A ONE-DIMENSIONAL KINETIC MODEL OF PLASMA DYNAMICS WITH A TRANSPORT FIELD

Charles Nguyen

Department of Mathematics
University of Texas at Arlington
Arlington, TX 76019, USA

Stephen Pankavich

Department of Applied Mathematics and Statistics
Colorado School of Mines
Golden, CO 80002, USA

(Communicated by Petronela Radu)

Abstract. Motivated by the fundamental model of a collisionless plasma, the Vlasov-Maxwell (VM) system, we consider a related, nonlinear system of partial differential equations in one space and one momentum dimension. As little is known regarding the regularity properties of solutions to the non-relativistic version of the (VM) equations, we study a simplified system which also lacks relativistic velocity corrections and prove local-in-time existence and uniqueness of classical solutions to the Cauchy problem. For special choices of initial data, global-in-time existence of these solutions is also shown. Finally, we provide an estimate which, independent of the choice of initial data, yields additional global-in-time regularity of the associated field.

1. **Introduction.** A plasma is a partially or completely ionized gas. Such a form of matter occurs if the velocity of individual particles in a material achieves an enormous magnitude, perhaps a sizable fraction of the speed of light. Plasmas are widely used in solid state physics since they are great conductors of electricity due to their free-flowing abundance of ions and electrons. When a plasma is of low density or the time scales of interest are sufficiently small, it is deemed to be "collisionless", as collisions between particles become infrequent. Many examples of collisionless plasmas occur in nature, including the solar wind, galactic nebulae, the Van Allen radiations belts, and comet tails.

The fundamental equations which describe the time evolution of a collisionless plasma are given by the Vlasov-Maxwell system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0 \\ \rho(t, x) = \int f(t, x, v) \ dv, \quad j(t, x) = \int v f(t, x, v) \ dv \\ \partial_t E = \nabla \times B - j, \qquad \nabla \cdot E = \rho \\ \partial_t B = -\nabla \times E, \qquad \nabla \cdot B = 0. \end{cases} \tag{VM}$$

Here, f represents the density of (positively-charged) ions in the plasma, while ρ and j are the charge and current density, and E and B represent electric and magnetic fields generated by the charge and current. The independent variables, $t > 0$ and $x, v \in \mathbb{R}^3$ represent time, position, and velocity, respectively, and physical constants, such as the charge and mass of particles, as well as, the speed of light, have been normalized to one. In the presence of large velocities, relativistic corrections become important and the corresponding system to consider is the relativistic analogue of (VM), denoted by (RVM) and constructed by replacing v with

$$\hat{v} = \frac{v}{\sqrt{1 + |v|^2}}$$

in the first equation of (VM), called the Vlasov equation, and in the integrand of the current j. General references concerning kinetic models of plasma dynamics, such as (VM) and (RVM), include [5] and [12].

Over the past twenty-five years significant progress has been made in the analysis of (RVM), specifically, the global existence of weak solutions (which also holds for (VM); see [3]) and the determination of sufficient conditions which ensure global existence of classical solutions (originally discovered in [7], and later in [8], and [1]) for the Cauchy problem. Additionally, a wide array of information has been discovered regarding the electrostatic versions of both (VM) and (RVM) - the Vlasov-Poisson and relativistic Vlasov-Poisson systems, respectively. These models do not include magnetic effects within their formulation, and the electric field is given by an elliptic, rather than a hyperbolic equation. This simplification has led to a great deal of progress concerning the electrostatic systems, including theorems regarding global existence, stability, and long-time behavior of solutions; though a global existence theorem for classical solutions with arbitrary data in the relativistic case has remained elusive. Independent of these advances, many of the most basic existence and regularity questions remain unsolved for (VM). The main difficulty which arises is the loss of strict hyperbolicity of the kinetic system due to the possibility that particle velocities v may travel faster than the propagation of signals from the electric and magnetic fields, which do so at the speed of light $c = 1$. As one can see, this difficulty is remedied by the inclusion of relativistic velocity corrections which uniformly constrain velocities $|\hat{v}| < 1$. In many physical systems one does not consider the effects of special (or general) relativity, but at the kinetic level such velocity corrections may play a fundamental role, even in the basic existence, uniqueness, and regularity properties of solutions. Hence, one of the primary goals of the current work is to understand how this difference in formulation affects such properties, and yield a partial answer to the question, "Are relativistic velocity corrections really necessary to ensure classical well-posedness?". It should be noted here that, whereas (RVM) is invariant under Lorentzian transformations, (VM) lacks invariance properties as it combines a Galilean-invariant equation for the particle distribution with a Lorentz-invariant field equation. To date, though, we are

unaware of any argument which truly utilizes the invariance properties of (RVM) or (VM) in order to arrive at an existence, uniqueness, or regularity theorem.

Often a remedy to the lack of progress on such a problem is to reduce the dimensionality of the system. Unfortunately, posing the problem in one-dimension (i.e., $x, v \in \mathbb{R}$) eliminates the relevance of the magnetic field as the Maxwell system decouples, yielding the one-dimensional Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v \partial_x f + E \partial_v f = 0 \\ \qquad \partial_x E = \int f \, dv. \end{cases} \tag{VP}$$

The lowest-dimensional reduction which includes magnetic effects is the so-called "one-and-one-half-dimensional" system which is constructed by taking $x \in \mathbb{R}$ but $v \in \mathbb{R}^2$ and considering

$$\begin{cases} \partial_t f + v \partial_x f + (E_1 + v_2 B) \partial_{v_1} f + (E_2 - v_1 B) \partial_{v_2} f = 0 \\ \qquad \partial_x E_1 = \int f \, dv, \qquad \partial_t E_1 = -\int v_1 f \, dv \\ \qquad \partial_t (E_2 + B) + \partial_x (E_2 + B) = -\int v_2 f \, dv \\ \qquad \partial_t (E_2 - B) - \partial_x (E_2 - B) = -\int v_2 f \, dv. \end{cases} \tag{1.5D VM}$$

Surprisingly, the question of classical regularity remains open even in this simplified case. One noticeable difference between (1.5D VM) and (VP) is the introduction of electric and magnetic fields E_2 and B that are solutions to transport equations. Thus, in order to study the existence and regularity questions, but keep the problem posed in a one-dimensional setting, we consider the following nonlinear system of PDE which couples the Vlasov equation to an advection equation for the associated field:

$$\begin{cases} \partial_t f + v \partial_x f + B \partial_v f = 0 \\ \qquad \partial_t B + \partial_x B = \int f \, dv. \end{cases} \tag{1.1}$$

The system (1.1) is supplemented by given initial data

$$f(0, x, v) = f_0(x, v), \quad B(0, x) = B_0(x). \tag{1.2}$$

Since the field equation in (1.1) is hyperbolic, we denote it by B so as to avoid confusion with the electric field E of (VP) which satisfies an elliptic equation. Notice that these equations retain the main difficulty of (VM), namely the interaction between characteristic particle velocities v and constant field velocities $c = 1$. Hence, we hope to analyze (1.1) and develop estimates or methods which can be generalized to deal with (1.5D VM). To our knowledge, this is the first analytic study of these kinetic equations, formed from a system of conservation laws coupled by a non-local field dependence on the particle densities. As such, the properties of solutions to (1.1) may also be of interest to mathematicians studying hyperbolic conservation laws in a two-dimensional phase space with nonlocal and nonlinear interaction. A related model, similar to (1.1), was previously studied [4] in an attempt to understand the nature of possible singularities generated by the intersection of Vlasov and field characteristics. Though a number of theorems were presented in [4], these results concerned a reduced system of ordinary differential equations rather than

partial differential equations, and hence the understanding of this phenomena remains far from complete. In the current article, we present existence uniqueness, and regularity theorems for the full system of PDEs given by (1.1) and the majority of our results also generalize to the original system of PDEs considered in [4]. We also mention the work [2] as it contains a discussion of computational methods for (1.1).

This paper proceeds as follows. In the next section, we will derive *a priori* estimates in order to simplify the proof of the local-in-time existence and uniqueness theorem for classical solutions of (1.1) with smooth initial data (1.2). The proof of this theorem then follows in Section 3. In Section 4, we present two results concerning global existence. In particular, we prove that initial particle distributions for which particle velocities travel strictly above or below the speed of light remain smooth globally in time as long as the initial field possesses a certain sign. The unfortunate detail of these theorems (4.1 and 4.3), however, is that they do not necessarily extend to arbitrary initial data or to (1.5D VM). Hence, it remains an open problem to show that any solution launched from smooth initial data remains smooth for all time. As an intermediate step, we prove in Section 5 an additional regularity result for the associated field B using a decomposition of derivatives similar to that of [7]. More specifically, we show *a priori* that for any $T > 0$, the field satisfies $B \in C^{0,1/2}([0,T] \times \mathbb{R})$. In fact, this result can be generalized to prove the same theorem for the electric and magnetic fields of (1.5D VM). Throughout the paper the value $C > 0$ will denote a generic constant that may change from line to line. When necessary, we will specifically identify the quantities upon which C may depend (e.g., C_T). Since we are interested in classical solutions, we will also assume the initial data are smooth, i.e. $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$, for the entirety of the paper.

2. **A priori estimates.** To begin, we will first prove a lemma that will allow us to represent and bound the particle density, its derivatives, and the associated field.

Lemma 2.1 (Estimates on f, B, ∂f, and velocity support)**.**

(a). *Let $T > 0$ be given and f be the solution of the Vlasov equation with given initial data*

$$\partial_t f + v \partial_x f + B \partial_v f = 0, \qquad t \in (0,T), \ x, v \in \mathbb{R}$$
$$f(0,x,v) = f_0(x,v), \qquad x, v \in \mathbb{R}$$

for some given $B \in C^1([0,T] \times \mathbb{R})$. Then, $f \in C^1([0,T]; C_c^1(\mathbb{R}^2))$ and for any $t \in [0,T]$ we have the estimates

$$\|f(t)\|_\infty \leq \|f_0\|_\infty,$$

$$\|\partial_x f(t)\|_\infty \leq \|\partial_x f_0\|_\infty + C \int_0^t \|\partial_x B(s)\|_\infty (1 + s \sup_{\tau \in [0,s]} \|\partial_x f(\tau)\|_\infty) \, ds,$$

$$\|\partial_v f(t)\|_\infty \leq \|\partial_v f_0\|_\infty + \int_0^t \|\partial_x f(s)\|_\infty \, ds$$

where C depends only upon the initial data.

(b). *Let $T > 0$ be given and B be the solution to the field equation with a given initial condition, namely*

$$\partial_t B + \partial_x B = \int f \, dv, \qquad t \in (0,T), \ x \in \mathbb{R}$$

$$B(0, x) = B_0(x), \qquad x \in \mathbb{R}$$

for some given $f \in C^1([0, T]; C_c^1(\mathbb{R}^2))$ and define

$$P(t) = \sup\{|v| : f(s, x, v) \neq 0, \ s \in [0, t], x \in \mathbb{R}\}.$$

Then, $B \in C^1([0, T] \times \mathbb{R})$ and for any $t \in [0, T]$, we have the estimate

$$\|B(t)\|_\infty \leq C \left(1 + \int_0^t P(s) \ ds \right)$$

where C depends only upon the initial data.

Proof. To prove the first result, we begin by introducing characteristics for the Vlasov equation. Define the curves $X(s, t, x, v)$ and $V(s, t, x, v)$ as solutions to the system of ODEs

$$\begin{cases} \dfrac{\partial X}{\partial s} = V(s, t, x, v), \\ \quad X(t, t, x, v) = x, \\ \dfrac{\partial V}{\partial s} = B(s, X(s, t, x, v)), \\ \quad V(t, t, x, v) = v \end{cases} \tag{2.1}$$

Often, the (t, x, v) dependence of these curves will be suppressed so, for example, $X(s, t, x, v)$ will be denoted by $X(s)$ for brevity. Then, the Vlasov equation can be expressed as a derivative along the characteristic curves by

$$\begin{aligned} \frac{d}{ds}\left(f(s, X(s), V(s)) \right) &= \partial_t f(s, X(s), V(s)) + \frac{\partial X}{\partial s} \partial_x f(s, X(s), V(s)) \\ &\quad + \frac{\partial V}{\partial s} \partial_v f(s, X(s), V(s)) \\ &= \partial_t f(s, X(s), V(s)) + V(s) \partial_x f(s, X(s), V(s)) \\ &\quad + B(s, X(s)) \partial_v f(s, X(s), V(s)) \\ &= 0. \end{aligned}$$

Thus, we find

$$f(t, X(t, t, x, v), V(t, t, x, v)) = f(0, X(0, t, x, v), V(0, t, x, v))$$

and

$$f(t, x, v) = f_0(X(0, t, x, v), V(0, t, x, v)) \leq \|f_0\|_\infty.$$

Finally, taking the supremum of this equality over $x, v \in \mathbb{R}$ yields the first estimate in (a). The remaining estimates involve derivatives, so differentiating the Vlasov equation in v yields

$$\left(\partial_t + v \partial_x + B \partial_v \right) \partial_v f = -\partial_x f$$

and upon integrating along characteristics we find

$$\partial_v f(t, x, v) = (\partial_v f_0)(X(0), V(0)) - \int_0^t (\partial_x f)(s, X(s), V(s)) \ ds. \tag{2.2}$$

Taking supremums in x and v gives the last estimate for (a). The second estimate is derived similarly. Differentiating with respect to x in the Vlasov equation, we find

$$\left(\partial_t + v \partial_x + B \partial_v \right) \partial_x f = -\partial_x B \partial_v f$$

and integrating as before along characteristics yields

$$\partial_x f(t, x, v) = (\partial_x f_0)(X(0), V(0)) - \int_0^t (\partial_x B \partial_v f)(s, X(s), V(s)) \, ds. \qquad (2.3)$$

Taking the supremum on the right side, inserting the estimate on $\|\partial_v f(s)\|_\infty$ above, and using the bounded data, we find

$$
\begin{aligned}
|\partial_x f(t, x, v)| &\leq \|\partial_x f_0\|_\infty + \int_0^t \|\partial_x B(s)\|_\infty \|\partial_v f(s)\|_\infty \, ds \\
&\leq \|\partial_x f_0\|_\infty + \int_0^t \|\partial_x B(s)\|_\infty \left(C + \int_0^s \|\partial_x f(\tau)\|_\infty \, d\tau \right) ds \\
&\leq \|\partial_x f_0\|_\infty + C \int_0^t \|\partial_x B(s)\|_\infty \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f(\tau)\|_\infty \right) ds.
\end{aligned}
$$

Finally, taking the supremum gives the estimate for $\|\partial_x f(t)\|_\infty$ in (a).

For the claim in (b), we may use the method of characteristics to solve for B in terms of f. First, we write the differential equation for B as a derivative along the curves $(s, x - t + s)$ so that

$$\frac{d}{ds}\left(B(s, x - t + s) \right) = \partial_t B(s, x - t + s) + \partial_x B(s, x - t + s) = \int f(s, x - t + s, v) \, dv.$$

Hence, integrating along these curves, we arrive at

$$B(t, x) = B_0(x - t) + \int_0^t \int f(s, x - t + s, v) \, dv \, ds. \qquad (2.4)$$

Using the bounded data, compact velocity support, and uniform bound on the particle density yields

$$
\begin{aligned}
|B(t, x)| &\leq C + \int_0^t \int_{\{|v| : f \neq 0\}} f(s, x - t + s, v) \, dv \, ds \\
&\leq C\left(1 + \int_0^t \|f(s)\|_\infty P(s) \, ds \right) \\
&\leq C\left(1 + \int_0^t P(s) \, ds \right).
\end{aligned}
$$

Finally, taking the supremum in x concludes the proof. $\qquad \square$

3. **Local-in-time existence of classical solutions.** With these *a priori* estimates and classical convergence theorems, we now have the necessary tools to prove the local-in-time existence theorem.

Theorem 3.1 (Existence of classical solutions)**.** *Let $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ be given. Then, there exists $T > 0$ and a unique classical solution*

$$f \in C^1([0, T]; C_c^1(\mathbb{R}^2)), \quad B \in C^1([0, T] \times \mathbb{R})$$

to (1.1) satisfying the initial conditions (1.2). Moreover, if we denote the maximal lifespan of the solution by T^* then for $T^* < \infty$ we must have

$$\limsup_{t \to T^*} \left(\|\partial_x f(t)\|_\infty + \|\partial_v f(t)\|_\infty \right) = \infty.$$

Proof. The outline of our proof generally follows the structure underlying [9], [10], and [11]. We begin with the existence argument, which utilizes the method of successive approximations. Hence, we define an iterative sequence of solutions to linear PDEs and show that it must converge to a solution of the nonlinear system (1.1). We take $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ and define

$$f^0(t, x, v) = f_0(x, v),$$
$$B^0(t, x) = B_0(x).$$

Additionally, for every $n \in \mathbb{N}$, define $f^n \in C^1([0, \infty); C_c^1(\mathbb{R}^2))$ and $B^n \in C^1([0, \infty) \times \mathbb{R})$ by solving the linear initial-value problems

$$\begin{cases} \partial_t f^n + v \partial_x f^n + B^{n-1} \partial_v f^n = 0 \\ \quad f^n(0, x, v) = f_0(x, v), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t B^n + \partial_x B^n = \int f^n dv \\ \quad B^n(0, x) = B_0(x) \end{cases} \tag{3.2}$$

respectively. Notice that if $f^n \to f$ and $B^n \to B$ in the appropriate sense as $n \to \infty$ then f and B will satisfy (1.1). Now as in Lemma 2.1, we further define the sequence of velocity support functions for every $t \geq 0, n \in \mathbb{N}$,

$$P^n(t) = \sup\{|v| : f^n(s, x, v) \neq 0, \ s \in [0, t], x \in \mathbb{R}\}.$$

Using these sequences the uniform convergence on some time interval $[0, T]$ follows using standard machinery. We briefly sketch the proof for clarity and completeness.

3.1. **Uniform boundedness.** For the first portion of the proof, we consider $T > 0$ given and estimate on the time interval $[0, T]$. In order to uniformly bound the sequence B^n, we utilize the estimates on the velocity support of f^n. First, by Lemma 2.1, we have the bound

$$\|B^n(t)\|_\infty \leq C \left(1 + \int_0^t P^n(s) \, ds\right) \tag{3.3}$$

for every $n \in \mathbb{N}$. To bound the velocity support in terms of the field, we express the solution of the Vlasov equation in terms of the associated characteristics. For every $n \in \mathbb{N}$ define the characteristic curves $X^n(s, t, x, v)$ and $V^n(s, t, x, v)$ by

$$\begin{cases} \dfrac{\partial X^n}{\partial s} = V^n(s, t, x, v), \\ \quad X^n(t, t, x, v) = x, \\ \dfrac{\partial V^n}{\partial s} = B^{n-1}(s, X^n(s, t, x, v)), \\ \quad V^n(t, t, x, v) = v. \end{cases} \tag{3.4}$$

As before, the (t, x, v) dependence of these curves will be suppressed for brevity. Then, using the argument in the proof of the first result of Lemma 2.1, we find

$$f^n(t, X^n(t, t, x, v), V^n(t, t, x, v)) = f^n(0, X^n(0, t, x, v), V^n(0, t, x, v))$$

and

$$f^n(t, x, v) = f_0(X^n(0, t, x, v), V^n(0, t, x, v))$$

for every $n \in \mathbb{N}$. Inverting the characteristics and using the identities

$$x = X^n\left(t, 0, X^n(0, t, x, v), V^n(0, t, x, v)\right)$$
$$v = V^n\left(t, 0, X^n(0, t, x, v), V^n(0, t, x, v)\right)$$

within this last equality, then utilizing the compact support of f_0 and the definition of P^n, it follows that

$$\sup_{x,v} |V^n(t, 0, x, v)| \leq P^n(t)$$

for every $t \geq 0$. From the velocity characteristic equation (3.4), we can integrate to find

$$V^n(t, 0, x, v) = v + \int_0^t B^{n-1}(\tau, X^n(\tau)) \, d\tau$$

and thus

$$|V^n(t)| \leq |v| + \int_0^t \|B^{n-1}(\tau)\|_\infty \, d\tau.$$

Taking the supremum over characteristics along which $f^n \neq 0$, we find

$$P^n(t) \leq P^n(0) + \int_0^t \|B^{n-1}(\tau)\|_\infty \, d\tau.$$

We can now use (3.3) to arrive at a recursive bound for P^n, namely

$$P^n(t) \leq P^n(0) + C \int_0^t \left(1 + \int_0^\tau P^{n-1}(s) \, ds\right) \, d\tau.$$

Since f_0 has compact support, we know $P^n(0)$ is finite and constant in n, thus for every $n \in \mathbb{N}$ and on every bounded time interval $[0, T]$,

$$P^n(t) \leq C_T \left(1 + \int_0^t P^{n-1}(\tau) \, d\tau\right)$$

where C_T depends upon f_0 and T. Using this recursive relation, we immediately deduce

$$P^n(t) \leq C_T \left(1 + \frac{t^n}{n!}\right) \leq C_T e^t \leq C_T.$$

Thus, on any bounded time interval $[0, T]$ the function $P^n(t)$ is uniformly bounded and from (3.3) so is $\|B^n(t)\|_\infty$.

3.2. **Uniform boundedness of derivatives.** Now we focus on obtaining uniform bounds on derivatives, sketching the proof for x derivatives, with t and v derivatives following similarly. From the definition of the iterates we can differentiate the representation for B^n (2.4) after integrating along characteristics with speed one, so that

$$\partial_x B^n(t, x) = B_0'(x - t) + \int_0^t \int \partial_x f^n(\tau, x - t + \tau, v) \, dv \, d\tau. \qquad (3.5)$$

Using the bound on the velocity support above, we arrive at

$$\|\partial_x B^n(t)\|_\infty \leq C_T \left(1 + \int_0^t \|\partial_x f^n(\tau)\|_\infty \, d\tau\right) \qquad (3.6)$$

for every $n \in \mathbb{N}$. Using Lemma 2.1(a) we also have the estimate

$$\|\partial_x f^n(t)\|_\infty \leq \|\partial_x f_0\|_\infty + C \int_0^t \|\partial_x B^{n-1}(s)\|_\infty \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^n(\tau)\|_\infty\right) \, ds.$$

We combine these inequalities to find

$$\|\partial_x f^n(t)\|_\infty$$
$$\leq C_T \left[1 + \int_0^t \left(1 + \int_0^s \|\partial_x f^{n-1}(\tau)\|_\infty \; d\tau \right) \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^n(\tau)\|_\infty \right) \; ds \right]$$
$$\leq C_T \left[1 + \int_0^t \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^{n-1}(\tau)\|_\infty \right) \left(1 + s \sup_{\tau \in [0,s]} \|\partial_x f^n(\tau)\|_\infty \right) \; ds \right].$$

Now, let

$$F^n(t) = \max_{1 \leq k \leq n} \sup_{\tau \in [0,t]} \|\partial_x f^k(\tau)\|_\infty.$$

With this definition, the previous inequality yields

$$F^n(t) \leq C_T \left(1 + \int_0^t [1 + sF^n(s)]^2 \; ds \right). \tag{3.7}$$

Hence, by induction $F^n(t) \leq F(t)$ for every $n \in \mathbb{N}$, $t \in [0, T^*)$, where $F(t)$ is the maximal solution of the integral equation corresponding to (3.7):

$$F(t) = C_{T^*} \left(1 + \int_0^t [1 + sF(s)]^2 \; ds \right). \tag{3.8}$$

This solution exists on some time interval $[0, T^*)$ with $T^* > 0$ determined by f_0 and B_0. This yields a uniform bound on $\|\partial_x f^n(t)\|_\infty$ on $[0, T]$ for every $n \in \mathbb{N}$ and $T < T^*$. Additionally, $\|\partial_x B^n(t)\|_\infty$ is bounded on the same interval by (3.6). The argument can be repeated in the same manner to bound $\|\partial_t B(t)\|_\infty$, $\|\partial_t f(t)\|_\infty$, and $\|\partial_v f(t)\|_\infty$ on the same time interval.

3.3. Convergence and properties of limiting functions.

Using the same tools, the uniform Cauchy property of the field, particle distribution, and their derivatives follows in a straightforward manner. We omit the details and direct the reader to [5] and [9] for further information. Assembling these steps, we prove that the sequences and their derivatives converge to solutions of (1.1). Let T^* again denote the maximal existence time of the solution to (3.8). Using the Cauchy property of f^n, X^n, V^n, B^n, and their derivatives, we conclude that each sequence of functions converges uniformly on the time interval $[0, T]$ for any $T < T^*$ and uniformly for $x, v \in \mathbb{R}, n \in \mathbb{N}$. Since the space of continuous functions is complete with respect to the norm of uniform convergence, we may conclude that these sequences converge to continuous functions. Therefore, let us define $f \in C([0, T] \times \mathbb{R}^2)$ by

$$f(t, x, v) = \lim_{n \to \infty} f^n(t, x, v) = \lim_{n \to \infty} f_0 \big(X^n(0, t, x, v), V^n(0, t, x, v) \big).$$

Then, we similarly define the field

$$B(t, x) = \lim_{n \to \infty} B^n(t, x) = \lim_{n \to \infty} \left(B_0(x - t) + \int_0^t \int f^n(s, x - t + s, v) \; dv \; ds \right).$$

Thus, using the uniform convergence of $f^n(t, x, v)$, we can pass the limit inside these integrals to find for every $t \in [0, T]$, $x \in \mathbb{R}$

$$B(t, x) = B_0(x - t) + \int_0^t \int f(s, x - t + s, v) \; dv \; ds. \tag{3.9}$$

Further, we define

$$X = \lim_{n \to \infty} X^n, \qquad V = \lim_{n \to \infty} V^n.$$

It follows from (3.4) and the uniform field bound that

$$\begin{cases} \dfrac{\partial X}{\partial s} = V(s,t,x,v), \\[2mm] \dfrac{\partial V}{\partial s} = B(s, X(s,t,x,v)), \\[2mm] X(t,t,x,v) = x, \\[1mm] V(t,t,x,v) = v, \end{cases} \qquad (3.10)$$

and by the continuity of f_0, we see that

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)),$$

whence for every $s \in [0,t]$,

$$f(t,x,v) = f(s, X(s,t,x,v), V(s,t,x,v)). \qquad (3.11)$$

Since the approximating sequence of derivatives (e.g., $\partial_x f^n$) of these functions converge uniformly, this implies that f and B are C^1 and their derivatives are necessarily the limits of the respective sequences, meaning

$$\partial_x f = \lim_{n\to\infty} \partial_x f^n, \quad \partial_v f = \lim_{n\to\infty} \partial_v f^n, \quad \partial_t f = \lim_{n\to\infty} \partial_t f^n,$$
$$\partial_x B = \lim_{n\to\infty} \partial_x B^n, \qquad \partial_t B = \lim_{n\to\infty} \partial_t B^n.$$

Furthermore, the uniform bound on $P^n(t)$ implies the compact x and v support of $f(t,x,v)$ for every $t \in [0,T]$. Thus, for every $T < T^*$, $f \in C^1([0,T]; C_c^1(\mathbb{R}^2))$. Using (3.9) and taking derivatives, we see that the field equation for B of (1.1) holds. Upon taking the derivative with respect to s in (3.11) and using (3.10), we see that the Vlasov equation of (1.1) holds. Additionally, the solutions (3.11) and (3.9) satisfy the initial conditions (1.2). Therefore, the continuously differentiable functions f and B satisfy (1.1) with (1.2). Hence, we have shown the existence of such a solution (f, B). Notice that this argument can be continued to a time interval of arbitrary size so long as $\|\partial_x f(t)\|_\infty$ and $\|\partial_v f(t)\|_\infty$ remain bounded.

3.4. **Uniqueness of solutions.** Finally, we turn to uniqueness. Let us first suppose that the functions $(f^{(1)}, B^{(1)})$ and $(f^{(2)}, B^{(2)})$ are two solutions to the system (1.1) on some time interval $[0,T]$ which satisfy (1.2). Also, for every $t \in [0,T]$ and $x, v \in \mathbb{R}$ define the difference of these solutions

$$f(t,x,v) = f^{(1)}(t,x,v) - f^{(2)}(t,x,v)$$
$$B(t,x) = B^{(1)}(t,x) - B^{(2)}(t,x).$$

Then, we subtract the first equation of (1.1) for $f^{(2)}$ from that for $f^{(1)}$ to find

$$\begin{aligned} 0 &= \partial_t f + v\partial_x f + B^{(1)}\partial_v f^{(1)} - B^{(2)}\partial_v f^{(2)} \\ &= \partial_t f + v\partial_x f + B^{(1)}\partial_v f^{(1)} - B^{(1)}\partial_v f^{(2)} + B^{(1)}\partial_v f^{(2)} - B^{(2)}\partial_v f^{(2)} \\ &= \partial_t f + v\partial_x f + B^{(1)}\partial_v f - B\partial_v f^{(2)} \end{aligned}$$

so that by rearranging terms this becomes

$$\partial_t f + v\partial_x f + B^{(1)}\partial_v f = B\partial_v f^{(2)}.$$

The left side of this equation can be expressed as a derivative along characteristic curves as

$$\frac{d}{ds} f\big(s, X^{(1)}(s), V^{(1)}(s)\big) = \left(B\partial_v f^{(2)} \right)\big(s, X^{(1)}(s), V^{(1)}(s)\big)$$

where the curves $X^{(1)}(s)$ and $V^{(1)}(s)$ are defined by the, now well-known, system of characteristic ordinary differential equations

$$\begin{cases} \dfrac{\partial X^{(1)}}{\partial s} = V^{(1)}(s,t,x,v), \\[2mm] \dfrac{\partial V^{(1)}}{\partial s} = B^{(1)}(s, X^{(1)}(s,t,x,v)), \\[2mm] X^{(1)}(t,t,x,v) = x, \\[2mm] V^{(1)}(t,t,x,v) = v. \end{cases} \tag{3.12}$$

Here, we have abbreviated $X^{(1)}(s,t,x,v)$ by $X^{(1)}(s)$ and similarly for $V^{(1)}(s)$. Now, integrating both sides of the above equation with respect to s, we find

$$f(t,x,v) - f\big(0, X^{(1)}(0), V^{(1)}(0)\big) = \int_0^t \big(B\partial_v f^{(2)}\big)\big(s, X^{(1)}(s), V^{(1)}(s)\big)ds$$

and since both solutions satisfy the same initial condition (1.2), as before this implies $f(0,x,v) \equiv 0$. Therefore, the equality simplifies to

$$f(t,x,v) = \int_0^t \big(B\partial_v f^{(2)}\big)\big(s, X^{(1)}(s), V^{(1)}(s)\big)ds.$$

Since $f^{(2)}$ is a solution, we know $\|\partial_v f^{(2)}(s)\|_\infty$ is bounded for $s \in [0,t]$ and we can bound the right side to find

$$\|f(t)\|_\infty \le C_T \int_0^t \|B(s)\|_\infty \ ds. \tag{3.13}$$

Now, by subtracting the $B^{(2)}$ equation from the $B^{(1)}$ equation, we arrive at

$$\partial_t B + \partial_x B = \int f \ dv$$

which we can write as a derivative along curves with slope one as

$$\frac{d}{ds} B(s, x - t + s) = \int f(s, x - t + s, v) \ dv.$$

Integrating in s and using (1.2) to conclude that $B(0,x) \equiv 0$, this becomes

$$B(t,x) = \int_0^t \int f(s, x - t + s, v) \ dvds.$$

Since both $f^{(1)}$ and $f^{(2)}$ are solutions, the velocity support of f is controlled and we can bound B by

$$\|B(t)\|_\infty \le C_T \sup_{s \in [0,t]} \|f(s)\|_\infty \tag{3.14}$$

Finally, combining (3.13) and (3.14) yields

$$||B(t)||_\infty \le C_T \int_0^t \sup_{\tau \in [0,s]} ||B(\tau)||_\infty ds$$

and

$$\sup_{s \in [0,t]} ||B(s)||_\infty \le C_T \int_0^t \sup_{\tau \in [0,s]} ||B(\tau)||_\infty ds$$

Again using Gronwall's Inequality, we deduce $||B(t)||_\infty \le 0$ for all $t \in [0,T]$ which implies that $B(t,x) = 0$ for every $t \in [0,T]$, $x \in \mathbb{R}$. Similarly, using (3.13) we see

that $f(t, x, v) = 0$ for every $t \in [0, T]$, $x, v \in \mathbb{R}$. Finally, this implies that $f^{(1)} \equiv f^{(2)}$ and $B^{(1)} \equiv B^{(2)}$, and hence there can be at most one such solution. $\qquad\square$

4. **Global existence.** Now that we know smooth solutions exist on some time interval, the next logical question is whether this can be extended for all times. Unfortunately, a complete answer remains unknown. The fundamental issue is that the Vlasov characteristics in the density equation propagate at an uncontrollable speed v and hence, are able to intersect the field characteristics which propagate with speed 1. Though we cannot currently prove that all initial data launch a global-in-time solution, we can provide an answer for certain classes of initial data. In what follows we will use the unidirectional nature of the transport operator in (1.1) to answer the question of global existence in the affirmative for a class of initial data (f_0, B_0). Then, we utilize a new invariance of the system to extend the global existence result to additional solutions. Such an approach may also work for (1.5D VM), but no ideas of this form currently exist within the literature. We begin with the former global existence result:

Theorem 4.1. *Let $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ be nonnegative with $supp(f_0(x, \cdot)) \subset (1, \infty)$ for all $x \in \mathbb{R}$. Then, for all $T > 0$ there exist $f \in C^1([0, T] \times \mathbb{R}^2)$ and $B \in C^1([0, T] \times \mathbb{R})$ satisfying (1.1) with (1.2) and $f(t, \cdot, \cdot)$ compactly supported for every $t \in [0, T]$.*

Proof. Let f and B be the local solution guaranteed by Theorem 3.1 and $T > 0$ be given. First, we represent B and use the sign of the data to find

$$
\begin{aligned}
B(t, x) &= B_0(x - t) + \int_0^t \int f(s, x + t - s, v) \, dv \, ds \\
&= B_0(x - t) + \int_0^t \int f_0(X(s, t, x + t - s, v), V(s, t, x + t - s, v)) \, dv \, ds \\
&\geq 0
\end{aligned}
$$

for every $t \in [0, T]$ and $x \in \mathbb{R}$. Thus, if we now let $(X(t), V(t))$ be characteristics along which f is nonzero. Then, a lower bound on velocity characteristics follows

$$
V(t) = V(0) + \int_0^t B(s, X(s)) \, ds \geq V(0).
$$

Since the velocity support of f_0 is strictly bounded below by $v = 1$, it follows that the velocity support of f satisfies this same property. Additionally, the compact velocity support of f implies

$$
|X(t)| = \left| \int_0^t V(s) \, ds \right| \leq tP(t) \leq C_T
$$

and thus the spatial support remains compact for $t \in [0, T]$.

With this, we may utilize the field representation of [7] to control derivatives of the density and field (as in the proof of (5.1)). Thus, we write the field derivative in terms of the derivative of the density, as in (3.5), so that

$$
\partial_x B(t, x) = B_0'(x - t) + \int_0^t \int \partial_x f(\tau, x - t + \tau, v) \, dv \, d\tau. \tag{4.1}
$$

Now, we would like to eliminate the x-derivative of the density in this equation, so similar to [6] we transform ∂_x into derivatives along the characteristic curves of the

system. Define the operators

$$\mathcal{S} = \partial_t + v\partial_x$$
$$\mathcal{T} = \partial_t + \partial_x.$$

Then, for $v \neq 1$ we may write the inverse transformation

$$\partial_x = \frac{1}{1-v}\left(\mathcal{T} - \mathcal{S}\right).$$

For $t \in [0, T]$, let

$$P(t) = \sup\{|v| : \exists x \in \mathbb{R} \text{ with } f(t,x,v) \neq 0\}$$

and recall $P(t) \leq C_T$ for all $t \in [0, T]$. Then, differentiation of the representation for the field (3.9) yields

$$
\begin{aligned}
\partial_x B(t,x) &= B_0'(x-t) + \int_0^t \int \frac{1}{1-v}\left[\mathcal{T}f(\tau, x-t+\tau, v)\right. \\
&\qquad \left. -\mathcal{S}f(\tau, x-t+\tau, v)\right]\, dv\, d\tau \\
&= B_0'(x-t) + \int_0^t \int \frac{1}{1-v}\frac{d}{d\tau}\left[f(\tau, x-t+\tau, v)\right]\, dv\, d\tau \\
&\quad + \int_0^t \int \frac{1}{1-v}\partial_v\left[B(\tau, x-t+\tau)f(\tau, x-t+\tau, v)\right]\, dv\, d\tau
\end{aligned}
$$

Here we have used the Vlasov equation of (1.1) to write the integrand as a pure v-derivative so that

$$\mathcal{S}f(\tau, y, v) = (\partial_t f + v\partial_x f)(\tau, y, v) = -\partial_v(Bf)(\tau, y, v).$$

Since the velocity support of f is bounded away from $v = 1$, we see that the integrands are non-singular and we integrate by parts to find

$$
\begin{aligned}
\partial_x B(t,x) &= B_0'(x-t) + \int \frac{1}{1-v}\left[f(t,x,v) - f_0(x-t,v)\right]\, dv \\
&\quad + \int_0^t \frac{1}{1-v}(Bf)(\tau, x-t+\tau, v)\Big|_{v=P(t)}\, d\tau \\
&\quad - \int_0^t \int \frac{1}{(1-v)^2}(Bf)(\tau, x-t+\tau, v)\, dv\, d\tau.
\end{aligned}
$$

Using the previously-derived L^∞ bounds on f, P, and B, which follow from the iterates, we find

$$\|\partial_x B(t)\|_\infty \leq C_T$$

Hence, using Lemma 2.1(a) we find

$$\|\partial_x f(t)\|_\infty \leq C_T\left(1 + \int_0^t \sup_{\tau \in [0,s]} \|\partial_x f(\tau)\|_\infty\, ds.\right)$$

Taking the supremum in t and using Gronwall's Inequailty implies

$$\|\partial_x f(t)\|_\infty \leq C_T$$

and using 2.1(a) again we find

$$\|\partial_v f(t)\|_\infty \leq C_T$$

for any $T > 0$ and $t \in [0, T]$. Thus, the local-in-time solution from Theorem 3.1 can be extended to arbitrarily large time. $\qquad\square$

Next, we utilize a new invariance of (1.1) to extend this result to additional solutions.

Lemma 4.2. *Let $T > 0$ be given. The functions $f \in C^1([0,T]; C_c^1(\mathbb{R}^2))$ and $B \in C^1([0,T] \times \mathbb{R})$ solve (1.1) for $t \in [0,T], x, v \in \mathbb{R}$ with initial data (1.2) if and only if the functions*

$$f^u(t, x, v) := f(t, (u+1)x - ut, (u+1)v - u) \tag{4.2}$$

and

$$B^u(t, x) := (u+1)^{-1} B(t, (u+1)x - ut) \tag{4.3}$$

satisfy (1.1) for $t \in [0,T], x, v \in \mathbb{R}$ with initial data

$$f_0^u(x, v) = f_0((u+1)x, (1+u)v - u)$$

and

$$B_0^u(x) = (1+u)^{-1} B_0((u+1)x)$$

for every $u \neq -1$.

Proof. Clearly, if (f^u, B^u) satisfies these properties then we may choose $u = 0$ and (f, B) will satisfy the same equations. Now, assume that (f, B) solve (1.1) and define (f^u, B^u) by (4.2) and (4.3) respectively. Then, a brief calculation shows that (f^u, B^u) satisfy (1.1) and the corresponding initial conditions. Denoting

$$t' = t, \qquad x' = (u+1)x - ut \qquad v' = (u+1)v - u$$

we find specifically

$$\partial_t f^u(t, x, v) + v \partial_x f^u(t, x, v) + B^u(t, x) \partial_v f^u(t, x, v)$$

$$= \left[\partial_t f(t', x', v') - u \partial_x f(t', x', v') \right] + v(u+1) \partial_x f(t', x', v')$$

$$+ (1+u)^{-1} B(t', x')(1+u) \partial_v f(t', x', v')$$

$$= \partial_t f(t', x', v') + v' \partial_x f(t', x', v') + B(t', x') \partial_v f(t', x', v') = 0$$

and

$$\begin{aligned}
\partial_t B^u(t, x) + \partial_x B^u(t, x) &= (u+1)^{-1} \left[\partial_t B(t', x') - u \partial_x B(t', x') \right] \\
&\quad + (u+1)^{-1} \cdot (u+1) \partial_x B(t', x') \\
&= (u+1)^{-1} \left(\partial_t B(t', x') + \partial_x B(t', x') \right) \\
&= (u+1)^{-1} \int f(t', x', v') \, dv' \\
&= \int f(t', x', v') \, dv \\
&= \int f^u(t, x, v) \, dv
\end{aligned}$$

since (f, B) satisfy (1.1) at every point $t \in [0, T]$ and $x, v \in \mathbb{R}$. Similarly the initial conditions are satisfied by inspection. $\square$

Theorem 4.3. *Let $f_0 \in C_c^1(\mathbb{R}^2)$ be nonnegative and $B_0 \in C^1(\mathbb{R})$ be nonpositive with the velocity support of f_0 contained in $(-\infty, 1)$. Then, for all $T > 0$ there exist $f \in C^1([0, T] \times \mathbb{R}^2)$ and $B \in C^1([0, T] \times \mathbb{R})$ satisfying (1.1) with (1.2) and $f(t, \cdot, \cdot)$ compactly supported for every $t \in [0, T]$.*

Proof. We assume $f_0(x,v) \geq 0$ for $x,v \in \mathbb{R}$ with $\mathrm{supp}(f(x,\cdot)) \subset (-\infty,1)$ for every $x \in \mathbb{R}$ and $B_0(x) \leq 0$ for $x \in \mathbb{R}$. Let (f,B) be the local solution guaranteed by Theorem 3.1 and $T > 0$ be given. By Lemma 4.2, there is a local-in-time solution of (1.1) denoted (f^u, B^u) with $u = -2$ and corresponding initial data

$$f_0^u(x,v) = f_0(-x, 2-v)$$
$$B_0^u(x) = -B_0(-x)$$

Hence, this solution satisfies $f_0^u(x,v), B_0^u(x) \geq 0$ for $x,v \in \mathbb{R}$ and $\mathrm{supp}(f_0^u(x,\cdot)) \subset (1,\infty)$ for $x \in \mathbb{R}$. Thus, by Theorem 4.1 the solution exists on $[0,T]$, and by Lemma 4.2 so too must the solution (f,B) launched by f_0 and B_0. Since $T > 0$ is arbitrary, the result follows. $\square$

5. **Additional field regularity.** Though we cannot currently prove the global existence of classical solutions to (1.1) for arbitrary initial data, we can provide an estimate which yields additional regularity of the field B. If one could show *a priori* for any $T > 0$ that $B \in C^1([0,T] \times \mathbb{R})$ then global existence would follow as this bound would imply smoothness of characteristics and ultimately $f \in C^1([0,T] \times \mathbb{R}^2)$. Hence, the local solution could be continued up to the arbitrary existence time T. Instead of the global result that B possesses a full derivative, however, we are only able to show that it possesses half of a derivative in space-time.

Theorem 5.1. *Let $f_0 \in C_c^1(\mathbb{R}^2)$ and $B_0 \in C^1(\mathbb{R})$ be given and let $f \in C^1([0,T_0] \times \mathbb{R}^2)$ and $B \in C^1([0,T_0] \times \mathbb{R})$ be the unique solution of (1.1) within the maximal interval of existence $T_0 \in (0,T^*)$. Then, for any $T > 0$ it follows that $B \in C^{0,1/2}([0,T] \times \mathbb{R})$.*

Proof. Let $T > 0$ be given. We will prove the result for the regularity in x, while a similar argument leads to the additional Hölder continuity in t. Let $h > 0$ be given. As in Theorem 4.1 we wish to use the operators

$$\mathcal{S} = \partial_t + v\partial_x$$
$$\mathcal{T} = \partial_t + \partial_x.$$

As before, for $v \neq 1$ we may write the inverse transformation

$$\partial_x = \frac{1}{1-v}\left(\mathcal{T} - \mathcal{S}\right).$$

For $t \in [0,T]$, let

$$P(t) = \sup\{|v| : \exists x \in \mathbb{R} \text{ with } f(t,x,v) \neq 0\} \leq C_T.$$

Since the $(\partial_t, \partial_x) \mapsto (\mathcal{S}, \mathcal{T})$ transformation is only valid for v bounded away from one, we decompose the v-integral over $[-P(t), P(t)]$ within our previous derivative estimate (4.1) into integrals over the disjoint sets

$$A^\epsilon = \{v : |1-v| < \epsilon\} \cap [-P(t), P(t)]$$

and

$$B^\epsilon = \{v : |1-v| > \epsilon\} \cap [-P(t), P(t)]$$

where $\epsilon > 0$ is to be chosen. Then, beginning with the representation for the field (3.9) we have

$$|B(t, x + h) - B(t, x)| \leq |B_0(x + h - t) - B_0(x - t)|$$

$$+ \left| \int_0^t \int \left[f(\tau, x + h - t + \tau, v) - f(\tau, x - t + \tau, v) \right] dv \, d\tau \right|$$

$$\leq I + II + III$$

where

$$I = \|B_0\|_{C^{0,1/2}} \cdot \sqrt{h},$$

$$II = \int_0^t \int_{A^\epsilon} |f(\tau, x + h - t + \tau, v) - f(\tau, x - t + \tau, v)| \, dv \, d\tau,$$

and

$$III = \left| \int_0^t \int_{B^\epsilon} f(\tau, x + h - t + \tau, v) - f(\tau, x - t + \tau, v) \, dv \, d\tau \right|.$$

Using the bound on f we crudely estimate II and find

$$II \leq 2t\|f_0\|_\infty \cdot \mu(A^\epsilon) \leq C\epsilon$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. As all velocities in B^ϵ are bounded away from 1, we can then use the transformation of derivatives for the estimate of III. We include the x-derivative of the density and transform ∂_x into terms involving $\mathcal{S}$ and $\mathcal{T}$ so that

$$\begin{aligned} III &= \left| \int_0^t \int_{B^\epsilon} \int_{x-t+\tau}^{x+h-t+\tau} \partial_x f(\tau, y, v) \, dy \, dv \, d\tau \right| \\ &= \left| \int_0^t \int_{B^\epsilon} \int_{x-t+\tau}^{x+h-t+\tau} \frac{1}{1-v} \left(\mathcal{T}f - \mathcal{S}f \right)(\tau, y, v) \, dy \, dv \, d\tau \right| \\ &\leq III_{\mathcal{T}} + III_{\mathcal{S}}, \end{aligned}$$

where the quantities

$$III_{\mathcal{T}} = \left| \int_0^t \int_{B^\epsilon} \int_{x-t+\tau}^{x+h-t+\tau} \frac{1}{1-v}(\partial_t f + \partial_x f)(\tau, y, v) \, dy \, dv \, d\tau \right|$$

and

$$III_{\mathcal{S}} = \left| \int_0^t \int_{B^\epsilon} \int_{x-t+\tau}^{x+h-t+\tau} \frac{1}{1-v}(\partial_t f + v\partial_x f)(\tau, y, v) \, dy \, dv \, d\tau \right|$$

separate the $\mathcal{T}$ and $\mathcal{S}$ terms. To estimate $III_{\mathcal{T}}$ we change variables in the y integral with $z = y - (x - t + \tau)$, switch the order of integration, and integrate by parts in τ to find

$$\begin{aligned} III_{\mathcal{T}} &= \left| \int_0^t \int_{B^\epsilon} \int_0^h \frac{1}{1-v} \frac{d}{d\tau} f(\tau, z + x - t + \tau, v) \, dz \, dv \, d\tau \right| \\ &= \left| \int_{B^\epsilon} \int_0^h \frac{1}{1-v}(f(t, z + x, v) - f(0, z + x - t, v) \, dz \, dv \right| \\ &\leq 2\|f_0\|_\infty \int_{B^\epsilon} \int_0^h \frac{1}{|1-v|} \, dz \, dv \\ &\leq Ch \int_{B^\epsilon} |1-v|^{-1} \, dv \end{aligned}$$

Notice that on the set B^ϵ, we have $|1-v| > \epsilon$ and thus $|1-v|^{-1} < \epsilon^{-1}$. In addition, $\mu(B^\epsilon) \le 2P(t) \le C_T$. Therefore,

$$III_T \le \frac{C_T h}{\epsilon}.$$

Estimating $III_\mathcal{S}$, we again use the Vlasov equation of (1.1) to write the integrand as a pure v-derivative so that

$$\mathcal{S}f(\tau,y,v) = (\partial_t f + v\partial_x f)(\tau,y,v) = -\partial_v(Bf)(\tau,y,v).$$

Then, we integrate by parts in v and use the bounds on f and B, yielding

$$
\begin{aligned}
III_\mathcal{S} &\le \int_0^t \int_{x-t+\tau}^{x+h-t+\tau} \left| \int_{B^\epsilon} \frac{1}{1-v} \partial_v(Bf)(\tau,y,v) \, dv \right| \, dy \, d\tau \\
&\le C_T \int_0^t \int_{x-t+\tau}^{x+h-t+\tau} \left| \left(\frac{Bf}{1-v}\right) \Big|_{(\tau,y,1+\epsilon)}^{(\tau,y,1-\epsilon)} - \int_{B^\epsilon} \frac{Bf}{(1-v)^2}(\tau,y,v) \, dv \right| \, dy \, d\tau \\
&\le C_T \int_0^t \int_{x-t+\tau}^{x+h-t+\tau} \left(\frac{1}{\epsilon} + \int_{B^\epsilon} \frac{1}{|1-v|^2} \, dv \right) \, dy \, d\tau \\
&\le \frac{C_T h}{\epsilon}.
\end{aligned}
$$

Finally, we combine the estimates to find

$$|B(t,x+h) - B(t,x)| \le C_T \left(\sqrt{h} + \epsilon + h/\epsilon \right). \tag{5.1}$$

We choose $\epsilon = \sqrt{h}$ optimally so that $\epsilon = h/\epsilon$, and finally

$$|B(t,x+h) - B(t,x)| \le C_T \sqrt{h}.$$

Thus, $B(t,\cdot) \in C^{0,1/2}(\mathbb{R})$ for every $t \in [0,T]$ and a similar argument can be used to establish the Hölder continuity in t for fixed $x \in \mathbb{R}$. $\qquad\square$

REFERENCES

[1] F. Bouchut, F. Golse and C. Pallard, Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system, *Arch. Ration. Mech. Anal.,* **170** (2003), 1–15.

[2] D. Brewer and S. Pankavich, Computational Methods for a One-dimensional Plasma Model with a Transport Field, *SIAM Undergraduate Research Online,* **4** (2011), 81–104.

[3] R. J. DiPerna and P. L. Lions, Global weak solutions of Vlasov-Maxwell systems, *Comm. Pure Appl. Math.,* **42** (1989), 729–757.

[4] P. Gerard and C. Pallard, A mean-field toy model for resonant transport, *Kinet. Relat. Models,* **3** (2010), 299–309.

[5] R. T. Glassey, *The Cauchy Problem in Kinetic Theory*, Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, 1996.

[6] R. T. Glassey and J. Schaeffer, On the "one and one-half dimensional" relativistic Vlasov-Maxwell system, *Math. Methods Appl. Sci.,* **13** (1990), 169–179.

[7] R. T. Glassey and W. A. Strauss, Singularity formation in a collisionless plasma could occur only at high velocities, *Arch. Rational Mech. Anal.,* **92** (1986), 59–90.

[8] S. Klainerman and G. Staffilani, A new approach to study the Vlasov-Maxwell system, *Commun. Pure Appl. Anal.,* **1** (2002), 103–125.

[9] M. Kunzinger, G. Rein, R. Steinbauer and G. Teschl, On classical solutions of the relativistic Vlasov-Klein-Gordon system, *Electron. J. Differential Equations (electronic - 17 pp.)*, **1** (2005), 17pp.

[10] S. Pankavich, Global existence for the Vlasov-Poisson system with steady spatial asymptotics, *Comm. Partial Differential Equations,* **31** (2006), 349–370.

[11] S. Pankavich, Local existence for the one-dimensional Vlasov-Poisson system with infinite mass, *Math. Methods Appl. Sci.,* **30** (2007), 529–548.

[12] N. G. van Kampen and B. U. Felderhof, *Theoretical Methods in Plasma Physics*, Wiley: New York, NY, 1967.

Received January 2014; revised April 2014.

E-mail address: charles.nguyen@mavs.uta.edu
E-mail address: pankavic@mines.edu