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A TOY MODEL FOR THE RELATIVISTIC VLASOV-MAXWELL SYSTEM

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Abstract. The global-in-time existence of classical solutions to the relativistic Vlasov-Maxwell (RVM) system in three space dimensions remains elusive after nearly four decades of mathematical research. In this note, a simplified "toy model" is presented and studied. This toy model retains one crucial aspect of the RVM system: the phase-space evolution of the distribution function is governed by a transport equation whose forcing term satisfies a wave equation with finite speed of propagation.

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1. **Introduction.** Let $f(t, x, v) \geq 0$ denote the one particle distribution in phase space of a monocharged plasma, where $x, v \in \mathbb{R}^3$ denote particle position and momentum, respectively, and $t \geq 0$ is the temporal variable. Taking relativistic effects

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into account, but neglecting collisions among the particles, f satisfies the relativistic Vlasov-Maxwell system:

$$\left.\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + (E + \hat{v}\wedge B)\cdot\nabla_v f = 0 \\ \partial_t E = \nabla\wedge B - 4\pi j, \quad \nabla\cdot E = 4\pi\rho, \\ \partial_t B = -\nabla\wedge E, \quad \nabla\cdot B = 0, \end{aligned}\right\} \tag{RVM}$$

where

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad j(t,x) = \int_{\mathbb{R}^3} \hat{v} f(t,x,v)\,dv$$

are the charge and current density of the plasma, respectively, while

$$\hat{v} = \frac{v}{\sqrt{1+|v|^2}}$$

is the relativistic velocity. Additionally, $E(t,x)$ and $B(t,x)$ are the self-consistent electric and magnetic fields generated by the charged particles, and we have chosen units such that the mass and charge of each particle, as well as the speed of light, are normalized to one.

The rigorous study of the relativistic Vlasov-Maxwell system largely dates back to the 1980s. A local-in-time existence and uniqueness result due to Wollman [14] was followed by the conditional result of Glassey and Strauss [6], which to this day remains the most significant step toward a complete existence and uniqueness theory. In [6] it is shown that solutions of (RVM) remain regular so long as one knows *a priori* that particle momenta are uniformly bounded in time. In other words, if $\sup\{|v| \,|\, \exists x \in \mathbb{R}^3 \text{ s.t. } f(t,x,v) \neq 0\} < +\infty$, then the solution can be continued to time $t+h$ for some small $h > 0$. This condition has been shown to hold for small [7] and nearly neutral [3] data, and also in lower dimensions [4, 5]. Using Fourier methods, Klainerman and Staffilani [8] provided an alternative method to prove the conditional result in [6]. Bouchut, Golse and Pallard [1] gave yet another proof which relied on the so-called "division lemma", which we use as well. More recently, Luk and Strain [9] were able to improve the conditional result by weakening some of the assumptions.

The problem of global existence in three dimensions remains elusive. It is for this reason that attempts have been made to solve various toy models, in the hope that those may provide further insight into the full problem. Prior to our efforts, two related mean-field systems modeling resonance between a coupled wave equation and a transport equation have been investigated. In particular, Gérard and Pallard [2] considered the one-dimensional relativistic problem

$$\left.\begin{aligned} \partial_t f + \hat{v}\partial_x f + E\partial_v f = 0, \\ \Box E = \partial_x \rho, \end{aligned}\right\}$$

where $\rho(t,x) = \int f(t,x,v)\,dv$ as in (RVM). Similarly, Nguyen and Pankavich [10] considered a related non-relativistic problem (also in one space and one momentum dimensions)

$$\left.\begin{aligned} \partial_t f + v\partial_x f + B\partial_v f = 0, \\ (\partial_t + \partial_x) B = \rho, \end{aligned}\right\}$$

with each arriving at global existence results under limited assumptions.

The purpose of this note is to prove a global-in-time existence and uniqueness result for the following toy model of (RVM) kindly proposed to us by C. Bardos and F. Golse:

$$\left.\begin{aligned}\partial_t f + \hat{v}\cdot\nabla_x f - \partial_t A\cdot\nabla_v f = 0,\\ \Box A = (\partial_{tt}-\Delta)A = j,\end{aligned}\right\} \tag{Toy}$$

with initial data $f(0,x,v) = f_0(x,v)$ that is smooth and compactly supported and consistent data for A satisfying $A(0,x) = A_0(x)$ and $\partial_t A(0,x) = A_1(x)$. Here, the current density $j(t,x)$ is given by

$$j(t,x) = \int \hat{v} f(t,x,v)\, dv.$$

We note that (Toy) also couples a relativistic transport equation to a mean-field model of particle interaction given be a wave equation. For this system the position and momentum x, v can be taken in $\mathbb{R}^d$ for any $d \geq 1$, though $d = 3$ is the primary case of interest. Our main result, similar to the two previous results, considers the case $d = 1$.

1.1. **Main results.** We prove local existence for bounded initial data and global existence for initial data that is once continuously differentiable and compactly supported.

Theorem 1.1 (Local existence). *Suppose that $(f_0, A_0, A_1) \in W^{1,\infty}(\mathbb{R}^2) \times W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})$ with compact support, then there exists $T > 0$ such that the Cauchy problem (Toy) has a unique solution*

$$(f, A) \in W^{1,\infty}([0,T)\times\mathbb{R}^2) \times W^{1,\infty}([0,T)\times\mathbb{R}).$$

If we denote the maximal lifespan of the solution by T^*, then $T^* < +\infty$ necessarily implies

$$\limsup_{t\to T^*}\Big(\|\partial_x f(t,x,v)\|_{L^\infty_{x,v}(\mathbb{R}^2)} + \|\partial_v f(t,x,v)\|_{L^\infty_{x,v}(\mathbb{R}^2)}\Big) = +\infty.$$

Theorem 1.2 (Global existence). *If $(f_0, A_0, A_1) \in \mathcal{C}^1_c(\mathbb{R}^2)\times\mathcal{C}^1_c(\mathbb{R})\times\mathcal{C}_c(\mathbb{R})$, then the Cauchy problem (Toy) has a unique global solution such that*

$$(f, A) \in \mathcal{C}^1_c([0,\infty)\times\mathbb{R}^2)\times\mathcal{C}^1_c([0,\infty)\times\mathbb{R}).$$

The proofs of these theorems are contained within Sections 2 and 3, respectively. Next, we provide a justification for the structure of (Toy) and discuss the fundamental issues in obtaining analogous results in three dimensions.

1.2. **Justification of the toy model.** As determined by classical theory, the electromagnetic field (E, B) in (RVM) is derived from potentials φ and A that are given by

$$E = -\nabla\varphi - \partial_t A, \qquad B = \nabla\wedge A.$$

In the Lorenz gauge, Maxwell's equations further reduce to the system of wave equations for the associated potentials, namely

$$\Box\varphi = \rho, \qquad \Box A = j.$$

It is therefore evident that the simplified model (Toy) is obtained from (RVM) by neglecting the potential φ and assuming that A is irrotational, i.e. $\nabla\wedge A = 0$. These are not physically justifiable assumptions, yet they reduce (RVM) to a simplified system that still retains the main obstacle preventing us from proving global existence: the interplay between the Vlasov equation (which is a transport equation describing the evolution of particles in the system) whose speed of propagation has no *a priori* bound, and the wave equations governing the fields which propagate at a constant and finite speed (normalized to $c = 1$ here).

An important feature of the system (Toy) is that it has a natural energy. While this conserved quantity is not used in the present note, it is an aspect of this toy model which makes it a natural 'sibling' of (RVM). Indeed, multiplying the transport equation by $v_0 := \sqrt{1+|v|^2}$ and integrating in phase space, one easily finds that the quantity

$$\mathcal{E}_{\mathrm{Toy}} := \iint v_0 f(t,x,v)\, dv\, dx + \frac{1}{2}\int \left(|\partial_t A|^2 + |\nabla A|^2\right) dx \tag{Energy}$$

remains constant in time. In particular, we note that performing this same operation within the previously studied toy problems does not appear to produce a conserved energy.

It is also illuminating to compare (Toy) with the Vlasov-Poisson system, which is the classical limit of the (RVM) system as the speed of light tends to infinity [12]:

$$\left.\begin{aligned} \partial_t f + v\cdot\nabla_x f - \nabla\varphi\cdot\nabla_v f = 0,\\ -\Delta\varphi = \rho. \end{aligned}\right\} \tag{VP}$$

The two systems – (VP) and (Toy) – are very similar, though the former features classical transport, while the latter is relativistic. The most important distinction arises in the equation for the potential, which is elliptic in (VP) and hyperbolic in (Toy). Additionally, we note that the conserved energy is a crucial ingredient within some of the known proofs [11, 13] of global-in-time existence for smooth solutions of (VP). Hence, one may expect (Energy) to be similarly important to the study of (Toy) in three dimensions.

2. **Local existence and uniqueness.**

We first establish the local-in-time existence and uniqueness result using a fixed-point argument similar to [2].

Proof of Theorem 1.1. Since the initial data is assumed to have compact support, we can fix $R > 0$ and $M > 0$ such that $f_0 \in \mathcal{C}_c((-R,R)\times(-M,M))$.

Definition 2.1 (The Set B_T). For a given $T > 0$, we define B_T to be the set of functions $g \in W^{1,\infty}([0,T]\times\mathbb{R}^2)$ that satisfy

(H1) $g(0,x,v) = f_0(x,v)$ and $\|g\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}$;
(H2) $\operatorname{supp} g \subset [0,T]\times(-R-1,R+1)\times(-M-1,M+1)$;
(H3) $\|g\|_{\mathrm{Lip}} \le 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}$, where

$$\|g\|_{\mathrm{Lip}} := \sup_{\substack{t\in[0,T]\\ x,v\in\mathbb{R}\\ h=(h_1,h_2)\neq 0}} \frac{|g(t,x+h_1,v+h_2) - g(t,x,v)|}{|h|};$$

(H4) $\|\partial_t g\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}(2 + \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})})$.

When endowed with the metric $d(g_1,g_2) := \|g_1 - g_2\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, the metric space (B_T, d) is complete. Next, for any given $g \in B_T$, we define A_g to be the solution to the linear wave equation

$$\Box A_g = (\partial_t^2 - \partial_x^2)A_g = \int_{\mathbb{R}} \hat{v} g(t,x,v)dv := j_g(t,x), \tag{2.1}$$

with initial conditions $A_g(0,x) = A_0(x)$ and $\partial_t A_g(0,x) = A_1(x)$ for any $x\in\mathbb{R}$.

Definition 2.2 (The Solution Map Φ). For any $g \in B_T$ we define the solution map $f = \Phi(g)$ where $f \in W^{1,\infty}((0,T)\times\mathbb{R}^2)$ to be the unique solution of the transport equation

$$\partial_t f + \hat{v}\partial_x f - \partial_t A_g(t,x)\partial_v f = 0, \quad (t,x,v) \in (0,T)\times\mathbb{R}^2, \tag{2.2}$$

with initial condition $f(0,x,v) = f_0(x,v)$.

For the fixed-point argument it suffices to show the following two properties hold for T sufficiently small:
(1) Φ maps B_T into itself, i.e. for every $g \in B_T$, $f = \Phi(g) \in B_T$;
(2) $\Phi : B_T \to B_T$ is a contraction, i.e. there is $0 < C < 1$ such that

$$\|\Phi(g_1) - \Phi(g_2)\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le C\|g_1 - g_2\|_{L^\infty([0,T]\times\mathbb{R}^2)}$$

for every $g_1, g_2 \in B_T$.

Step 1: Preliminary estimates. Throughout we will use the fact that the mapping $v \mapsto \hat{v} = v/\sqrt{1+v^2}$ and its derivative are both bounded above by one. The function A_g of (2.1) is given by the solution of the wave equation, namely

$$\begin{aligned} &A_g(t,x) \\ &= (\partial_t Y(t,\cdot) *_x A_0)(t,x) + (Y(t,\cdot) *_x A_1)(t,x) + (Y(\cdot,\cdot) *_{t,x} (j_g \mathbb{1}_{t>0}(t)))(t,x) \end{aligned} \tag{2.3}$$

where $Y(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|\le t\}}$ is the forward fundamental solution of the one-dimensional wave operator. Note that the derivatives of Y satisfy

$$\partial_x Y(t,x) = \frac{1}{2}\delta_{x=-t} - \frac{1}{2}\delta_{x=t}, \quad \partial_t Y(t,x) = \frac{1}{2}\delta_{x=-t} + \frac{1}{2}\delta_{x=t} \tag{2.4}$$

(these expressions will be used later). More explicitly, the d'Alembert formula gives

$$A_g(t,x) = \frac{1}{2}[A_0(x+t) + A_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} A_1(s)\,ds + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} j_g(s,y)\,dy\,ds.$$

Thus, for $t \in [0,T]$ we find

$$\begin{aligned} \|A_g(t,\cdot)\|_{L^\infty(\mathbb{R})} &\le \|A_0\|_{L^\infty(\mathbb{R})} + t\|A_1\|_{L^\infty(\mathbb{R})} + \frac{1}{2}t^2\|j_g\|_{L^\infty([0,T]\times\mathbb{R})}, \\ \|\partial_t A_g(t,\cdot)\|_{L^\infty(\mathbb{R})} &\le \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + t\|j_g\|_{L^\infty([0,T]\times\mathbb{R})}, \\ \|\partial_x\partial_t A_g(t,\cdot)\|_{L^\infty(\mathbb{R})} &\le \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})} + t\|\partial_x j_g\|_{L^\infty([0,T]\times\mathbb{R})}. \end{aligned} \tag{2.5}$$

Taking $g \in B_T$, we have by (H1), (H2) and (H3)

$$\begin{aligned} \|j_g\|_{L^\infty([0,T]\times\mathbb{R})} &= \left\| \int_{\mathbb{R}} \hat{v} g(\cdot,\cdot,v)dv \right\|_{L^\infty([0,T]\times\mathbb{R})} \le 2(M+1)\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}, \\ \|\partial_x j_g\|_{L^\infty([0,T]\times\mathbb{R})} &= \left\| \int_{\mathbb{R}} \hat{v}\partial_x g(\cdot,\cdot,v)dv \right\|_{L^\infty([0,T]\times\mathbb{R})} \le 6(M+1)\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}. \end{aligned}$$

Hence, taking T sufficiently small we obtain the estimates

$$\begin{aligned} \|A_g\|_{L^\infty([0,T]\times\mathbb{R})} &\le \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + 1, \\ \|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} &\le \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + 1, \end{aligned} \tag{2.6}$$

$$\|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} \le \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})} + 1. \tag{2.7}$$

Step 2: Φ maps B_T into itself. Denote by $(X(s;t,x,v), V(s;t,x,v))$ the characteristic curves of (2.2). They satisfy the system of ODEs

$$\left.\begin{aligned} &\frac{dX}{ds}(s;t,x,v) = \hat{V}(s;t,x,v) = \frac{V(s;t,x,v)}{\sqrt{1+V^2(s;t,x,v)}}, \\ &\frac{dV}{ds}(s;t,x,v) = -(\partial_t A_g)(s, X(s;t,x,v)), \end{aligned}\right\} \tag{2.8}$$

with the initial conditions $X(t;t,x,v) = x$ and $V(t;t,x,v) = v$. It is well-known that the solution of the transport equation (2.2) can be expressed as

$$f(t,x,v) = f_0(X(0;t,x,v), V(0;t,x,v)).$$

From this, we immediately find that $f = \Phi(g)$ satisfies $f(0,x,v) = f_0(x,v)$ and

$$\|f\|_{L^\infty([0,T])\times\mathbb{R}^2)} = \|f_0(X(0;\cdot,\cdot,\cdot), V(0;\cdot,\cdot,\cdot)))\|_{L^\infty([0,T])\times\mathbb{R}^2)} = \|f_0\|_{L^\infty(\mathbb{R}^2)}$$

so that (H1) is satisfied. Additionally, using (2.6) we have for $T > 0$ sufficiently small

$$|x| = |X(t;t,x,v)| \le |X(0;t,x,v)| + \int_0^t |\hat{V}(s;t,x,v)| ds \le |X(0;t,x,v)| + 1$$

and

$$\begin{aligned} |v| = |V(t;t,x,v)| &\le |V(0;t,x,v)| + \int_0^t |(\partial_t A_g)(s, X(s;t,x,v))| ds \\ &\le |V(0;t,x,v)| + 1. \end{aligned}$$

for $t \in [0,T]$. Now, for $(x,v) \in \mathrm{supp}(f(t,\cdot,\cdot))$ one has $0 \ne f(t,x,v) = f_0(X(0;t,x,v), V(0;t,x,v))$ and consequently $|X(0;t,x,v)| \le R, |V(0;t,x,v)| \le M$. Therefore, $|x| \le R+1$ and $|v| \le M+1$ and (H2) is satisfied.

Next, we verify (H3). By the definition of the Lipschitz norm, we have

$$\begin{aligned} \|f\|_{\mathrm{Lip}} &= \sup_{\substack{t\in[0,T] \\ (x,v)\ne(y,p)\in\mathbb{R}^2}} \frac{|f(t,x,v) - f(t,y,p)|}{|(x-y, v-p)|} \\ &= \sup_{\substack{t\in[0,T] \\ (x,v)\ne(y,p)\in\mathbb{R}^2}} \frac{|f_0(X(0;t,x,v), V(0;t,x,v)) - f_0(X(0;t,y,p), V(0;t,y,p))|}{|(x-y, v-p)|} \\ &\le \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} (\|X\|_{\mathrm{Lip}} + \|V\|_{\mathrm{Lip}}) \end{aligned}$$

where the Lipschitz norms of the characteristics are defined by

$$\|X\|_{\mathrm{Lip}} := \sup_{\substack{s,t\in[0,T] \\ x,v\in\mathbb{R} \\ h=(h_1,h_2)\ne 0}} \frac{|X(s;t,x+h_1,v+h_2) - X(s;t,x,v)|}{|h|}$$

and analogously for $\|V\|_{\mathrm{Lip}}$. Integrating the characteristics of (2.8) yields

$$\begin{aligned} X(\tau;t,x,v) &= x + \int_t^\tau \hat{V}(s;t,x,v) ds, \\ V(\tau;t,x,v) &= v - \int_t^\tau (\partial_t A_g)(s, X(s;t,x,v)) ds, \end{aligned}$$

which provides the following bounds on the Lipschitz norms:

$$\|X\|_{\text{Lip}} \leq 1 + T\|V\|_{\text{Lip}},$$
$$\|V\|_{\text{Lip}} \leq 1 + T\|\partial_x \partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})}\|X\|_{\text{Lip}}.$$

Summing and using (2.7) then gives

$$\|X\|_{\text{Lip}} + \|V\|_{\text{Lip}} \leq 2 + \frac{1}{3}\left(\|V\|_{\text{Lip}} + \|X\|_{\text{Lip}}\right),$$

for T sufficiently small, which implies $\|X\|_{\text{Lip}} + \|V\|_{\text{Lip}} \leq 3$. Inserting this into the estimate on $\|f\|_{\text{Lip}}$, we conclude

$$\|f\|_{\text{Lip}} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} \tag{2.9}$$

and (H3) is satisfied.

Finally, we verify (H4). Computing the time derivative of f, we find

$$\begin{aligned}\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)} &= \|\partial_t[f_0(X(0;\cdot,\cdot,\cdot), V(0;\cdot,\cdot,\cdot))]\|_{L^\infty([0,T]\times\mathbb{R}^2)} \\ &\leq \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}\left(|\partial_t X(0;\cdot,\cdot,\cdot)|_{L^\infty([0,T]\times\mathbb{R}^2)} + |\partial_t V(0;\cdot,\cdot,\cdot)|_{L^\infty([0,T]\times\mathbb{R}^2)}\right).\end{aligned}$$

To bound the two terms on the right hand side, we first estimate

$$\begin{aligned}|\partial_t X(\tau;t,x,v)| &= \left|\partial_t\left(x + \int_t^\tau \hat{V}(s;t,x,v)ds\right)\right| \\ &= \left|-\hat{v} - \int_\tau^t \partial_v(\hat{V}(s;t,x,v))\partial_t V(s;t,x,v)ds\right| \\ &\leq 1 + \int_0^t |\partial_t V(s;t,x,v)|ds\end{aligned}$$

and

$$\begin{aligned}|\partial_t V(\tau;t,x,v)| &= \left|\partial_t\left(v - \int_t^\tau (\partial_t A_g)(s, X(s;t,x,v))ds\right)\right| \\ &= \left|\partial_t A_g(t,x) + \int_\tau^t (\partial_x\partial_t A_g)(s,X(s;t,x,v))\partial_t X(s;t,x,v)ds\right| \\ &\leq \|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})}\int_0^t |\partial_t X(s;t,x,v)|ds.\end{aligned}$$

Therefore, using (2.6) we obtain

$$\begin{aligned}&\sup_{\tau\in[0,t]}\left(|\partial_t X(\tau,t,x,v)| + |\partial_t V(\tau,t,x,v)|\right) \leq 1 + \|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} \\ &\quad + (1 + \|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t (|\partial_t V(s;t,x,v)| + |\partial_t X(s;t,x,v)|)ds \\ &\quad \leq 2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} \\ &\quad + (2 + \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})})\int_0^t \sup_{\tau\in[0,s]}\left(|\partial_t X(\tau;t,x,v)| + |\partial_t V(\tau;t,x,v)|\right)ds.\end{aligned}$$

Invoking Grönwall's inequality now yields

$$\begin{aligned}&\sup_{\tau\in[0,t]}\left(|\partial_t X(\tau;t,x,v)| + |\partial_t V(\tau;t,x,v)|\right) \\ &\qquad\qquad \leq \left(2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})}\right)e^{t(2+\|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})})}.\end{aligned}$$

Using this estimate we ultimately find

$$|\partial_t X(0;t,x,v)| + |\partial_t V(0;t,x,v)| \leq 3\left(2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})}\right)$$

for all $t \in [0,T]$, $x, v \in \mathbb{R}$ and $T > 0$ sufficiently small. Taking the supremum over $x, v \in \mathbb{R}$ and combining this with the estimate of $\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$ yields

$$\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}(2 + \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})})$$

and (H4) is satisfied.

Step 3: Φ is a contraction. Let $g, \tilde{g} \in B_T$ and $f = \Phi(g), \tilde{f} = \Phi(\tilde{g})$. Then, subtracting the respective Vlasov equations yields

$$\partial_t(f - \tilde{f}) + \hat{v}\partial_x(f - \tilde{f}) - \partial_t A_g(t,x)\partial_v(f - \tilde{f}) - \big(\partial_t A_g(t,x) - \partial_t A_{\tilde{g}}(t,x)\big)\partial_v \tilde{f} = 0$$

with $(f - \tilde{f})(0,x,v) = 0$. Consequently

$$\begin{aligned}&(f - \tilde{f})(t, X(t;0,x,v), V(t;0,x,v)) \\ &= -\int_0^t \big(\partial_t A_g - \partial_t A_{\tilde{g}}\big)(s, X(s;0,x,v)) \cdot \partial_v \tilde{f}(s, X(s;0,x,v), V(s;0,x,v))ds.\end{aligned} \tag{2.10}$$

Using (2.5) and (2.9), we have the following estimates for the right side of (2.10):

$$\begin{aligned}&\|\partial_v \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}, \\ &\|\partial_t A_g - \partial_t A_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R})} \leq 2T\|j_g - j_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R})}.\end{aligned}$$

Therefore, we find from (2.10) that

$$\|f - \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} T^2 \|j_g - j_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R})},$$

which implies

$$\|f - \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq \frac{1}{2}\|g - \tilde{g}\|_{L^\infty([0,T]\times\mathbb{R}^2)}$$

provided that T is sufficiently small. Thus, we obtain a unique local solution to the Cauchy problem (Toy) on $[0,T]$ for T sufficiently small. Furthermore, we can extend the lifespan of the solution as long as derivatives remain finite, namely for any $t \in [0,T]$ such that

$$\|\partial_x f(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2)} + \|\partial_v f(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2)} < +\infty.$$

This completes the proof. □

3. **Global existence.** With the existence of a local-in-time solution established, we now extend the solution globally by uniformly bounding the momentum support of the distribution function and the derivatives of the field.

Proof of Theorem 1.2. We assume that the maximal life span is $[0, T^*)$ for some $T^* > 0$, and shall prove that $T^* = +\infty$, hence the solution is global. We need to show

$$(f, \partial_t A) \in W^{1,\infty}([0,T^*) \times \mathbb{R}^2) \times W^{1,\infty}([0,T^*) \times \mathbb{R}). \tag{3.1}$$

Step 1: Bounds on f and $\partial_t A$. From the proof of local existence, we know that f has compact support for any $t \in [0,T^*)$. In particular, the v support of f is uniformly bounded for any fixed time $t \in [0,T^*)$ but may tend to $+\infty$ as $t \to T^*$. We therefore define the following crucial quantity

$$P(t) := \sup\{|v| : \exists\, x \in \mathbb{R}\, \text{such that}\, f(t,x,v) \neq 0\}$$

and prove the following result.

Proposition 3.1. *Let T^* be the maximal lifespan of the solution and let $T \in (0, T^*)$. Then there exists $C > 0$ independent of T such that*

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \le C \tag{3.2}$$

and

$$P(T) \le C. \tag{3.3}$$

Proof. Our strategy is classical: we generate a Grönwall-type inequality by first using the wave equation $\Box A = j$ to show that $\partial_t A$ is controlled by j, then showing that j can be controlled by $P(t)$, and finally bounding $P(t)$ by a time-integral of $\partial_t A$.

Define the functions

$$B^\pm(t,x) = \partial_t A(t,x) \pm \partial_x A(t,x).$$

Using the relationship $\Box A = j$, we have

$$(\partial_t \mp \partial_x) B^\pm(t,x) = j(t,x),$$

so that, for any $h > 0$,

$$\begin{aligned}\partial_\tau [B^\pm(\tau, x \pm (t+h-\tau))] &= [(\partial_t \mp \partial_x)B^\pm](\tau, x \pm (t+h-\tau)) \\ &= j(\tau, x \pm (t+h-\tau)).\end{aligned}$$

Integrating with respect to $\tau \in [t, t+h]$, we obtain

$$B^\pm(t+h, x) = B^\pm(t, x \pm h) + \int_t^{t+h} j(\tau, x \pm (t+h-\tau))d\tau.$$

Taking $t = 0$ and replacing h by t, we can represent $B^\pm$ as:

$$B^\pm(t,x) = B^\pm(0, x \pm t) + \int_0^t j(\tau, x \pm (t-\tau))d\tau.$$

This allows us to represent $\partial_t A$ as follows:

$$\begin{aligned}\partial_t A(t,x) =& \frac{1}{2}(B^+(t,x) + B^-(t,x)) \\ =& \frac{1}{2}\left(A_0'(x+t) - A_0'(x-t) + A_1(x+t) + A_1(x-t)\right) \\ &+ \frac{1}{2}\int_0^t \left(j(\tau, x-(t-\tau)) + j(\tau, x+(t-\tau))\right) d\tau.\end{aligned} \tag{3.4}$$

We are now ready to prove (3.2). Using (3.4) and the properties of the initial data, for $T \in [0, T^*)$ we can estimate

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \le C_1 \left(1 + \sum_\pm \sup_{\substack{x\in\mathbb{R} \\ t\in[0,T]}} \left|\int_0^t j(\tau, x \pm (t-\tau))d\tau\right|\right)$$

where C_1 only depends on the initial data. We therefore turn to bounding $\|j\|_{L^\infty([0,T]\times\mathbb{R})}$. Because the relativistic velocity is bounded above by $|\hat{v}| < 1$ and using the definition of $P(t)$, we have

$$|j(t,x)| = \left|\int_{\mathbb{R}} \hat{v} f(t,x,v)dv\right| \le \|f_0\|_{L^\infty} P(t)$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$. Due to the characteristic equations (2.8), the change in velocity is governed by $\partial_t A$ so that

$$P(t) \leq P(0) + \int_0^t \|\partial_t A(\tau,\cdot)\|_{L^\infty} d\tau.$$

Inserting the last three estimates into one another, we find

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \leq C_2 \left(1 + \int_0^T \|\partial_t A(\tau,\cdot)\|_{L^\infty([0,T]\times\mathbb{R})} d\tau\right) \tag{3.5}$$

where C_2 also only depends on the initial data. A standard Grönwall argument applied to (3.5) yields

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \leq C_3 e^T \leq C_3 e^{T^*} \leq C$$

where C_3 again only depends on the initial data and $C < +\infty$ depends only on the initial data and T^*, but not on T. Therefore

$$P(T) \leq P(0) + CT^*.$$

This completes the proof of Proposition 3.1. □

Step 2: Bounds on the derivatives of f and $\partial_t A$. The transport equation for f in (Toy) takes the following form for the derivatives of f:

$$(\partial_t + \hat{v}\partial_x + \partial_t A(t,x)\partial_v)\begin{pmatrix}\partial_x f\\ \partial_v f\end{pmatrix} = -\begin{pmatrix}0 & \partial_x\partial_t A(t,x)\\ (1+|v|^2)^{-3/2} & 0\end{pmatrix}\begin{pmatrix}\partial_x f\\ \partial_v f\end{pmatrix}.$$

We therefore need to bound $\partial_x\partial_t A$.

Proposition 3.2. *Let T^* be the maximal lifespan of the solution and let $T \in (0,T^*)$. Then there exists $C > 0$ independent of T such that*

$$\|\partial_x\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \leq C.$$

Proof. Since A satisfies $\Box A = j$, inverting the wave operator means that A is obtained from j and the initial data via the expression (2.3). Assuming, without loss of generality, that the initial data for the field is trivial, i.e., $A_0 = A_1 = 0$, (2.3) reduces to

$$A_f(t,x) = (Y(\cdot,\cdot) *_{t,x} (j_f \mathbb{1}_{t>0})(t,x),$$

where we recall that $Y = \frac{1}{2}\mathbb{1}_{\{|x|\leq t\}}$ is the forward fundamental solution of the one-dimensional wave operator. Therefore

$$\partial_x\partial_t A_f(t,x) = \partial_x\partial_t \left(Y(\cdot,\cdot) *_{t,x} (j_f \mathbb{1}_{t>0})\right)(t,x) = \left(\partial_t Y(\cdot,\cdot) *_{t,x} (j_{\partial_x f}\mathbb{1}_{t>0})\right)(t,x),$$

and using the Vlasov equation $\partial_t f + \hat{v}\partial_x f - \partial_t A_f \partial_v f = 0$ in the term $j_{\partial_x f} = \int \hat{v}\partial_x f\, dv$, we have

$$\partial_x\partial_t A_f(t,x) = \left[\partial_t Y(\cdot,\cdot) *_{t,x} (\partial_t \rho_f \mathbb{1}_{t>0})\right](t,x).$$

Integrating by parts in the convolution (and henceforth dropping the subscript f for brevity), we transfer the time derivative from ρ to Y so that

$$\partial_x\partial_t A(t,x) = \left[\partial_{tt} Y(\cdot,\cdot) *_{t,x} \rho\mathbb{1}_{t>0}\right](t,x) + \left[\partial_t Y(t,\cdot) *_x \rho(0,\cdot)\right](x).$$

As the fundamental solution Y satisfies $\Box Y = \delta_{(t,x)=(0,0)}$, this further yields

$$\begin{aligned}\partial_x\partial_t A(t,x) &= \left[\left(\partial_{xx}Y + \delta_{(t,x)=(0,0)}\right) *_{t,x} \rho\mathbb{1}_{t>0}\right](t,x) + \left[\partial_t Y(t,\cdot) *_x \rho(0,\cdot)\right](x) \\ &= \underbrace{\left[\partial_{xx}Y *_{t,x} \rho\mathbb{1}_{t>0}\right](t,x)}_{I} + \underbrace{\left[\partial_t Y(t,\cdot) *_x \rho(0,\cdot)\right](x)}_{II} + \rho(t,x).\end{aligned}$$

Let us consider the term I. Using the division lemma (Lemma A.1) with $a(v) = \hat{v} = \frac{v}{\sqrt{1+v^2}}$ we write $\partial_{xx}Y$ as

$$\partial_{xx}Y = (\partial_t + \hat{v}\partial_x)\left(\frac{x}{\hat{v}x - t}\partial_x Y\right) + (1+v^2)\delta_{(t,x)=(0,0)}.$$

Because this holds for every v, in the term I we replace ρ with $\int f\,dv$ and get

$$\begin{aligned}I &= \int\left[(\partial_t + \hat{v}\partial_x)\left(\frac{x}{\hat{v}x - t}\partial_x Y\right) *_{t,x} f(\cdot,\cdot,v)\mathbb{1}_{t>0}\right](t,x)\,dv \\ &\quad + \int\left[(1+v^2)\delta_{(t,x)=(0,0)} *_{t,x} f(\cdot,\cdot,v)\mathbb{1}_{t>0}\right](t,x)\,dv \\ &= \int\left[\frac{x}{\hat{v}x - t}\partial_x Y *_{t,x} (\partial_t + \hat{v}\partial_x)(f(\cdot,\cdot,v)\mathbb{1}_{t>0})\right](t,x)\,dv + \int(1+v^2)f(t,x,v)\,dv \\ &= \underbrace{\int\left[\frac{x}{\hat{v}x - t}\partial_x Y *_{t,x} \partial_t A\partial_v f(\cdot,\cdot,v)\mathbb{1}_{t>0}\right](t,x)\,dv}_{I_a} \\ &\quad + \underbrace{\int\left[\frac{x}{\hat{v}x - t}\partial_x Y *_{t,x} f(\cdot,\cdot,v)\delta_{t=0}\right](t,x)\,dv}_{I_b} + \int(1+v^2)f(t,x,v)\,dv.\end{aligned}$$

Using the properties of the derivatives of Y (see (2.4)), we can simplify the terms I_a and I_b. Let us first consider the term I_a:

$$\begin{aligned}I_a &= \int\left[\frac{x}{\hat{v}x - t}\partial_x Y *_{t,x} \partial_t A\partial_v f(\cdot,\cdot,v)\mathbb{1}_{t>0}\right](t,x)\,dv \\ &= \frac{1}{2}\int\left[\frac{x}{\hat{v}x - t}(\delta_{x=-t} - \delta_{x=t}) *_{t,x} \partial_t A\partial_v f(\cdot,\cdot,v)\mathbb{1}_{t>0}\right](t,x)\,dv \\ &= \frac{1}{2}\int\int_0^t\int\frac{y}{\hat{v}y - s}(\delta_{y=-s} - \delta_{y=s})\partial_t A(t-s,x-y)\partial_v f(t-s,x-y,v)\,dy\,ds\,dv \\ &= \frac{1}{2}\sum_{\pm}\int\int_0^t\frac{1}{1\pm\hat{v}}\partial_t A(t-s,x\pm s)\partial_v f(t-s,x\pm s,v)\,ds\,dv.\end{aligned}$$

To integrate by parts in v, we observe that

$$\frac{d}{dv}\left(\frac{1}{1\pm\hat{v}}\right) = \frac{v\mp\sqrt{1+v^2}}{1+v^2\pm v\sqrt{1+v^2}}$$

and we obtain

$$I_a = \frac{1}{2}\sum_{\pm}\mp\int\int_0^t\frac{v\mp\sqrt{1+v^2}}{1+v^2\pm v\sqrt{1+v^2}}\partial_t A(t-s,x\pm s)f(t-s,x\pm s,v)\,ds\,dv.$$

Turning to the term I_b, and again using (2.4), we have

$$\begin{aligned} I_b &= \frac{1}{2}\int \left[\frac{x}{\hat{v}x - t}(\delta_{x=-t} - \delta_{x=t}) *_{t,x} f(\cdot,\cdot,v)\delta_{t=0}\right](t,x)\,dv \\ &= \frac{1}{2}\int\int_0^t\int \frac{y}{\hat{v}y - s}(\delta_{y=-s} - \delta_{y=s}) f(t-s, x-y, v)\delta_{t-s=0}\,dy\,ds\,dv \\ &= \frac{1}{2}\sum_{\pm} \pm \int \frac{1}{\hat{v} \pm 1} f(0, x \pm t, v)\,dv. \end{aligned}$$

Now we can consider the term II which easily simplifies to

$$II = [\partial_t Y(t,\cdot) *_x \rho(0,\cdot)](x) = \frac{1}{2}\int (\delta_{y=-t} + \delta_{y=t})\rho(0, x-y)\,dx = \frac{1}{2}\sum_{\pm}\rho(0, x \pm t).$$

Collecting these terms, we have

$$\begin{aligned} &\partial_x\partial_t A(t,x) \\ &= \frac{1}{2}\sum_{\pm} \mp \int\int_0^t \frac{v \mp \sqrt{1+v^2}}{1+v^2 \pm v\sqrt{1+v^2}} \partial_t A(t-s, x \pm s) f(t-s, x \pm s, v)\,ds\,dv \\ &\quad + \underbrace{\frac{1}{2}\sum_{\pm} \pm \int \frac{1}{\hat{v} \pm 1} f(0, x \pm t, v)\,dv + \frac{1}{2}\sum_{\pm}\rho(0, x \pm t)}_{\text{data}} \\ &\quad + \int (2+v^2) f(t,x,v)\,dv. \end{aligned}$$

To estimate $\partial_x\partial_t A_f(t,x)$ we need to bound the first and last terms; the "data" terms depend on the initial data and are therefore finite and independent of t. The integrand within the first term can be controlled by $1 + P(T)$. Indeed, letting $v_0 = \sqrt{1+v^2}$ yields

$$\left|\frac{v \mp \sqrt{1+v^2}}{1+v^2 \pm v\sqrt{1+v^2}}\right| = v_0^{-1}\left|\frac{v \mp v_0}{v_0 \pm v}\right| = v_0^{-1}|v_0 \mp v|^2 \le 2v_0,$$

and thus

$$\sup_{|v|\le P(T)} \left|\frac{v \mp \sqrt{1+v^2}}{1+v^2 \pm v\sqrt{1+v^2}}\right| \le 2\sqrt{1+P(T)^2} \le 2(1+P(T)).$$

Hence, allowing C to be a constant independent of T that may change from line to line, we have

$$\begin{aligned} \|\partial_x\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} &\le \text{data} \\ &\quad + CTP(T)\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})}\|f_0\|_{L^\infty(\mathbb{R})} \sup_{|v|\le P(T)} \left|\frac{v \mp \sqrt{1+v^2}}{1+v^2 \pm v\sqrt{1+v^2}}\right| \\ &\quad + \|f_0\|_{L^\infty(\mathbb{R})}(2+P(T)^2)P(T) \\ &\le \text{data} + CP(T)\|f_0\|_{L^\infty(\mathbb{R})}\left(T^*(1+P(T))\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} + 2 + P(T)^2\right). \end{aligned}$$

Inserting here the uniform bounds on $\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})}$ and $P(T)$ from (3.2) and (3.3), respectively, we conclude a uniform bound independent of T on $\|\partial_x\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})}$, and the proof is complete. □

The bound for $\|\partial_x \partial_t A\|_{L^\infty([0,T]\times\mathbb{R})}$ proved in Proposition 3.2 leads immediately to a uniform bound for the derivatives of f, so that the condition (3.1) is verified,

$$(f, \partial_t A) \in W^{1,\infty}([0,T^*) \times \mathbb{R}^2) \times W^{1,\infty}([0,T^*) \times \mathbb{R}),$$

and the solution is global. This completes the proof of Theorem 1.2. □

Appendix A. Division lemma.

Lemma A.1 (Division lemma, originally in [1], appearing in this particular form in [2]). *Let $Y(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|\le t\}}$ be the forward fundamental solution of the 1d wave operator and $a(v) \in (-1,1)$. Then the equality*

$$\partial_x^2 Y = (\partial_t + a(v)\partial_x)\left(\frac{x}{a(v)x - t}\partial_x Y\right) + \frac{1}{a(v)^2-1}\delta_{(t,x)=(0,0)} \tag{A.1}$$

holds in $\mathcal{D}'(\mathbb{R}^2)$.

Proof. Denoting $m(t,x) = \frac{x}{ax-t}$ and $T = \partial_t + a\partial_x$ the identity (A.1) which we seek to prove can be rewritten as

$$T(m\partial_x Y) = -\frac{1}{a(v)^2-1}\delta_{(t,x)=(0,0)} + \partial_x^2 Y. \tag{A.2}$$

Recalling that $\partial_x Y(t,x) = \frac{1}{2}\delta_{x=-t} - \frac{1}{2}\delta_{x=t}$, one needs to be clear about the meaning of the left side of (A.2). Having the terms $\delta_{x=-t}$ and $\delta_{x=t}$, combined with the restriction $|a| \neq 1$, means that $m\partial_x Y \in \mathcal{D}'(\mathbb{R}^2\backslash\{(0,0)\})$ is a well-defined distribution which is homogeneous of order -1. It admits a unique extension as a homogeneous distribution of order -1 in $\mathcal{D}'(\mathbb{R}^2)$ which we still denote $m\partial_x Y$. Working with test functions, we observe that for every $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$\langle m\partial_x Y, \varphi\rangle = \frac{1}{2}\frac{1}{a+1}\int_0^\infty \varphi(t,-t)\,dt - \frac{1}{2}\frac{1}{a-1}\int_0^\infty \varphi(t,t)\,dt.$$

We can now compute $\langle T(m\partial_x Y), \varphi\rangle$ by integrating by parts, and using the observation that

$$T = \partial_t + a\partial_x = \partial_t + \partial_x + (a-1)\partial_x = \partial_t - \partial_x + (a+1)\partial_x.$$

Specifically, we obtain

$$\begin{aligned}\langle T(m\partial_x Y), \varphi\rangle &= \frac{1}{2}\frac{1}{a+1}\int_0^\infty (-T\varphi)(t,-t)\,dt - \frac{1}{2}\frac{1}{a-1}\int_0^\infty (-T\varphi)(t,t)\,dt \\ &= -\frac{1}{a^2-1}\varphi(0,0) + \frac{1}{2}\int_0^\infty (\partial_x\varphi(t,t) - \partial_x\varphi(t,-t))\,dt.\end{aligned}$$

The proof is complete by observing that the last term is precisely $\langle \partial_x^2 Y, \varphi\rangle$. Indeed,

$$\langle \partial_x^2 Y, \varphi\rangle = \frac{1}{2}\int_0^\infty \int_{-t}^t \partial_x^2\varphi(t,x)\,dx\,dt = \frac{1}{2}\int_0^\infty (\partial_x\varphi(t,t) - \partial_x\varphi(t,-t))\,dt.$$

□

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