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TIME DECAY FOR SOLUTIONS TO ONE-DIMENSIONAL TWO COMPONENT PLASMA EQUATIONS

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Dedicated to Professor Walter Strauss on his 70th birthday

1. Dedication and introduction. We represent three generations of students: Bob Glassey, Walter's student finishing at Brown in 1972, Jack Schaeffer, Bob's student finishing at Indiana University in 1983, and Steve Pankavich, Jack's student finishing at Carnegie Mellon in 2005. We have all thrived professionally from our association with Walter and are delighted to dedicate this note to him on the occasion of his 70th birthday. The problem we study below concerns the asymptotic behavior of solutions, an area to which Walter has contributed greatly.

The motion of a collisionless plasma is described by the Vlasov–Maxwell system. If we neglect magnetic effects we then have the Vlasov–Poisson system (VP). We can also consider the effect of large velocities and solutions to the relativistic Vlasov–Poisson system (RVP). We will study both systems in one space and one momentum dimension with two species of oppositely charged particles. We further assume that each system is *neutral*, which means that the average value of the density ρ vanishes (see below). The

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Vlasov–Poisson system (VP) is

$$\begin{cases} \partial_t f + v\ \partial_x f + E(t,x)\ \partial_v f = 0, \\ \partial_t g + \dfrac{v}{m}\ \partial_x g - E(t,x)\ \partial_v g = 0, \\ \rho(t,x) = \displaystyle\int (f(t,x,v) - g(t,x,v))\ dv, \\ E(t,x) = \dfrac{1}{2}\left(\displaystyle\int_{-\infty}^{x} \rho(t,y)\ dy - \int_{x}^{\infty} \rho(t,y)\ dy\right). \end{cases} \tag{1.1}$$

Here $t \geq 0$ is time, $x \in \mathbb{R}$ is position, $v \in \mathbb{R}$ is momentum, f is the number density in phase space of particles with mass one and positive unit charge, while g is the number density of particles with mass $m > 0$ and negative unit charge. The effect of collisions is neglected. The initial conditions

$$f(0,x,v) = f_0(x,v) \geq 0$$

and

$$g(0,x,v) = g_0(x,v) \geq 0$$

for $(x,v) \in \mathbb{R}^2$ are prescribed. We assume that f_0, $g_0 \in C^1(\mathbb{R}^2)$ are nonnegative, compactly supported and satisfy the neutrality condition

$$\iint f_0\, dv\, dx = \iint g_0\ dv\, dx. \tag{1.2}$$

Using the notation

$$\hat{v}_m = \frac{v}{\sqrt{m^2 + v^2}},$$

we can write the relativistic Vlasov–Poisson system (abbreviated RVP) as

$$\begin{cases} \partial_t f + \hat{v}_1\ \partial_x f + E\ \partial_v f = 0, \\ \partial_t g + \hat{v}_m\ \partial_x g - E\ \partial_v g = 0, \\ \rho(t,x) = \displaystyle\int (f - g)\ dv, \\ E(t,x) = \dfrac{1}{2}\left(\displaystyle\int_{-\infty}^{x} \rho\ dy - \int_{x}^{\infty} \rho\ dy\right). \end{cases} \tag{1.3}$$

Global existence and regularity are known for solutions of (1.1) and (1.3). Both $f(t,\cdot,\cdot)$ and $g(t,\cdot,\cdot)$ are compactly supported for all $t \geq 0$. There is scant literature regarding the large time behavior of solutions. Some time decay is known for the three-dimensional analogue of (1.1) ([6], [7], [9]). Also, there are time decay results for (1.1) (in dimension one) when the plasma is monocharged (set $g \equiv 0$) ([1], [2], [12]). In this work two species of particles with opposite charge are considered; thus the methods used in these references do not apply. References [3], [4], and [5] are also mentioned since they deal with time-dependent rescalings and time decay for other kinetic equations. We will take $m = 1$ below. A full description of these results will appear in [10].

First we sketch the derivation of an identity for solutions to (1.1) from which we can conclude that certain positive quantities are integrable in t on the interval $[0,\infty)$. This identity also extends to (1.3), but the results are weaker. Unfortunately, these identities are very "one-dimensional"; that is, they do not seem to easily generalize to higher dimension. Moreover, it is not clear if there is an extension which allows for more than two species of particles.

Here are the results we have obtained. The classical equations for (VP) are

$$f_t + vf_x + Ef_v = 0, \quad g_t + vg_x - Eg_v = 0,$$

where $E_x = \rho = \int (f - g)\,dv$. Let

$$F(t,x) = \int f(t,x,v)\,dv, \quad G(t,x) = \int g(t,x,v)\,dv.$$

Then $\rho = F - G$. We will show that

$$\int_0^\infty \int E^2(F+G)\,dx\,dt < \infty.$$

From this it will follow in the nonrelativistic case that

$$\int_0^\infty \int (F^4 + G^4)\,dx\,dt < \infty,$$

while the corresponding result for solutions to (RVP) is

$$\int_0^\infty \left(\int \left(F(t,x)^{\frac{7}{4}} + G(t,x)^{\frac{7}{4}} \right) \, dx \right)^4 \, dt < \infty.$$

The local charges for solutions to both systems will satisfy for any fixed $R > 0$

$$\lim_{t\to\infty} \int_{|x|<R} F(t,x)\,dx = \lim_{t\to\infty} \int_{|x|<R} G(t,x)\,dx = 0.$$

Finally, for solutions to (1.1) or (1.3) we can show that

$$\lim_{t\to\infty} \|E(t,\cdot)\|_\infty = 0.$$

2. Results. We first derive a general identity which holds for both (VP) and (RVP). From the above definitions and (VP) we have

$$F_t = -\int (vf_x + Ef_v)\,dv = -\partial_x \int vf\,dv$$

and thus

$$\partial_t \int_{-\infty}^x F(t,y)\,dy = -\int vf(t,x,v)\,dv,$$

with a similar result for g. Multiply the f equation in (VP) by $v \cdot \int_{-\infty}^x F(t,y)\,dy$ and integrate over v:

$$\int vf_t \int_{-\infty}^x F(t,y)\,dy\,dv + \int v^2 f_x \int_{-\infty}^x F(t,y)\,dy\,dv + \int vf_v E \int_{-\infty}^x F(t,y)\,dy\,dv = 0.$$

Write this as $I + II + III = 0$. Then

$$\begin{aligned} I &= \partial_t \left[\int vf \int_{-\infty}^x F(t,y)\,dy\,dv \right] - \int vf\,dv \int_{-\infty}^x F_t(t,y)\,dy & (2.1) \\ &= \partial_t \left[\int vf \int_{-\infty}^x F(t,y)\,dy\,dv \right] + \left(\int vf\,dv \right)^2, & (2.2) \\ II &= \partial_x \left[\int v^2 f \int_{-\infty}^x F(t,y)\,dy\,dv \right] - \int v^2 f \cdot F(t,x)\,dv, \end{aligned}$$

and after integrating by parts in v

$$III = -\partial_x \left[\frac{1}{2} E(t,x) \left(\int_{-\infty}^x F(t,y)\,dy \right)^2 \right] + \frac{1}{2}\rho(t,x) \left[\int_{-\infty}^x F(t,y)\,dy \right]^2.$$

Now integrate over x:

$$\begin{aligned} &\frac{d}{dt} \int \left[\int vf \int_{-\infty}^x F(t,y)\,dy\,dv \right] dx + \int \left(\int vf\,dv \right)^2 dx - \int F(t,x) \int v^2 f\,dv\,dx \\ &\quad + \frac{1}{2} \int \rho(t,x) \left[\int_{-\infty}^x F(t,y)\,dy \right]^2 dx = 0. \end{aligned}$$

Now, repeat this calculation with f replaced by g and add the two results to derive

$$\begin{aligned} &\frac{d}{dt} \int \left[\int vf \int_{-\infty}^x F(t,y)\,dy\,dv + \int vg \int_{-\infty}^x G(t,y)\,dy\,dv \right] dx \\ &\quad + \int \left(\int vf\,dv \right)^2 dx \\ &\quad - \int F(t,x) \int v^2 f\,dv\,dx \\ &\quad + \int \left(\int vg\,dv \right)^2 dx \\ &\quad - \int G(t,x) \int v^2 g\,dv\,dx \\ &\quad + \frac{1}{2} \int \rho(t,x) \left(\left[\int_{-\infty}^x F(t,y)\,dy \right]^2 - \left[\int_{-\infty}^x G(t,y)\,dy \right]^2 \right) dx = 0. \end{aligned}$$

The first line is bounded when integrated in time. The second and third lines are non-positive. Call L the last term above. Then because $\rho = \int (f-g)\,dv = F - G$ and $E = \int_{-\infty}^x \rho(t,y)\,dy = \int_{-\infty}^x (F-G)\,dy$, we get after a brief calculation

$$L = -\frac{1}{4} \int E^2 (F+G)\,dx.$$

Thus in particular

$$\int_0^\infty \int E^2 (F+G)\,dx\,dt < \infty$$

and

$$\int_0^\infty \int \left[F(t,x) \int v^2 f\, dv - \left(\int vf\, dv \right)^2 \right] dx\, dt < \infty$$

$$\int_0^\infty \int \left[G(t,x) \int v^2 g\, dv - \left(\int vg\, dv \right)^2 \right] dx\, dt < \infty.$$

We can use these inequalities directly to establish the L^4 estimate. Write

$$F(t,x) \int v^2 f\, dv - \left(\int vf\, dv \right)^2$$

as

$$\frac{1}{2} \int \int (w-v)^2 f(v) f(w)\, dv\, dw.$$

Then, from above we know that the quantity

$$k(t,x) \equiv \int \int (w-v)^2 f(t,x,v) f(t,x,w)\, dv\, dw$$

is integrable over all x, t. To get the L^4 bound we split the integral for $F(t,x)^2$ in the usual manner:

$$F(t,x)^2 = \int \int f(v) f(w)\, dv\, dw = \int_{|v-w|<R} + \int_{|v-w|>R} \equiv I_1 + I_2.$$

Therefore, $I_2 \le R^{-2} k(t,x)$, and in I_1

$$\int_{|v-w|<R} f(w)\, dw = \int_{v-R}^{v+R} f(w)\, dw \le cR.$$

Thus

$$I_1 \le c \cdot R \cdot F.$$

Set $R = k^{1/3} F^{-1/3}$. Then $F^4 \le ck$ so F^4 is integrable over all x, t. The result for G is exactly the same.

Our final results will show that for solutions to the classical VP system (1.1) and RVP system (1.3), the electric field E tends to 0 in the maximum norm. Here we consider the former only.

Theorem 2.1. Under the above assumptions consider solutions f, g to (1.1). Then

$$\int_0^\infty \|E(t)\|_\infty^3\, dt < \infty.$$

Proof. This will follow immediately from the result that

$$Q(t) := \int_{-\infty}^{\infty} E^2(t,x) \Big[F(t,x) + G(t,x) \Big]\, dx$$

is integrable in time. Indeed because $E_x = \rho = \int (f-g)\, dv = F - G$, we have

$$\frac{\partial}{\partial x} E^3 = 3E^2 \rho = 3E^2 (F-G).$$

Integrate in x to get

$$E^3(t,x) = \int_{-\infty}^{x} 3E^2(F-G)\,dx$$

so that

$$|E(t,x)|^3 \leq \int_{-\infty}^{\infty} 3E^2(F+G)\,dx = 3Q(t), \tag{2.3}$$

and the result follows as claimed. This can now be exploited to show that the electric field tends uniformly to 0 as $t \to \infty$.

THEOREM 2.2. Let the previous assumptions hold and consider solutions f, g to the classical (VP) system (1.1). Then

$$\lim_{t\to\infty} \|E(t)\|_\infty = 0.$$

Proof. We will show that

$$\lim_{t\to\infty} Q(t) = 0.$$

The conclusion will then follow from (2.3). Since $Q(t)$ is integrable over $[0,\infty)$, $\liminf Q(t) = 0$. If we can show that $\dot{Q}(t)$ is bounded, we may then conclude the statement of the theorem. (An alternate proof is given in [10].)

Using $E_x = \rho = F - G$ and $E_t = -j = -\int v(f-g)\,dv$ we compute

$$\frac{dQ}{dt} = -2\int jE(F+G)\,dx + 2\int \rho E \int v(f+g)\,dv\,dx.$$

Now, E is uniformly bounded, because by definition in (1.1)

$$|E(t,x)| \leq \int_{-\infty}^{x} (F+G)(t,x)\,dx \leq \int_{-\infty}^{\infty} (F+G)(t,x)\,dx \leq \text{const.},$$

where the last inequality follows by conservation of mass. Therefore

$$\left|\frac{dQ}{dt}\right| \leq c\int (F+G)\int |v|(f+g)\,dv\,dx.$$

Define e to be the kinetic energy density,

$$e(t,x) := \int v^2(f+g)\,dv.$$

Then by splitting the v integral into sets $|v| < R$ and its complement we get

$$\int |v|(f+g)\,dv \leq ce^{\frac{2}{3}}(t,x).$$

Similarly

$$F \equiv \int f\,dv \leq ce^{\frac{1}{3}}(t,x)$$

and therefore

$$\left|\frac{dQ}{dt}\right| \leq c\int (F+G)e^{\frac{2}{3}}\,dx \leq c\Big(\int (F+G)^3\,dx\Big)^{\frac{1}{3}} \leq \text{const.}$$

This concludes the proof. Complete details may be found in [10].

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