain the integral form of characteristics:

$$V(s) = v + \int_s^t E(\tau, X(\tau, t, x, v))\, \mathrm{d}\tau$$

DOI: 10.1002/mma

and

$$X(s) = x - \int_s^t \left(v + \int_\tau^t E(\bar{s}, X(\bar{s}, t, x, v))\, \mathrm{d}\bar{s} \right) \mathrm{d}\tau$$

Recall,

$$R(t) := R + tQ_g(t) + \int_0^t \int_\tau^t \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau$$

Thus, for $|x| > R(t)$ and $s \in [0, t]$,

$$\begin{aligned}
|X(s,t,x,v)| &\geqslant |x| - |v|(t-s) - \int_s^t \int_\tau^t |E(\bar{s}, X(\bar{s}, t, x, v))|\,\mathrm{d}\bar{s}\,\mathrm{d}\tau \\
&\geqslant |x| - Q_g(t)(t-s) - \int_s^t \int_\tau^t \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau \\
&> R + tQ_g(t) - (t-s)Q_g(t) + \int_0^t \int_\tau^t \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau - \int_s^t \int_\tau^t \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau \\
&= R + sQ_g(t) + \int_0^s \int_\tau^t \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau \\
&\geqslant R + sQ_g(s) + \int_0^s \int_\tau^s \|E(\bar{s})\|_\infty \,\mathrm{d}\bar{s}\,\mathrm{d}\tau \\
&= R(s)
\end{aligned}$$

Thus, the proof of Lemma 2 is complete. □

REFERENCES

1. Pankavich S. Local existence for the one-dimensional Vlasov–Poisson system with infinite mass. *Mathematical Methods in the Applied Sciences* 2007; **30**:529–548.
2. Batt J, Kunze M, Rein G. On the asymptotic behavior of a one-dimensional, monocharged plasma and a rescaling method. *Advances in Differential Equations* 1998; **3**:271–292.
3. Burgan JR, Feix MR, Fijalkow E, Munier A. Self-similar and asymptotic solutions for a one-dimensional Vlasov beam. *Journal of Plasma Physics* 1983; **29**:139–142.
4. Pfaffelmoser K. Global classical solution of the Vlasov–Poisson system in three dimensions for general initial data. *Journal of Differential Equations* 1992; **95**(2):281–303.
5. Lions PL, Pertham B. Propagation of moments and regularity for the three dimensional Vlasov–Poisson system. *Inventiones Mathematicae* 1991; **105**:415–430.
6. Schaeffer J. Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. *Communications in Partial Differential Equations* 1991; **16**(8/9):1313–1335.
7. Batt J. Global symmetric solutions of the initial-value problem of stellar dynamics. *Journal of Differential Equation* 1977; **25**:342–364.
8. Horst E. On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov-equation, Parts I and II. *Mathematical Methods in the Applied Sciences* 1981; **3**:229–248; 1982; **4**:19–32.
9. Glassey R. *The Cauchy Problem in Kinetic Theory*. SIAM: Philadelphia, PA, 1996.

DOI: 10.1002/mma

10. Rein G. Collisionless kinetic equations from astrophysics—the Vlasov–Poisson system. In *Handbook of Differential Equations*, *Evolutionary Equations*, Dafermos CM, Feireisl E (eds), vol. 3. Elsevier: Amsterdam, 2007.
11. Jabin P-E. The Vlasov–Poisson system with infinite mass and energy. *Journal of Statistical Physics* 2001; **103**(5/6):1107–1123.
12. Perthame B. Time decay, propagation of low moments and dispersive effects for kinetic equations. *Communications in Partial Differential Equations* 1996; **21**(3/4):659–686.
13. Schaeffer J. The Vlasov–Poisson system with steady spatial asymptotics. *Communications in Partial Differential Equations* 2003; **28**(5/6):1057–1084.
14. Schaeffer J. Steady spatial asymptotics for the Vlasov–Poisson system. *Mathematical Methods in the Applied Sciences* 2003; **26**:273–296.
15. Pankavich S. Global existence for the radial Vlasov–Poisson system with steady spatial asymptotics. 2005, submitted.
16. Pankavich S. Global existence for the Vlasov–Poisson system with steady spatial asymptotics. *Communications in Partial Differential Equations* 2006; **31**:249–270.

DOI: 10.1002/mma