WELL-POSEDNESS OF A ONE-DIMENSIONAL PLASMA MODEL WITH A HYPERBOLIC FIELD

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1. Introduction

A plasma is a partially or completely ionized gas. Nearly all (approximately 99.9%) of the matter in the universe exists in the state of plasma, as opposed to a solid, fluid, or a gaseous state. Such a form of matter occurs if the velocity of individual particles in a material achieves an enormous magnitude, perhaps a sizable fraction of the speed of light. Hence all matter, if heated to a significantly great temperature, will enter a plasma state. In terms of practical use, plasmas are of great interest to the energy, aeronautical, and aerospace industries among others, as they are used in the production of electronics, (plasma) engines, and lasers, as well as, in harnessing the power of nuclear energy. Plasmas are widely used in solid state physics since they are great conductors of electricity due to their free-flowing abundance of ions and electrons. When a plasma is of low density or the time scales of interest are sufficiently small, it is deemed to be "collisionless", as collisions between particles become infrequent. Many examples of collisionless plasmas occur in nature, including the solar wind, galactic nebulae, the Van Allen radiations belts, and comet tails.

The fundamental equations which describe the time evolution of a collisionless plasma are given by the Vlasov-Maxwell system:

$$
\text{(VM)} \quad
\begin{cases}
\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0 \\
\rho(t,x) = \displaystyle\int f(t,x,v) \ dv, \quad j(t,x) = \int v f(t,x,v) \ dv \\
\partial_t E = \nabla \times B - j, \qquad \nabla \cdot E = \rho \\
\partial_t B = -\nabla \times E, \qquad \nabla \cdot B = 0.
\end{cases}
$$

Here, f represents the density of (positively-charged) ions in the plasma, while ρ and j are the charge and current density, and E and B represent electric and magnetic fields generated by the charge and current. The independent variables, $t \geq 0$ and $x, v \in \mathbb{R}^3$ represent time, position, and velocity, respectively, and physical constants, such as the charge and mass of particles, as well as, the speed of light, have been normalized to one. In the presence of large velocities, relativistic corrections become important and the corresponding system to consider is the relativistic analogue of (VM), denoted by (RVM) and constructed by replacing v with

$$
\hat{v} = \frac{v}{\sqrt{1 + |v|^2}}
$$

in the first equation of (VM), called the Vlasov equation, and in the integrand of the current j. General references on the kinetic equations of plasma dynamics, such as (VM)

and (RVM), include [4] and [8]. Over the past twenty-five years significant progress has been made in the analysis of (RVM), specifically, the global existence of weak solutions (which also holds for (VM); see [2]) and the determination of conditions which ensure global existence of classical solutions (originally discovered in [5], and later in [6], and [1]) for the Cauchy problem. Additionally, a wide array of information has been discovered regarding the electrostatic versions of both (VM) and (RVM) - the Vlasov-Poisson and relativistic Vlasov-Poisson systems, respectively. These models do not include magnetic effects within their formulation, and the electric field is given by an elliptic, rather than a hyperbolic equation. This simplification has led to a great deal of progress concerning the electrostatic systems, including theorems regarding global existence, stability, and long-time behavior of solutions; though a global existence theorem for classical solutions in the relativistic case has remained elusive. Independent of these advances, many of the most basic existence and regularity questions remain unsolved for (VM). The main difficulty which arises is the loss of strict hyperbolicity of the kinetic system due to the possibility that particle velocities v may travel faster than the propagation of signals from the electric and magnetic fields, which do so at the speed of light $c = 1$.

Often a remedy to the lack of progress on such a problem is to reduce the dimensionality of the system. Unfortunately, posing the problem in one-dimension (i.e., $x, v \in \mathbb{R}$) eliminates the relevance of the magnetic field as the Maxwell system decouples, yielding the one-dimensional Vlasov-Poisson system:

$$\text{(VP)} \qquad \begin{cases} \partial_t f + v \partial_x f + E \partial_v f = 0 \\ \partial_x E = \int f \, dv. \end{cases}$$

The lowest-dimensional reduction which includes magnetic effects is the so-called "one-and-one-half-dimensional" system which is constructed by taking $x \in \mathbb{R}$ but $v \in \mathbb{R}^2$. Surprisingly, the question of existence remains open even in this case. Thus, in order to study this question, but keep the problem posed in a one-dimensional setting, we consider the following nonlinear system of hyperbolic PDE:

$$\text{(SVM)} \qquad \begin{cases} \partial_t f + v \partial_x f + B \partial_v f = 0 \\ \partial_t B + \partial_x B = - \int f(t,x,v) dv. \end{cases}$$

Since the field equation in (SVM) is hyperbolic, we denote it with a magnetic field variable B, as opposed to the electric field E of (VP) which satisfies an elliptic equation. Notice that these equations retain the main difficulty of (VM), namely the interaction between characteristic particle velocities v and constant field velocities $c = 1$. The system (SVM) is supplemented by given initial data

$$\text{(IC)} \qquad \begin{aligned} f(0,x,v) &= f_0(x,v), \\ B(0,x) &= B_0(x) \end{aligned}$$

where $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in Lip(\mathbb{R})$. Here, $t \geq 0$ is again time, $x \in \mathbb{R}$ is space, $v \in \mathbb{R}$ is momentum (and velocity since the mass has been normalized), $f = f(t,x,v)$ represents the density of ions in phase space (x,v) over time, and $B = B(t,x)$ is effectively a magnetic field. We wish to prove the local-in-time existence of a unique solution in the space of Lipschitz functions.

Theorem 1.1. *There exists $T > 0$ and unique functions $f \in Lip([0,T] \times \mathbb{R}^2)$, $B \in Lip([0,T] \times \mathbb{R} \to \mathbb{R})$ such that f and B satisfy (SVM) and (IC).*

1.1. **Notation.** In order to prove the theorem, we begin by defining the following norms. For $f \in C(\mathbb{R}^2)$, we define

$$\|f\|_\infty = \sup_{x,v} |f(x,v)|.$$

Similarly for $B \in C(\mathbb{R})$, we use

$$\|B\|_\infty = \sup_x |B(x)|$$

where the norms on the domains $\mathbb{R}^2$ and $\mathbb{R}$ are distinguished by context. Furthermore, we will use

$$\|f\|_{Lip} = \inf\{c : |f(x,v) - f(y,u)| \leq c\sqrt{|x-y|^2 + |v-u|^2}, \ \forall x, y, v, u \in \mathbb{R}\}$$

for any $f \in Lip(\mathbb{R}^2)$ and the analogous definition for function which have domain $\mathbb{R}$. Finally, for any $T > 0$ and $f \in C([0,T]; Lip(\mathbb{R}^2))$ we define the norm

$$\|f\|_L = \sup_{t \in [0,T]} \|f(t)\|_{Lip}.$$

Each will be utilized in the local existence proof to follow. Additionally, we define the supremum of the velocity support of a given ion distribution by

$$P_f(t) := \sup\{|v| : \exists x \text{ such that } f(t,x,v) \neq 0\}.$$

Before beginning the proof of Theorem 1.1, we first derive some *a priori* estimates on functions which satisfy (SVM) with (IC).

2. A PRIORI BOUNDS

We begin by estimating the magnetic field, assuming the ion distribution and initial data are known in advance.

Lemma 2.1. *Let $T > 0$, $B_0 \in Lip(\mathbb{R})$, and $f \in Lip([0,T] \times \mathbb{R}^2)$ be given. Consider $B \in Lip([0,T] \times \mathbb{R})$ to be the unique solution of the linear Cauchy problem*

$$\begin{cases} \partial_t B + \partial_x B = -\int f \, dv \\ B(0,x) = B_0(x). \end{cases}$$

then, for every $t \in [0,T]$

(1)
$$\|B(t)\|_\infty \leq \|B_0\|_\infty + 2\int_0^t P_f(s)\|f(s)\|_\infty ds$$

and

(2)
$$\|B\|_L \leq \|B_0\|_{Lip} + 2\|f\|_L \int_0^T P_f(s) ds.$$

Proof. We will use the method of characteristics. Let $\chi(s,t,x) : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\begin{cases} \dfrac{\partial}{\partial s}\chi(s,t,x) = 1 \\ \chi(t,t,x) = x. \end{cases}$$

Then χ is linear in s and so it must satisfy $\chi(s,t,x) = s - t + x$. Using this we see

$$\frac{d}{ds}B(s, s - t + x) = \partial_t B(s, s - t + x) + \partial_x B(s, s - t + x)$$

$$= -\int f(s, s - t + x, v)dv$$

$$= -\int_{-P_f(s)}^{P_f(s)} f(s, s - t + x, v)dv$$

Integrating with respect to s from 0 to t yields

(3) $$B(t, x) = B(0, x - t) - \int_0^t \int_{-P_f(s)}^{P_f(s)} f(s, s - t + x, v)dvds.$$

Taking the supremum over all $x \in \mathbb{R}$, we conclude

$$\sup_x |B(t, x)| \leq \sup_x |B(0, x)| + 2 \int_0^t P_f(s) \sup_{x,v} f(s, x, v)ds$$

or

$$\|B(t)\|_\infty \leq \|B_0\|_\infty + 2 \int_0^t P_f(s)\|f(s)\|_\infty ds$$

which proves (1). Further, from (3)

$$B(t,x) - B(t,y) = B_0(x-t) - B_0(y-t) - \int_0^t \int_{-P_f(s)}^{P_f(s)} \left(f(s, s - t + x, v) - f(s, s - t + y, v) \right) dvds$$

for every $x, y \in \mathbb{R}$. So, taking $x \neq y$, we have

$$\frac{|B(t,x) - B(t,y)|}{|x - y|} = \frac{|B_0(x - t) - B_0(y - t)|}{|x - y|} + \int_0^t \int_{-P_f(s)}^{P_f(s)} \frac{|f(s, s - t + x, v) - f(s, s - t + y, v)|}{|x - y|} dvds.$$

Since

$$|x - y| = |(x - t) - (y - t)| = |(s - t + x) - (s - t + y)|,$$

this yields

$$\frac{|B(t,x) - B(t,y)|}{|x - y|} \leq \|B_0\|_{Lip} + \int_0^t \int_{-P_f(s)}^{P_f(s)} \|f(s)\|_{Lip}dvds$$

$$\leq \|B_0\|_{Lip} + 2 \int_0^t P_f(s)\|f\|_L ds$$

$$\leq \|B_0\|_{Lip} + 2\|f\|_L \int_0^T P_f(s)ds.$$

Taking the infemum over such values of x and y shows (2).   □

With this result, we can now estimate the Lipschitz norms of the resulting characteristic curves induced by the magnetic field.

Lemma 2.2. *Let $T > 0$ and $B \in Lip([0,T] \times \mathbb{R})$ be given and define $X, V \in Lip([0,T]^2 \times \mathbb{R}^2)$ as the unique solutions to the system of ODEs with Cauchy data*

$$\begin{cases} \dfrac{\partial}{\partial s} X(s,t,x,v) = V(s,t,x,v) \\[2mm] \dfrac{\partial}{\partial s} V(s,t,x,v) = B(s, X(s,t,x,v)) \\[2mm] \qquad\qquad X(t,t,x,v) = x \\[1mm] \qquad\qquad V(t,t,x,v) = v \end{cases}$$

then

(4) $$\|X\|_L + \|V\|_L \le 2\left[1 + T \max\{1, \|B\|_L\}(\|X\|_L + \|V\|_L)\right].$$

Proof. First observe from the characteristic ODEs

$$X(s,t,x,v) = x - \int_s^t V(\tau,t,x,v)d\tau$$

and

$$V(s,t,x,v) = v - \int_s^t B(\tau, X(\tau,t,x,v))d\tau.$$

Thus, we have

$$X(s,t,x,v) - X(s,t,y,v) = x - y - \int_s^t \left(V(\tau,t,x,v) - V(\tau,t,y,v)\right) d\tau,$$

$$X(s,t,x,v) - X(s,t,x,u) = -\int_s^t \left(V(\tau,t,x,v) - V(\tau,t,x,u)\right) d\tau,$$

$$V(s,t,x,v) - V(s,t,y,v) = -\int_s^t \left(B(\tau, X(\tau,t,x,v)) - B(\tau, X(\tau,t,y,v))\right) d\tau,$$

and

$$V(s,t,x,v) - V(s,t,x,u) = v - u - \int_s^t \left(B(\tau, X(\tau,t,x,v)) - B(\tau, X(\tau,t,x,u))\right) d\tau.$$

Taking absolute values yields,

$$|X(s,t,x,v) - X(s,t,y,v)| = |x - y| + \int_s^t |V(\tau,t,x,v) - V(\tau,t,y,v)| \, d\tau$$

$$\le |x-y|\left(1 + \int_s^t \|V\|_L d\tau\right)$$

$$\le |x-y|(1 + T\|V\|_L)$$

and similarly for the difference in v

$$|X(s,t,x,v) - X(s,t,x,u)| = \int_s^t |V(\tau,t,x,v) - V(\tau,t,x,u)| \, d\tau$$

$$\le |v-u| \int_s^t \|V\|_L d\tau$$

$$\le |v-u| T\|V\|_L.$$

We proceed analogously for the velocity characteristics, so that

$$|V(s,t,x,v) - V(s,t,y,v)| = \int_s^t |B(\tau, X(\tau,t,x,v)) - B(\tau, X(\tau,t,y,v))|\, d\tau$$

$$= \int_s^t \frac{|B(\tau, X(\tau,t,x,v)) - B(\tau, X(\tau,t,y,v))|}{|X(\tau,t,x,v) - X(\tau,t,y,v)|}$$
$$\cdot |X(\tau,t,x,v) - X(\tau,t,y,v)|\, d\tau$$

$$\leq |x-y| \int_s^t \|B\|_L \|X\|_L\, d\tau$$

$$\leq T|x-y|\|B\|_L \|X\|_L,$$

and

$$|V(s,t,x,v) - V(s,t,x,u)| \leq |v-u| + \int_s^t |B(\tau, X(\tau,t,x,v)) - B(\tau, X(\tau,t,x,u))|\, d\tau$$

$$\leq |v-u|(1 + \int_s^t \|B\|_L \|X\|_L\, d\tau)$$

$$\leq |v-u|(1 + T\|B\|_L \|X\|_L).$$

Adding the estimates on X together we find

$$\|X(s,t)\|_{Lip} \leq 1 + 2T\|V\|_L,$$

and taking the supremum over s and t,

$$\|X\|_L \leq 1 + 2T\|V\|_L.$$

We do the same for V, which gives us

$$\|V(s,t)\|_{Lip} \leq 1 + 2T\|B\|_L \|X\|_L,$$

and thus

$$\|V\|_L \leq 1 + 2T\|B\|_L \|X\|_L.$$

Moreover, we combine the estimates on characteristics so that

$$\|X\|_L + \|V\|_L \leq 2\left[1 + T\max\{1, \|B\|_L\}(\|X\|_L + \|V\|_L)\right]$$

and this completes the proof. $\square$

Here, we note that characteristics will be abbreviated in the future as $X(s)$ and $V(s)$ respectively. Thus, we will suppress their dependence on t, x, and v for brevity. Next, we estimate that particle distribution assuming that the field is known.

Lemma 2.3. *Let $T > 0$ and $B \in Lip([0,T] \times \mathbb{R})$ be given. Define the characteristics X and V as in Lemma 2.2 and let $f \in Lip([0,T] \times \mathbb{R}^2)$ be the unique solution to*

$$\begin{cases} \partial_t f + v\partial_x f + B\partial_v f = 0 \\ f(0,x,v) = f_0(x,v) \end{cases}$$

then for all $t \in [0,T]$

(5) $$\|f(t)\|_\infty = \|f_0\|_\infty$$

and

(6) $$\|f\|_L \leq \|f_0\|_{Lip}(\|X\|_L + \|V\|_L).$$

Proof. The former result follows from the conservation of ions along characteristics. Using X and V, we may write the Vlasov equation as

$$\frac{d}{ds}f(s,X(s),V(s)) = \partial_t f(s,X(s),V(s)) + V(s)\partial_x f(s,X(s),V(s)) + B(s,X(s))\partial_v f(s,X(s),V(s)) = 0.$$

Integrating with respect to s from 0 to t we obtain,

$$f(t,x,v) = f(0,X(0),V(0)) = f_0(X(0),V(0)).$$

Hence it follows,

$$\|f(t)\|_\infty = \|f_0\|_\infty$$

Further, let us denote for fixed $x,y,v,u \in \mathbb{R}$ the quantity

$$R(t) = \sqrt{|X(0,t,x,v) - X(0,t,y,u)|^2 + |V(0,t,x,v) - V(0,t,y,u)|^2}.$$

Then, estimating the Lipschitz norm of f, we find

$$\frac{|f(t,x,v) - f(t,y,u)|}{\sqrt{|x-y|^2 + |v-u|^2}}$$

$$= \frac{|f(0,X(0,t,x,v),V(0,t,x,v)) - f(0,X(0,t,y,u),V(0,t,y,u))|}{R(t)}\frac{R(t)}{R(0)}$$

$$\leq \|f_0\|_{Lip}\frac{|X(0,t,x,v) - X(0,t,y,u)| + |V(0,t,x,v) + V(0,t,y,u)|}{R(0)}$$

$$\leq \|f_0\|_{Lip}\frac{|X(0,t,x,v) - X(0,t,y,u)|}{\sqrt{|x-y|^2 + |v-u|^2}} + \|f_0\|_{Lip}\frac{|V(0,t,x,v) + V(0,t,y,u)|}{\sqrt{|x-y|^2 + |v-u|^2}}$$

$$\leq \|f_0\|_{Lip} \cdot \|X(0,t)\|_{Lip} + \|f_0\|_{Lip} \cdot \|V(0,t)\|_{Lip}$$

whence

$$\|f(t)\|_{Lip} \leq \|f_0\|_{Lip}\|X\|_L + \|f_0\|_{Lip}|V\|_L.$$

Taking the supremum over $t \in [0,T]$ gives (6). $\qquad\square$

3. Local existence and proof of Theorem 1.1

Using the *a priori* estimates of the previous section, we can now begin to prove the existence theorem. Since, f_0 is given and has compact support, let $R_0, R_1 > 0$ be such that $supp(f_0) \subseteq [-R_0, R_0] \times [-R_1, R_1]$. For $f \in Lip([0,T] \times \mathbb{R}^2)$ we will refer to the following properties:

(P1) $$f(0,x,v) = f_0(x,v)$$

(P2) $$supp(f(t,\cdot,\cdot)) \subseteq [-R_0 - 1, R_0 + 1] \times [-R_1 - 1, R_1 + 1], \quad \forall t \in [0,T]$$

(P3) $$\|f\|_L \leq 4\|f_0\|_{Lip}.$$

For any $T > 0$, we define the function space

$$G_T := \left\{ f \in Lip([0,T] \times \mathbb{R}^2) : f \text{ satisfies } (P1) - (P3) \right\}.$$

Notice that G_T is nonempty as the function $f(t,x,v) = f_0(x,v)$ for all $t \in [0,T]$ is certainly an element of G_T. We claim that when equipped with the norm, $\|f\|_{G_T} = \sup_{t\in[0,T]} \|f(t)\|_\infty$, G_T is a complete, closed subset of the Banach space $Lip([0,T] \times \mathbb{R}^2)$ for any $T > 0$. To show this, all that must be demonstrated is that G_T is complete. If we consider a Cauchy sequence of functions f_n in G_T, because we are using the $C^0([0,T] \times \mathbb{R}^2)$ norm and that space is complete, there exists a limit $f \in C([0,T] \times \mathbb{R}^2)$ such that $f_n \to f$ uniformly (i.e.,

in the supremum norm). Certainly, f satisfies conditions (P1) - (P3). In addition, it follows from Arzelá-Ascoli (cf. [7]) that the uniform limit of Lipschitz functions is also Lipschitz and will share the same uniform Lipschitz constant as those in the sequence. Thus, $f \in G_T$ and the subset is complete.

Next, we will construct a map $\phi : G_T \to Lip([0, T] \times \mathbb{R}^2)$ as follows. Given $f \in G_T$ define $B \in Lip([0, T] \times \mathbb{R})$ to be the unique solution of

$$\begin{cases} \partial_t B + \partial_x B = -\int f dv \\ B(0, x) = B_0(x). \end{cases}$$

From B, define $g \in Lip([0, T] \times \mathbb{R}^2)$ as the unique solution to

$$\begin{cases} \partial_t g + v \partial_x g + B \partial_v g = 0 \\ g(0, x, v) = f_0(x, v) \end{cases}$$

and let the mapping ϕ be defined by $\phi(f) = g$. In order to use the Contraction Mapping Principle, we must prove that for every sufficiently small $T > 0$, we have $\phi : G_T \to G_T$ and ϕ is a contraction on this space. The following two lemmas do just this.

Lemma 3.1. *There exists $T_1^* > 0$ such that for any $T \in (0, T_1^*)$ we have $\phi : G_T \to G_T$.*

Proof. To show this we must show, $\|g\|_L \leq 4\|f_0\|_{Lip}$ and $supp(f(t\cdot, \cdot)) \subseteq [-R_0 - 1, R_0 + 1] \times [-R_1 - 1, R_1 + 1]$. Notice that condition (P1) holds by construction of ϕ. From Lemma 2.3 we know

$$\|g\|_L \leq \|f_0\|_{Lip}(\|X\|_L + \|V\|_L)$$

where X and V are the characteristics defined from B in Lemma 2.2. Using Lemma 2.2 we can reduce the right side to knowledge of the Lipschitz norm of B since

$$\|X\|_L + \|V\|_L \leq 2\left[1 + T \max\{1, \|B\|_L\}(\|X\|_L + \|V\|_L)\right].$$

Now, utilizing Lemma 2.1 and the fact that $f \in G_T$ for some $T > 0$, we find

$$\begin{aligned} \|B\|_L &\leq \|B_0\|_{Lip} + 2\|f\|_L \int_0^T P_f(s)ds \\ &\leq \|B_0\|_{Lip} + 2\|f\|_L T(R_1 + 1) \\ &\leq \|B_0\|_{Lip} + 8\|f_0\|_{Lip} T(R_1 + 1) \end{aligned}$$

Let

$$T_1 = \min\left\{ \frac{1}{4}, \frac{1}{4}\left[\|B_0\|_{Lip} + 2\|f_0\|_{Lip}(R_1 + 1)\right]^{-1} \right\}.$$

Then, it follows from the bound on $\|B\|_L$ that

$$\|X\|_L + \|V\|_L \leq 2 + \frac{1}{2}(\|X\|_L + \|V\|_L)$$

on the interval $[0, T_1]$. This implies $\|X\|_L + \|V\|_L \leq 4$ and we have,

$$\|g\|_L \leq 4\|f_0\|_{Lip}$$

where the $\|\cdot\|_L$ norm is taken over the interval $[0, T_1]$.

To show our support constraint is also satisfied, we compute for characteristics along which $g \not\equiv 0$,

$$P_g(t) := \sup\{|v| : \exists x \in \mathbb{R} \text{ with } g(t, x, v) \neq 0\}$$
$$= \sup\{|v| : \exists x \in \mathbb{R} \text{ with } g(0, X(0, t, x, v), V(0, t, x, v)) \neq 0\}$$
$$= \sup\{|v| : \exists x \in \mathbb{R} \text{ with } f_0(X(0, t, x, v), V(0, t, x, v)) \neq 0\}$$

If we invert our characteristics (see [4] for more details) and allow

$$\begin{cases} y = X(0, t, x, v) \\ w = V(0, t, x, v) \end{cases}$$

then it follows that

$$\begin{cases} x = X(t, 0, y, w) \\ v = V(t, 0, y, w). \end{cases}$$

Thus we conclude,

$$P_g(t) = \sup\{|V(t, 0, y, w)| : \exists y \text{ with } f_0(y, w) \neq 0\}.$$

Using the definition of characteristics and integrating over $[s, t]$ gives us

$$V(t, t, x, v) - V(s, t, x, v) = \int_s^t B(\tau, X(\tau, t, x, v))d\tau$$

Taking $t = 0$ we have

$$V(0, 0, x, v) - V(s, 0, x, v) = \int_s^0 B(\tau, X(\tau, 0, x, v))d\tau.$$

If we then let $s = t$ this becomes

$$V(0, 0, x, v) - V(t, 0, x, v) = -\int_0^t B(\tau, X(\tau, 0, x, v))d\tau$$

or

$$V(t, 0, x, v) - V(0, 0, x, v) = \int_0^t B(\tau, X(\tau, 0, x, v))d\tau \leq \int_0^t \|B(\tau)\|_\infty d\tau$$

Hence, for all $x, v \in \mathbb{R}$ and any characteristics along which $f_0 \neq 0$, we have

$$|V(t, 0, x, v)| \leq |V(0, 0, x, v)| + \int_0^t \|B(\tau)\|_\infty d\tau$$

If we take the supremum over all characteristics such that $f_0 \neq 0$ then we have

$$P_g(t) \leq P_g(0) + \int_0^t \|B(\tau)\|_\infty d\tau$$

Utilizing (1) from Lemma 2.1 we find

$$\|B(t)\|_\infty \leq \|B_0\|_\infty + 2 \int_0^t P_f(s)\|f(s)\|_\infty ds.$$

So, we consider $t \in [0, T_2]$ for some $T_2 > 0$ to be determined and estimate the integral above using (5), (P2), and (P3) since $f \in G_T$

$$\int_0^t \|B(\tau)\|_\infty d\tau \leq \int_0^t \left(\|B_0\|_\infty + 2 \int_0^\tau P_f(s)\|f(s)\|_\infty ds \right) d\tau$$

$$\leq \int_0^{T_2} \left(\|B_0\|_\infty + 2 \int_0^{T_2} P_f(s)\|f(s)\|_\infty ds \right) d\tau$$

$$\leq \int_0^{T_2} \left(\|B_0\|_\infty + 2 \int_0^{T_2} (R_1 + 1)\|f_0\|_\infty ds \right) d\tau$$

$$\leq T_2 \|B_0\|_\infty + 2(T_2)^2 (R_1 + 1)\|f_0\|_\infty.$$

We can then take T_2 small enough such that $T_2\|B_0\|_\infty + 2T_2(R_1 + 1)\|f_0\| < 1$, and because $g(0) = f_0$, we have

$$P_g(t) \leq P_g(0) + \int_0^t \|B(\tau)\|_\infty d\tau$$

$$\leq R_1 + 1.$$

Lastly, let $T_3 < \frac{1}{R_1 + 1}$. Since

$$\frac{d}{ds} X(s, 0, x, v) = V(s, 0, x, v),$$

we take a characteristic $X(s, t, x, v)$ along which $g \neq 0$ and find

$$X(t, 0, x, v) = X(0, 0, x, v) + \int_0^t V(s, 0, x, v) ds$$

$$\leq x + \int_0^{T_3} (R_1 + 1) ds$$

$$= x + T_3(R_1 + 1)$$

$$\leq R_0 + 1.$$

Finally, if we consider $T_1^* < \min\{T_1, T_2, T_3\}$ then all of the above statements must hold on $[0, T_1^*]$. Thus, conditions (P2) and (P3) hold for $g = \phi(f)$ and for any $T \in (0, T_1^*)$, we have $\phi : G_T \to G_T$. $\qquad \square$

Next, we show that the mapping we have constructed is indeed a contraction on G_T for $T > 0$ sufficiently small.

Lemma 3.2. *There exists $T_2^* > 0$ such that for any $T \in (0, T_2^*)$, ϕ is a contraction on G_T.*

Proof. Let $f_1, f_2 \in G_T$, Let B_i be the solution of

$$\begin{cases} \partial_t B_i + \partial_x B_i = - \int f_i dv \\ B_i(0, x) = B_0(x) \end{cases}$$

and define $g_i = \phi(f_i)$ for $i = 1, 2$. Then, as in Lemma 2.1, we can write

$$B_1(t, x) - B_2(t, x) = B_1(0, x-t) - B_2(0, x-t) - \int_0^t \int_{-R_1-1}^{R_1+1} (f_1(s, s-t+x, v) - f_2(s, s-t+x, v)) \, dv ds.$$

Taking the absolute value and applying the triangle inequality,

$$|B_1(t,x) - B_2(t,x)| \leq |B_0(x-t) - B_0(x-t)| + \int_0^t \int_{-R_1-1}^{R_1+1} |f_1(s, s-t+x, v) - f_2(s, s-t+x, v)| \, dvds$$

$$\leq \int_0^t \int_{-R_1-1}^{R_1+1} \|(f_1 - f_2)(s)\|_\infty dvds$$

$$\leq 2(R_1 + 1) \int_0^t \|(f_1 - f_2)(s)\|_\infty ds$$

and we conclude the uniform bound

$$\tag{7} \|(B_1 - B_2)(t)\|_\infty \leq 2(R_1 + 1)T \sup_{s \in [0,T]} \|(f_1 - f_2)(s)\|_\infty$$

for every $t \in [0, T]$. Subtracting the g equations we see $g_1 - g_2$ satisfies,

$$\partial_t(g_1 - g_2) + v\partial_x(g_1 - g_2) + B_1 \partial_v g_1 - B_2 \partial_v g_2 = 0.$$

Adding and subtracting $B_2 \partial_v g_1$,

$$\partial_t(g_1 - g_2) + v\partial_x(g_1 - g_2) + B_1 \partial_v g_1 - B_2 \partial_v g_1 + B_2 \partial_v g_1 - B_2 \partial_v g_2 = 0.$$

Rearranging we now have,

$$\partial_t(g_1 - g_2) + v\partial_x(g_1 - g_2) + B_2 \partial_v(g_1 - g_2) = -(B_1 - B_2)\partial_v g_1.$$

Consider, the characteristics $X_2(s, t, x, v), V_2(s, t, x, v)$ that satisfy

$$\begin{cases} \dfrac{\partial}{\partial s} X_2(s, t, x, v) = V_2(s, t, x, v) \\ \dfrac{\partial}{\partial s} V_2(s, t, x, v) = B_2(s, X_2(s, t, x, v)) \\ X_2(t, t, x, v) = x \\ V_2(t, t, x, v) = v. \end{cases}$$

Then we write the equation for $g_1 - g_2$ as

$$\frac{d}{ds}(g_1 - g_2)(s, X_2(s, t, x, v), V_2(s, t, x, v)) = -(B_1 - B_2)\partial_v g_1(s, X_2(s, t, x, v), V_2(s, t, x, v))$$

Integrating over $[0, t]$, this becomes

$$(g_1 - g_2)(t, X_2(t, t, x, v), V_2(t, t, x, v)) = (g_1 - g_2)(0, X_2(0, t, x, v), V_2(0, t, x, v))$$

$$- \int_0^t (B_1 - B_2)\partial_v g_1(s, X_2(s, t, x, v), V_2(s, t, x, v))ds$$

But $g_1(0, x, v) = g_2(0, x, v) = f_0(x, v)$ so $(g_1 - g_2)(0, x, v) = 0$. Using this, the bound on $\|g_1\|_L$ from (P3), and (7) we have

$$|(g_1 - g_2)(t, x, v)| \leq \int_0^t |(B_1 - B_2)(s, X_2(s, t, x, v))| \cdot \|g_1(s)\|_{Lip} ds$$

$$\leq \int_0^T \|(B_1 - B_2)(s)\|_\infty \cdot \|g_1\|_L ds$$

$$\leq 2T^2(R_1 + 1)\|g_1\|_L \sup_{s \in [0,T]} \|(f_1 - f_2)(s)\|_\infty$$

$$\leq 8T^2(R_1 + 1)\|f_0\|_{Lip}\|f_1 - f_2\|_{G_T}$$

If we take $T_2^* < \frac{1}{\sqrt{8(R_1+1)\|f_0\|_{Lip}}}$, then we see for $T < T_2^*$, $t \in [0,T]$, and for all $x,v \in \mathbb{R}$,

$$|(g_1 - g_2)(t,x,v)| \le A\|f_1 - f_2\|_{G_T}$$

where $A < 1$. Thus it follows that

$$\|g_1 - g_2\|_{G_T} \le A\|f_1 - f_2\|_{G_T}$$

and $\phi : G_T \to G_T$ is a contraction for all $T \in (0, T_2^*)$.   □

The last lemma is devoted to showing that no solutions of (SVM) other than the ones that we construct can exist in $Lip([0,T] \times \mathbb{R}^2)$.

Lemma 3.3. *There exists at most one solution to (SVM), for $(f,B) \in Lip([0,T] \times \mathbb{R}^2) \times Lip([0,T] \times \mathbb{R})$ which satisfy the given initial conditions (IC).*

Proof. Suppose that $(f_1, B_1), (f_2, B_2) \in Lip([0,T] \times \mathbb{R}^2) \times Lip([0,T] \times \mathbb{R})$ are two solutions to our system. Define

$$f(t,x,v) := (f_1 - f_2)(t,x,v)$$

and

$$B(t,x,v) := (B_1 - B_2)(t,x,v).$$

Then we see

$$
\begin{aligned}
0 &= \partial_t(f_1 - f_2) + v\partial_x(f_1 - f_2) + B_1\partial_v f_1 - B_2\partial_v f_2 \\
&= \partial_t f + v\partial_x f + B_1\partial_v f_1 - B_2\partial_v f_1 + B_2\partial_v f_2 \\
&= \partial_t f + v\partial_x f + B\partial_v f_1 + B_2\partial_v f
\end{aligned}
$$

If we let X_2, V_2 be characteristics satisfying

$$
\begin{cases}
\dfrac{\partial}{\partial s}X_2(s,t,x,v) = V_2(s,t,x,v) \\
\dfrac{\partial}{\partial s}V_2(s,t,x,v) = B_2(s, X_2(s,t,x,v)) \\
\qquad\qquad X_2(t,t,x,v) = x \\
\qquad\qquad V_2(t,t,x,v) = v,
\end{cases}
$$

then we have

$$\frac{d}{ds}(f(s, X_2(s,t,x,v), V_2(s,t,x,v)) = -(B\partial_v f_1)(s, X_2(s,t,x,v), V_2(s,t,x,v)).$$

Integrating and using $f(0,x,v) = f_1(0,x,v) - f_2(0,x,v) = 0$ for every $x,v \in \mathbb{R}$, we have

$$
\begin{aligned}
f(t,x,v) &= f(0, X_2(0,t,x,v), V_2(0,t,x,v)) \\
&\quad - \int_0^t B(s, X_2(s,t,x,v))\partial_v f_1(s, X_2(s,t,s,v), V_2(s,t,x,v))ds \\
&\le \|f_1\|_L \int_0^t \|B(s)\|_\infty ds
\end{aligned}
$$

Taking the supremum of both sides we see,

$$
(8) \qquad\qquad\qquad \|f(t)\|_\infty \le \|f\|_L \int_0^t \|B(s)\|_\infty ds.
$$

Subtracting the B equations we observe,

$$\partial_t B + \partial_x B = -\int f dv$$

and thus
$$\frac{d}{ds}B(s, s - t + x) = -\int f(s, s - t + x, v)dv.$$

Integrating yields,
$$B(t, x) = B(0, x - t) - \int_0^t \int_{-P_f(s)}^{P_f(s)} f(s, s - t + x, v)dvds.$$

Because $B(0, x) = B_1(0, x) - B_2(0, x) = B_0(x) - B_0(x) = 0$ we find

(9)
$$\|B(t)\|_\infty \le 2 \int_0^t P_f(s)\|f(s)\|_\infty \, ds.$$

Adding (8) and (9) we have,
$$\|f(t)\|_\infty + \|B(t)\|_\infty \le \int_0^t \max\{\|f\|_L, 2P_f(s)\} \left(\|f(s)\|_\infty + \|B(s)\|_\infty\right) ds$$

Since $f \in Lip([0, T] \times \mathbb{R}^2)$ is uniformly bounded and both f_1 and f_2 inherit the compact velocity support of f_0, there is $C > 0$ such that
$$\|f(t)\|_\infty + \|B(t)\|_\infty \le C \int_0^t \left(\|f(s)\|_\infty + \|B(s)\|_\infty\right) \, ds.$$

Applying Grönwall's inequality (cf. [3]), we arrive at
$$\|f(t)\|_\infty + \|B(t)\|_\infty \le 0.$$

So it must follow that $f \equiv 0$ and $B \equiv 0$, and thus $f_1 = f_2$ with $B_1 = B_2$. Hence, we see that there can be at most one solution to (SVM) within the space of Lipschitz functions. $\square$

We are now at the point where we can apply Lemmas 3.1, 3.2, and 3.3 in order to conclude by the contraction mapping principal that Theorem 1.1 is true.

Proof of Theorem 1.1. Let $T = \min\{T_1^*, T_2^*\}$ and define $G = G_T$. Then, we know from Lemmas 3.1 and 3.2 that ϕ is a contraction on G. Because G is a complete, closed subset of a Banach space, we may apply the contraction mapping principal and conclude that there exist a unique $f = f(t, x, v) \in G$ such that $\phi(f) = f$. Let $B = B(t, x) \in Lip([0, T] \times \mathbb{R})$ be the unique solution to

(10)
$$\begin{cases} \partial_t B + \partial_x B = -\int f dv \\ B(0, x) = B_0(x). \end{cases}$$

Notice that by Lemma 2.2, since $f \in Lip([0, T] \times \mathbb{R}^2)$, it follows that B is also Lipschitz. Then, because $g = \phi(f) = f$ satisfies
$$\begin{cases} \partial_t g + v\partial_x g + B\partial_v g = 0 \\ g(0, x, v) = f_0(x, v), \end{cases}$$

we see that (f, B) satisfies

(SVM)
$$\begin{cases} \partial_t f + v\partial_x f + B\partial_v f = 0 \\ \partial_t B + \partial_x B = -\int f(t, x, v)dv \end{cases}$$

and

(IC)
$$\begin{cases} f(0, x, v) = f_0(x, v) \\ B(0, x) = B_0(x) \end{cases}$$

where f is the fixed point and B is defined by (10). Further from Lemma 3.3 we know that (f, B) is the only pair of Lipschitz functions that satisfy (SVM) and (IC). Thus, our constructed solution is unique in this space. □

4. Conclusion

Future work for this topic includes justifying the existence and uniqueness of classical solutions, $f, B \in C^1$. Also of interest, is the question of global-in-time solutions. Finally, we are also interested in showing that there are no steady state solutions except $f = B \equiv 0$ and determining the stability of the zero solution in different L^p norms.

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