LARGE TIME BEHAVIOR OF THE RELATIVISTIC VLASOV MAXWELL SYSTEM IN LOW SPACE DIMENSION

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Abstract. When particle speeds are large the motion of a collisionless plasma is modeled by the relativistic Vlasov Maxwell system. Large time behavior of solutions which depend on one position variable and two momentum variables is considered. In the case of a single species of charge it is shown that there are solutions for which the charge density ($\rho = \int f dv$) does not decay in time. This is in marked contrast to results for the non-relativistic Vlasov Poisson system in one space dimension. The case when two oppositely charged species are present and the net total charge is zero is also considered. In this case, it is shown that the support in the first component of momentum can grow at most as $t^{\frac{3}{4}}$.

1. Introduction

Consider the relativistic Vlasov-Maxwell system:

$$\begin{cases} \partial_t f^\alpha + \hat{v}_1^\alpha \partial_x f^\alpha + e^\alpha \left(E_1 + \hat{v}_2^\alpha B\right) \partial_{v_1} f^\alpha + e^\alpha \left(E_2 - \hat{v}_1^\alpha B\right) \partial_{v_2} f^\alpha = 0, \\ \rho(t,x) = \int \sum_\alpha e^\alpha f^\alpha(t,x,v) dv, \quad j(t,x) = \int \sum_\alpha e^\alpha f^\alpha(t,x,v) \hat{v}^\alpha dv, \\ E_1(t,x) = \frac{1}{2} \int_{-\infty}^x \rho(t,y) dy - \frac{1}{2} \int_x^\infty \rho(t,y) dy, \\ \partial_t E_2 + \partial_x B = -j_2, \quad \partial_t B + \partial_x E_2 = 0, \end{cases} \quad (1.1)$$

for $\alpha = 1, \ldots, N$. Here, $t \geq 0$ is time, $x \in \mathbb{R}$ is the first component of position, and $v = (v_1, v_2) \in \mathbb{R}^2$ contains the first two components of momentum. Hence, $dv = dv_2 dv_1$ and the v integrals are understood to be over $\mathbb{R}^2$. f^α gives the number density in phase space of particles of mass m^α and charge e^α. Velocity is given by

$$\hat{v}^\alpha = \frac{v}{\sqrt{(m^\alpha)^2 + |v|^2}} ,$$

where the speed of light has been normalized to one. The effects of collisions are neglected.

The initial conditions

$$\begin{cases} f^\alpha(0, x, v) = f_0^\alpha(x, v) \geq 0, & \alpha = 1, \ldots, N, \\ E_2(0, x) = E_{20}(x), & B(0, x) = B_0(x), \end{cases}$$

are given where it is assumed throughout the paper that $f_0^\alpha \in C_0^1(\mathbb{R}^3)$ is nonnegative and compactly supported and that $E_{20}, B_0 \in C_0^1(\mathbb{R})$ are compactly supported. When the neutrality condition,

$$\iint \sum_\alpha e^\alpha f_0^\alpha \, dv \, dx = 0,$$

holds, we will refer to this as the neutral case. A major goal of this paper is to compare the neutral case with the monocharge case, which may be obtained from (1.1) by setting $N = 1$. In the monocharge case we will drop α and write, for example, $f = f^\alpha = f^1$ and take $e^\alpha = e^1 = 1$, $m^\alpha = m^1 = 1$.

Choose C_0 such that f_0^α, E_{20}, B_0 vanish (for all α) if $|x| \geq C_0$. The letter C will denote a positive generic constant which may depend on the initial data (but not t, x, v) and may change from line to line, whereas a numbered constant (such as C_0) has a fixed value. We also define the characteristics, $(X^\alpha(s, t, x, v), V^\alpha(s, t, x, v))$, of f^α by

$$\begin{cases} \dfrac{dX^\alpha}{ds} = \hat{V}_1^\alpha, & X^\alpha(t, t, x, v) = x \\[2mm] \dfrac{dV_1^\alpha}{ds} = e^\alpha \left(E_1(x, X^\alpha) + \hat{V}_2^\alpha B(s, X^\alpha) \right), & V_1^\alpha(t, t, x, v) = v_1 \\[2mm] \dfrac{dV_2^\alpha}{ds} = e^\alpha \left(E_2(s, X^\alpha) - \hat{V}_1^\alpha B(s, X^\alpha) \right), & V_2^\alpha(t, t, x, v) = v_2. \end{cases} \qquad (1.2)$$

Theorem 1.1. *In the neutral case there is a constant, C, such that*

$$C \geq \int_0^t \left[E_1^2 + (E_2 - B)^2 + \int \sum_\alpha f^\alpha \left(\sqrt{(m^\alpha)^2 + |v|^2} - v_1 \right) dv \right] \Big|_{(\tau, x-t+\tau)} d\tau$$

$$+ \int_0^t \left[E_1^2 + (E_2 + B)^2 + \int \sum_\alpha f^\alpha \left(\sqrt{(m^\alpha)^2 + |v|^2} + v_1 \right) dv \right] \Big|_{(\tau, x+t-\tau)} d\tau, \tag{1.3}$$

for all $t \geq 0, x \in \mathbb{R}$. In the monocharge case there is a constant, C, such that

$$C(C_0 + t - x) \geq \int_0^t \left[(E_2 - B)^2 + f \frac{(\sqrt{1 + |v|^2} - v_1)^2}{\sqrt{1 + |v|^2}} dv \right] \Big|_{(\tau, x-t+\tau)} d\tau \tag{1.4}$$

for $x < C_0 + t$ and,

$$C(C_0 + t + x) \geq \int_0^t \left[(E_2 + B)^2 + f \frac{(\sqrt{1 + |v|^2} + v_1)^2}{\sqrt{1 + |v|^2}} dv \right] \Big|_{(\tau, x+t-\tau)} d\tau \tag{1.5}$$

for $-C_0 - t < x$.

The proof of this theorem relies on conservation of energy and of momentum (in the monocharge case) and is contained in Section 2.

Theorem 1.2. *In the neutral case there is a constant, C, such that*

$$|v_2| \leq C + C\sqrt{t - |x| + C_0},$$

on the support of f^α for every α. In the monocharge case there is a positive constant, C, such that

$$|v_2| \leq C + C\sqrt{(t + C_0)^2 - x^2},$$

on the support of f.

The proof of Theorem 1.2 is in Section 3.

Theorem 1.3. *There are solutions of the monocharge problem for which there exist $x_0 \in \mathbb{R}$ and $C > 0$ such that*

$$\int_{x_0+t}^\infty \rho(t, x) dx > C, \tag{1.6}$$

for all $t \geq 0$. Furthermore, there exists $C > 0$ such that

$$\|\rho(t, \cdot)\|_{L^p(\mathbb{R})} > C,$$

for all $t \geq 0$ and $p \in [1, \infty]$.

The second assertion of Theorem 1.3 follows from (1.6) by using Hölder's inequality:

$$C < \int_{x_0+t}^{C_0+t} \rho(t, x) dx \leq \|\rho(t, \cdot)\|_{L^p(\mathbb{R})} (C_0 - x_0)^{1 - \frac{1}{p}}.$$

The proof of Theorem 1.3 is contained in Section 4. In [8] an analogous, but more detailed, result is obtained for the relativistic Vlasov Poisson system (which may be obtained from (1.1) by setting $E_2 = B = 0$).

Theorem 1.4. *For the neutral problem there is a constant, C, such that*

$$|v_1| \leq C + Ct^{\frac{1}{2}}(t - |x| + 2C_0)^{\frac{1}{4}}, \tag{1.7}$$

on the support of f^α for every α.

The proof is in Section 5. A similar, but different, estimate is obtained in [8] for the relativistic Vlasov Poisson system. Also, note that (1.7) rules out an estimate like (1.6). If (1.6) held, then there would be characteristics for which $f^\alpha \neq 0$ and

$$X^\alpha(t, 0, x, v) \geq x_0 + t, \tag{1.8}$$

for all $t \geq 0$. Then by (1.7) and (1.8)

$$|V_1^\alpha(t, 0, x, v)| \leq C + Ct^{\frac{1}{2}},$$

so

$$
\begin{aligned}
1 - \hat{V}_1^\alpha(t, 0, x, v) &= \frac{1 + (V_2^\alpha)^2}{\sqrt{1 + |V^\alpha|^2}(\sqrt{1 + |V^\alpha|^2} + V_1^\alpha)} \\
&\geq \frac{1}{2(1 + |V^\alpha|^2)} \geq \frac{C}{1 + t},
\end{aligned}
$$

and

$$C \ln(1 + t) \leq \int_0^t \left(1 - \hat{V}_1^\alpha(s, 0, x, v)\right) ds = t - X^\alpha(t, 0, x, v) + x \leq x - x_0$$

for all $t \geq 0$.

Finally, Section 6 contains the proof of the following.

Theorem 1.5. *In both the neutral and monocharge cases there are no non-trivial steady solutions with f^α, E_2 and B compactly supported.*

The global existence in time of smooth solutions to (1.1) is shown in [9] when a neutralizing background density is included. Adaptation of the essential estimate from [9] to the current situation is briefly discussed in Section 2. Global existence has been shown in two dimensions, [11], and two and one-half dimensions, [10], but is open for large data in three dimensions. Some time decay is known for the classical Vlasov Poisson system in three dimensions ([12], [14], [15]). Additionally there are time decay results for the classical Vlasov Poisson system in one dimension ([1], [2], [7], [17]). For decay results on the relativistic Vlasov Poisson system, see [7] and [13]. References

[3], [4], and [5] are also mentioned since they deal with time dependent rescalings and time decay for other kinetic equations. We also cite [6] and [16] as general references on mathematical kinetic theory.

A main point to this article is that the non-decay stated in Theorem 1.3 is in marked contrast to the decay found in [1], [2], and [17]. In [1], [2] and [17] the problem studied is non-relativistic. Hence, there is no a priori upper bound on particle speed and this leads to dispersion. In this paper (and also [8]), particle speeds are bounded by the speed of light and this limits the dispersion.

2. Conservation Laws

Define

$$e = \int \sum_\alpha f^\alpha \sqrt{(m^\alpha)^2 + |v|^2} dv + \frac{1}{2}|E|^2 + \frac{1}{2}B^2,$$

$$m = \int \sum_\alpha f^\alpha v_1 dv + E_2 B,$$

$$\ell = \int \sum_\alpha f^\alpha v_1 \hat{v}_1^\alpha dv - \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 + \frac{1}{2}B^2.$$

A short computation reveals that

$$\partial_t e + \partial_x m = 0, \tag{2.1}$$

and

$$\partial_t m + \partial_x \ell = 0. \tag{2.2}$$

Using (2.1), the divergence theorem yields

$$0 = \int_0^t \int_{x-t+\tau}^{x+t-\tau} (\partial_\tau e + \partial_y m) \, dy \, d\tau \tag{2.3}$$

$$= \int_0^t (e+m)\Big|_{(\tau,x+t-\tau)} d\tau + \int_0^t (e-m)\Big|_{(\tau,x-t+\tau)} d\tau - \int_{x-t}^{x+t} e(0,y) dy.$$

Note that

$$e \pm m = \int \sum_\alpha f^\alpha \left(\sqrt{(m^\alpha)^2 + |v|^2} \pm v_1 \right) dv + \frac{1}{2}E_1^2 + \frac{1}{2}(E_2 \pm B)^2 \geq 0,$$

and that

$$|j_2| \leq \int \sum_\alpha f^\alpha \frac{|v_2|}{\sqrt{(m^\alpha)^2 + |v|^2}} \, dv \leq C \int \sum_\alpha f^\alpha \left(\sqrt{(m^\alpha)^2 + |v|^2} \pm v_1 \right) \, dv.$$

In the neutral case (2.3) yields

$$C \geq \int_{x-t}^{x+t} e(0,y)dy = \int_0^t (e+m)\Big|_{(\tau,x+t-\tau)} d\tau + \int_0^t (e-m)\Big|_{(\tau,x-t+\tau)} d\tau. \quad (2.4)$$

In the monocharge case, since E_1 is not compactly supported, (2.3) only yields

$$Ct \geq \int_0^t (e+m)\Big|_{(\tau,x+t-\tau)} d\tau + \int_0^t (e-m)\Big|_{(\tau,x-t+\tau)} d\tau.$$

It follows that

$$|E_2| + |B| \leq C + C \int_0^t |j_2(\tau, x+t-\tau)| \, d\tau + C \int_0^t |j_2(\tau, x-t+\tau)| \, d\tau$$
$$\leq C + Ct^p, \quad (2.5)$$

where $p = 0$ in the neutral case and $p = 1$ in the monocharge case. Global existence of smooth solutions follows in both cases as in [9].

Consider the monocharge case now. Bounds independent of t may be obtained by also using (2.2). For $x_0 < C_0$ the divergence theorem yields

$$0 = \int_0^t \int_{x_0+\tau}^{C_0+\tau} [\partial_\tau(e-m) + \partial_y(m-\ell)] \, dy \, d\tau$$
$$= \int_0^t [e - 2m + \ell]\Big|_{(\tau,x_0+\tau)} d\tau + \int_{x_0+t}^{C_0+t} (e-m)\Big|_{(t,y)} dy$$
$$- \int_0^t [e - 2m + \ell]\Big|_{(\tau,C_0+\tau)} d\tau - \int_{x_0}^{C_0} (e-m)\Big|_{(0,y)} dy.$$

Note that

$$e - 2m + \ell = \int \sum_\alpha f^\alpha \frac{(\sqrt{(m^\alpha)^2 + |v|^2} - v_1)^2}{\sqrt{(m^\alpha)^2 + |v|^2}} dv + (E_2 - B)^2$$

is nonnegative and vanishes on $y = C_0 + \tau$ (since E_1 canceled). Hence,

$$C(C_0 - x_0) \geq \int_{x_0}^{C_0} (e-m)\Big|_{(0,y)} dy \quad (2.6)$$
$$= \int_0^t (e - 2m + \ell)\Big|_{(\tau,x_0+\tau)} d\tau + \int_{x_0+t}^{C_0+t} (e-m)\Big|_{(t,y)} dy.$$

Similarly,

$$e + 2m + \ell = \int \sum_\alpha f^\alpha \frac{(\sqrt{(m^\alpha)^2 + |v|^2} + v_1)^2}{\sqrt{(m^\alpha)^2 + |v|^2}} dv + (E_2 + B)^2$$

is nonnegative and vanishes on $y = -C_0 - \tau$ and

$$0 = \int_0^t \int_{-C_0-\tau}^{x_0-\tau} [\partial_\tau(e+m) + \partial_y(m+\ell)] \, dy \, d\tau;$$

this leads to

$$C(x_0 + C_0) \geq \int_{-C_0}^{x_0} (e+m)\Big|_{(0,y)} dy \tag{2.7}$$

$$= \int_0^t (e + 2m + \ell)\Big|_{(\tau,x_0-\tau)} d\tau + \int_{-C_0-t}^{x_0-t} (e+m)\Big|_{(t,y)} dy,$$

for $x_0 > -C_0$. Theorem 1.1 now follows from (2.4), (2.6), and (2.7).

3. Bounds on v_2 Support

Define

$$A(t,x) = \int_{-\infty}^{x} B(t,y)dy,$$

and note that

$$\partial_t A + \partial_x A = -(E_2 - B),$$

$$\partial_t A - \partial_x A = -(E_2 + B),$$

so

$$A(t,x) = A(0, x-t) - \int_0^t (E_2 - B)\Big|_{(\tau,x-t+\tau)} d\tau \tag{3.1}$$

$$= A(0, x+t) - \int_0^t (E_2 + B)\Big|_{(\tau,x+t-\tau)} d\tau.$$

For $|x| \geq C_0 + t$, $|A(t,x)| = |A(0, x-t)| \leq C$, so consider $|x| < C_0 + t$. Then (3.1) becomes

$$A(t,x) = A(0, x-t) - \int_{\max(0,\frac{t-x-C_0}{2})}^{t} (E_2 - B)\Big|_{(\tau,x-t+\tau)} d\tau$$

$$= A(0, x+t) - \int_{\max(0,\frac{x+t-C_0}{2})}^{t} (E_2 + B)\Big|_{(\tau,x+t-\tau)} d\tau.$$

In the neutral case, (1.3) and the Cauchy Schwartz inequality yield

$$|A(t,x)| \leq C + \sqrt{t - \max(0, \frac{t-x-C_0}{2})}\sqrt{C} \leq C + C\sqrt{t+x+C_0},$$

and

$$|A(t,x)| \leq C + \sqrt{t - \max(0, \frac{x + t - C_0}{2})}\sqrt{C} \leq C + C\sqrt{t - x + C_0}.$$

Hence,

$$|A(t,x)| \leq C + C\sqrt{t - |x| + C_0} \tag{3.2}$$

follows. For the monocharge case, (1.4) is used in place of (1.3) to obtain

$$|A(t,x)| \leq C + \sqrt{t - \max(0, \frac{t - x - C_0}{2})}\sqrt{C(C_0 + t - x)} \tag{3.3}$$

$$\leq C + C\sqrt{(t + C_0)^2 - x^2}.$$

From (1.2) we have $f^\alpha(s, X^\alpha(s,t,x,v), V^\alpha(s,t,x,v)) = f^\alpha(t,x,v)$ and

$$V_2^\alpha(s,t,x,v) + e^\alpha A(s, X^\alpha(s,t,x,v)) = v_2 + e^\alpha A(t,x),$$

for all s,t,x,v. If $f^\alpha(t,x,v) \neq 0$, then

$$|v_2 + e^\alpha A(t,x)| = |V_2^\alpha(0,t,x,v) + e^\alpha A(0, X^\alpha(0,t,x,v))| \leq C.$$

In the neutral case, (3.2) yields

$$|v_2| \leq C + C\sqrt{t - |x| + C_0}. \tag{3.4}$$

In the monocharge case, (3.3) yields

$$|v_2| \leq C + C\sqrt{(t + C_0)^2 - x^2} \tag{3.5}$$

on the support of f^α.

Theorem 1.2 follows from (3.4) and (3.5) but we make one further observation. On the support of f^α $|v_2 + e^\alpha A(t,x)| \leq C$ so $v_2 \in (-e^\alpha A(t,x) - C, -e^\alpha A(t,x) + C)$. Thus, the v_2 support has bounded measure.

4. NON-DECAY OF ρ IN THE MONOCHARGE CASE

In this section only the monocharge case is considered. Let

$$M = \int \rho(t,x)dx,$$

and note that

$$E_1 = \frac{1}{2}\int_{-\infty}^{x} \rho\,dy - \frac{1}{2}\int_{x}^{\infty} \rho\,dy = \frac{1}{2}M - \int_{x}^{C_0 + t} \rho\,dy.$$

For some $x_0 \in (-C_0, C_0)$ define

$$\mu(t) = \int_{x_0 + t}^{C_0 + t} \rho(t,y)dy \quad \text{and} \quad \mathcal{E}(t) = \int_{x_0 + t}^{C_0 + t} (e - m)\Big|_{(t,y)} dy.$$

Then

$$\mu'(t) = j_1(t, x_0 + t) - \rho(t, x_0 + t) \leq 0,$$

and by (2.1) and (2.2)

$$\begin{aligned}
\mathcal{E}'(t) &= -\int_{x_0+t}^{C_0+t} \partial_y(m - \ell) dy + (e - m)\Big|_{(t,C_0+t)} - (e - m)\Big|_{(t,x_0+t)} \\
&= (e - 2m + \ell)\Big|_{(t,C_0+t)} - (e - 2m + \ell)\Big|_{(t,x_0+t)} \\
&= -(e - 2m + \ell)\Big|_{(t,x_0+t)} \leq 0.
\end{aligned}$$

Suppose that

$$\frac{1}{2}M \geq \mu(0), \tag{4.1}$$

and

$$\frac{1}{2}(C_0 - x_0)(\frac{1}{2}M)^2 > \mathcal{E}(0). \tag{4.2}$$

Then, for $y \geq x_0 + t$,

$$E_1(t, y) \geq E_1(t, x_0 + t) = \frac{1}{2}M - \mu(t) \geq \frac{1}{2}M - \mu(0) \geq 0,$$

so

$$\mathcal{E}(0) \geq \mathcal{E}(t) \geq \frac{1}{2}\int_{x_0+t}^{C_0+t} E_1^2 dy \geq \frac{1}{2}(C_0 - x_0)(\frac{1}{2}M - \mu(t))^2,$$

and hence

$$\sqrt{\frac{2\mathcal{E}(0)}{C_0 - x_0}} \geq \frac{1}{2}M - \mu(t),$$

and

$$\int_{x_0+t}^{C_0+t} \rho\, dy = \mu(t) \geq \frac{1}{2}M - \sqrt{\frac{2\mathcal{E}(0)}{C_0 - x_0}} > 0.$$

Hence, Theorem 1.3 follows from (4.1) and (4.2).

To see that there are initial conditions for which (4.1) and (4.2) hold consider the following: Let $f_0^L, f_0^R \in C_0^1(\mathbb{R}^3)$ be nonnegative and compactly supported with

$$f_0^L(x, v) = 0 \quad \text{if } x \geq -1,$$
$$f_0^R(x, v) = 0 \quad \text{if } x \notin (-1, 0),$$

and

$$\frac{1}{2}\iint f_0^L dv\, dx \geq \iint f_0^R dv\, dx > 0.$$

Let
$$C_0 = \sup \left\{ |x| : f_0^L(x,v) \neq 0 \text{ for some } v \right\},$$
and
$$f(0,x,v) = f_0^L(x,v) + f_0^R(x - C_0, v_1 - W, v_2),$$
for $W > 1$. Taking $x_0 = -1$ we have
$$\mu(0) = \iint f_0^R dv\, dx \leq \frac{1}{2} M,$$
which is (4.1). Taking
$$E_2(0,y) = B(0,y) = 0$$
(and using $x_0 = -1$), we have
$$
\begin{aligned}
\mathcal{E}(0) &= \int_{x_0}^{C_0} \left[\int f(\sqrt{1+|v|^2} - v_1) dv + \frac{1}{2} E_1^2 \right] \Big|_{(0,y)} dy \\
&= \int_{x_0}^{C_0} \left[\int f_0^R(y - C_0, v_1 - W, v_2) \frac{1 + v_2^2}{\sqrt{1+|v|^2} + v_1} dv + \frac{1}{2} E_1^2 \right] dy \\
&\leq \frac{C}{W} + \frac{1}{2} \int_{x_0}^{C_0 - 1} \left(\frac{1}{2} M - \mu(0) \right)^2 dy + \frac{1}{2} \int_{C_0 - 1}^{C_0} (\frac{1}{2} M)^2 dy \\
&= \frac{C}{W} + \frac{C_0 - 1 - x_0}{2} \left(\frac{M^2}{4} - M\mu(0) + \mu^2(0) \right) + \frac{1}{8} M^2 \\
&= \frac{C}{W} + \frac{C_0}{2} \left(\frac{M^2}{4} - M\mu(0) + \mu^2(0) \right) - \frac{x_0}{8} M^2 \\
&= \frac{C}{W} + \frac{C_0 - x_0}{8} M^2 - \frac{C_0}{2} \mu(0)(M - \mu(0)).
\end{aligned}
$$
Now, taking W sufficiently large yields (4.2) completing the proof.

5. BOUNDS ON v_1 SUPPORT IN THE NEUTRAL CASE

In this section we consider only the neutral case. Define
$$k = \int \sum_\alpha f^\alpha \sqrt{(m^\alpha)^2 + |v|^2} dv,$$
and
$$\sigma_\pm = \int \sum_\alpha f^\alpha \left(\sqrt{(m^\alpha)^2 + |v|^2} \pm v_1 \right) dv.$$
Then (2.1) yields
$$\int k dx \leq \int e dx = \int e(0,x) dx = C.$$

Also (1.3) yields

$$\int_0^t \left[\sigma_-(\tau, x - t + \tau) + \sigma_+(\tau, x + t - \tau) \right] d\tau \leq C.$$

These bounds are used in the following.

Lemma 5.1. *For all $t \geq 0$ and $x \in \mathbb{R}$*

$$\int \sum_\alpha f^\alpha dv \leq C\sqrt{k\sigma_-}, \tag{5.1}$$

and

$$\int \sum_\alpha f^\alpha dv \leq C\sqrt{k\sigma_+}. \tag{5.2}$$

Proof. We will show (5.1); the proof of (5.2) is similar. For any $R \geq 0$

$$\int \sum_\alpha f^\alpha dv \leq \int_{|v| \leq R} \sum_\alpha f^\alpha \, dv + \frac{Ck}{\sqrt{1 + R^2}}.$$

For $|v| \leq R$,

$$\sqrt{(m^\alpha)^2 + |v|^2} - v_1 = \frac{(m^\alpha)^2 + v_2^2}{\sqrt{(m^\alpha)^2 + |v|^2} + v_1} \geq \frac{(m^\alpha)^2}{2\sqrt{(m^\alpha)^2 + |v|^2}} \geq \frac{C}{\sqrt{1 + R^2}}$$

so

$$\int \sum_\alpha f^\alpha dv \leq \int_{|v| \leq R} \sum_\alpha f^\alpha C\sqrt{1 + R^2} \left(\sqrt{(m^\alpha)^2 + |v|^2} - v_1 \right) dv + \frac{Ck}{\sqrt{1 + R^2}}$$

$$\leq C\sqrt{1 + R^2}\sigma_- + \frac{Ck}{\sqrt{1 + R^2}}.$$

If $0 < \sigma_- \leq k$, taking $R = \sqrt{\frac{k}{\sigma_-} - 1}$ leads to (5.1).

If $k < \sigma_-$, then

$$\int \sum_\alpha f^\alpha dv < Ck < C\sqrt{k\sigma_-},$$

and if $\sigma_- = 0$, then

$$\int \sum_\alpha f^\alpha dv = \sqrt{k\sigma_-} = 0.$$

In all cases (5.1) holds so the proof is complete. $\qquad\qquad\square$

Consider a characteristic

$$(X(s), V(s)) = (X^\alpha(s, 0, \overline{x}, \overline{v}), V^\alpha(s, 0, \overline{x}, \overline{v})),$$

of f^α (defined in (1.2)) along which $f^\alpha(s, X(s), V(s)) \neq 0$. The idea of the following estimate is that as long as V_1 is large, the integration

$$V_1(t) = V_1(t - \Delta) + \int_{t-\Delta}^{t} e^\alpha \left(E_1 + \hat{V}_2(s)B\right)\Big|_{(s, X(s))} ds$$

is nearly integration on a light cone and (1.3) can be used to obtain an improved estimate. Define

$$C_1 = \sup \left\{ |v_1| : \exists t \in [0, 1], \ x \in \mathbb{R}, v_2 \in \mathbb{R} \text{ with } \sum_\alpha f^\alpha(t, x, v) \neq 0 \right\},$$

and suppose that $t > 0$ and $V_1(t) > 2C_1$. Define

$$\Delta = \sup \left\{ \tau \in (0, t] : V_1(s) \geq \frac{1}{2} V_1(t) \text{ for all } s \in [t - \tau, t] \right\}.$$

Note that

$$V_1(t - \Delta) \geq \frac{1}{2} V_1(t) > C_1,$$

so $t - \Delta > 1$ and

$$V_1(t - \Delta) = \frac{1}{2} V_1(t) \tag{5.3}$$

follows. Define $X_C(s) = X(t) + s - t$. Using Theorem 1.2, we have

$$\left| \frac{d}{ds} (X_C(s) - X(s)) \right| = 1 - \hat{V}_1(s)$$

$$= \frac{(m^\alpha)^2 + V_2^2(s)}{\sqrt{(m^\alpha)^2 + |V(s)|^2} \left(\sqrt{(m^\alpha)^2 + |V(s)|^2} + V_1(s) \right)}$$

$$\leq \frac{C + C\left(s - |X(s)| + C_0\right)}{V_1^2(s)}.$$

Since $s - |X(s)|$ is increasing, for $t - \Delta \leq s \leq t$ we have

$$\left| \frac{d}{ds} (X_C(s) - X(s)) \right| \leq \frac{C + C\left(t - |X(t)| + C_0\right)}{\left(\frac{1}{2} V_1(t)\right)^2},$$

and hence

$$|X_C(s) - X(s)| \leq \frac{C\Delta(t - |X(t)| + 2C_0)}{V_1^2(t)}. \tag{5.4}$$

By (1.3), the Cauchy Schwartz inequality, and (5.1), we have

$$\Big| \int_{t-\Delta}^{t} E_1(s, X(s)) ds \Big|$$

$$= \Big| \int_{t-\Delta}^{t} E_1(s, X_C(s)) ds + \int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} \int \sum_\alpha e^\alpha f^\alpha dv \, dx \, ds \Big|$$

$$\leq C\sqrt{\Delta} + \int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} C\sqrt{k\sigma_-} dx \, ds.$$

Now

$$\int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} k dx \, ds \leq C\Delta,$$

and letting

$$S(t) = \frac{C\Delta(t - |X(t)| + 2C_0)}{V_1^2(t)},$$

(5.4) and (1.3) yield

$$\int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} \sigma_- dx \, ds \leq \int_{t-\Delta}^{t} \int_{X_C(s)}^{X_C(s)+S(t)} \sigma_- dx \, ds$$

$$= \int_{t-\Delta}^{t} \int_{X(t)-t}^{X(t)-t+S(t)} \sigma_-(s, y+s) dy \, ds$$

$$= \int_{X(t)-t}^{X(t)-t+S(t)} \int_{t-\Delta}^{t} \sigma_-(s, y+s) ds \, dy \leq CS(t).$$

Hence, the Cauchy Schwartz inequality yields

$$\int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} C\sqrt{k\sigma_-} \, dx \, ds \leq C\sqrt{\Delta S(t)},$$

and hence

$$\Big| \int_{t-\Delta}^{t} E_1(s, X(s)) ds \Big| \leq C\sqrt{\Delta} + C\sqrt{\Delta S(t)}. \tag{5.5}$$

Next consider

$$\int_{t-\Delta}^{t} \hat{V}_2(s) B(s, X(s)) ds.$$

Using Theorem 1.2 we have, for $t - \Delta \leq s \leq t$,

$$|\hat{V}_2(s)| \leq C\frac{|V_2(s)|}{V_1(s)} \leq \frac{C + C\sqrt{s - |X(s)| + C_0}}{V_1(s)} \leq \frac{C\sqrt{t - |X(t)| + 2C_0}}{\frac{1}{2}V_1(t)}.$$

Hence, by (2.5)

$$\int_{t-\Delta}^{t} \left| \hat{V}_2(s)B(s, X(s)) \right| ds \leq \frac{C\Delta\sqrt{t - |X(t)| + 2C_0}}{V_1(t)}. \qquad (5.6)$$

Collecting (5.5) and (5.6) yields

$$V_1(t) = V_1(t - \Delta) + \int_{t-\Delta}^{t} e^{\alpha} \left(E_1 + \hat{V}_2(s)B \right) \Big|_{(s, X(s))} ds$$

$$\leq V_1(t - \Delta) + C\sqrt{\Delta} + C\frac{\Delta\sqrt{t - |X(t)| + 2C_0}}{V_1(t)},$$

and with (5.3) this becomes

$$V_1(t) \leq C\sqrt{\Delta} + C\frac{\Delta\sqrt{t - |X(t)| + 2C_0}}{V_1(t)}.$$

Hence,

$$V_1^2(t) - C\sqrt{\Delta}V_1(t) \leq C\Delta\sqrt{t - |X(t)| + 2C_0},$$

$$\left(V_1(t) - \frac{C\sqrt{\Delta}}{2} \right)^2 \leq \Delta \left(C\sqrt{t - |X(t)| + 2C_0} + \frac{C^2}{4} \right),$$

and

$$V_1(t) \leq \frac{C\sqrt{\Delta}}{2} + \sqrt{\Delta \left(C\sqrt{t - |X(t)| + 2C_0} + \frac{C^2}{4} \right)}$$

$$\leq C\sqrt{\Delta} \left(1 + (t - |X(t)| + 2C_0)^{\frac{1}{4}} \right) \leq Ct^{\frac{1}{2}}(t - |X(t)| + 2C_0)^{\frac{1}{4}}.$$

Similar estimates may be derived if $V_1(t) < -2C_1$, thus

$$|V_1(t)| \leq 2C_1 + Ct^{\frac{1}{2}}(t - |X(t)| + 2C_0)^{\frac{1}{4}}$$

in all cases. Theorem 1.4 follows.

6. NONEXISTENCE OF STEADY STATES

Consider the monocharge case first. The dilation identity is

$$\frac{d}{dt} \left(\iint fxv_1 dv\, dx + \int xE_2 B dx \right) = \iint f\left(v_1\hat{v}_1 + x\left(E_1 + \hat{v}_2 B \right) \right) dv\, dx$$

$$+ \int x \left[\left(-\partial_x B - j_2 \right) B + E_2 \left(-\partial_x E_2 \right) \right] dx$$

$$= \iint fv_1\hat{v}_1 dv\, dx + \int x\left(\rho E_1 + j_2 B \right) dx - \int x \left[\partial_x\left(\frac{B^2 + E_2^2}{2} \right) + j_2 B \right] dx$$

$$= \iint f v_1 \hat{v}_1 dv \, dx + \int x \rho E_1 dx + \frac{1}{2} \int \left(B^2 + E_2^2 \right) dx.$$

Let

$$M = \iint f dv \, dx,$$

then for $R > C_0 + t$, we have

$$\frac{-M}{2} = E_1(t, -R) \leq E_1(t, x) \leq E_1(t, R) = \frac{M}{2},$$

for all x. Hence

$$\int x \rho E_1 dx = \frac{1}{2} \int_{-R}^{R} x \partial_x E_1^2 dx = \frac{1}{2} \left(R \left(\frac{M}{2} \right)^2 - (-R) \left(\frac{M}{2} \right)^2 - \int_{-R}^{R} E_1^2 dx \right)$$

$$= \frac{1}{2} \int_{-R}^{R} \left(\left(\frac{M}{2} \right)^2 - E_1^2 \right) dx \geq 0.$$

Hence, for f not identically zero,

$$\frac{d}{dt} \left(\iint f x v_1 dv \, dx + \int x E_2 B dx \right) \geq \iint f v_1 \hat{v}_1 dv \, dx > 0,$$

and f cannot be a steady solution.

Next, consider a steady solution in the neutral case. Note that from (1.1) we have $\partial_x E_2 = 0$ so $E_2 = 0$ for all x follows. Next note that

$$\frac{d}{dx} \left(\int \sum_\alpha f^\alpha v_1 \hat{v}_1^\alpha dv - \frac{1}{2} E_1^2 + \frac{1}{2} B^2 \right)$$

$$= \int v_1 \sum_\alpha \hat{v}_1^\alpha \partial_x f^\alpha dv - \rho E_1 - j_2 B$$

$$= - \int v_1 \sum_\alpha e^\alpha \left[(E_1 + \hat{v}_2^\alpha B) \, \partial_{v_1} f^\alpha + (E_2 - \hat{v}_1^\alpha B) \, \partial_{v_2} f^\alpha \right] dv - \rho E_1 - j_2 B$$

$$= \int \sum_\alpha e^\alpha f^\alpha \left(E_1 + \hat{v}_2^\alpha B \right) dv - \rho E_1 - j_2 B = 0,$$

and hence

$$2 \int \sum_\alpha f^\alpha v_1 \hat{v}_1^\alpha dv = E_1^2 - B^2, \tag{6.1}$$

for all x. If $E_1(x) = 0$ for some x then, since $f^\alpha \geq 0$,

$$\int \sum_\alpha f^\alpha v_1 \hat{v}_1^\alpha dv = 0 \tag{6.2}$$

follows and thus $B(x) = 0$ and $f^\alpha(x, v) = 0$ for all v. Suppose $E_1(x_0) \neq 0$ for some x_0. A contradiction will be derived from this and the proof will be complete.

Choose $a < x_0$ and $b > x_0$ such that $E_1(x) \neq 0$ on (a, b) and $E_1(a) = E_1(b) = 0$. Consider $E_1(x) > 0$ on (a, b). Choose $d \in (a, b)$ with

$$0 < E_1'(d) = \int \sum_\alpha e^\alpha f^\alpha(d, v) dv.$$

Choose $\alpha \in \{1, \ldots, N\}$ and $w \in \mathbb{R}^2$ such that $f^\alpha(d, w) > 0$ and $e^\alpha > 0$. By continuity we may take $w_1 \neq 0$. Let

$$(X(s), V(s)) = (X^\alpha(s, 0, d, w), V^\alpha(s, 0, d, w)) .$$

If $w_1 > 0$ define

$$T = \sup \left\{ t > 0 : V_1(s) \geq 0 \text{ and } X(s) \leq b \text{ for all } s \in [0, t] \right\} .$$

On $[0, T), X(s) \in [a, b]$ so $E_1(X(s)) \geq 0$. From (6.1) it follows that

$$|B(X(s))| \leq E_1(X(s)),$$

and hence that

$$\dot{V}_1(s) = e^\alpha(E_1(X(s)) + \hat{V}_2(s)B(X(s))) \geq 0,$$

and $V_1(s) \geq w_1 > 0$. It follows that T is finite and that $X(T) = b$. Hence,

$$f^\alpha(b, V(T)) = f^\alpha(d, w) > 0,$$

which contradicts (6.2). If $w_1 < 0$ define

$$T = \inf \left\{ t < 0 : V_1(s) \leq 0 \text{ and } X(s) \leq b \text{ for all } s \in [t, 0] \right\} .$$

It may be shown that T is finite and that $X(T) = b$, which again contradicts (6.2).

A contradiction may be reached in a similar manner if $E_1 < 0$ on (a, b) so the proof is complete.

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