

## ERRATUM TO “GLOBAL CLASSICAL SOLUTIONS OF THE ‘ONE AND ONE-HALF’ DIMENSIONAL VLASOV-MAXWELL-FOKKER-PLANCK SYSTEM”

STEPHEN PANKAVICH\* AND JACK SCHAEFFER†

**Abstract.** This note is an erratum to [S. Pankavich and J. Schaeffer, *Comm. Math. Sci.*, 14(1):209–232, 2016] and corrects an  $L^2$  estimate concerning derivatives of a Green’s function for the linear Vlasov–Fokker–Planck operator. Here, the proof that relies on this estimate is corrected using alternative means.

**Key words.** Kinetic Theory, Vlasov, Fokker–Planck equation, global existence.

**AMS subject classifications.** 35L60, 35Q83, 82C22, 82D10.

### 1. Introduction

The purpose of this article is to correct an error in [1]. The notation and assumptions used in [1] will be used here, and it is assumed that the reader is familiar with [1]. In particular, the estimate

$$\left( \iint |\nabla_w \mathcal{G}(t, x, v, \tau, y, w)|^2 dw dy \right)^{\frac{1}{2}} \leq C(t - \tau)^{-1/2}, \quad (1.1)$$

which appears just above line (4.28) of [1], is incorrect. What follows completes the proof of Theorem 1.1 of [1] without using the estimate (1.1) by proving that the solution possesses the stated regularity.

### 2. Correction to Proof of Theorem 1.1

Proceeding as in (4.16) of [1] we have

$$\begin{aligned} & \frac{d}{dt} \iint v_0^a (f^{n+1} - f^{\ell+1})^2 dv dx \\ &= -2 \iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 dv dx \\ &+ \iint (f^{n+1} - f^{\ell+1})^2 (\Delta_v v_0^a + E^n \cdot \nabla_v v_0^a) dv dx \\ &+ 2 \iint f^{\ell+1} (K^n - K^\ell) \cdot (v_0^a \nabla_v (f^{n+1} - f^{\ell+1})) (f^{n+1} - f^{\ell+1}) \nabla_v v_0^a dv dx. \end{aligned} \quad (2.1)$$

We will use the notation

$$\sigma^{n,\ell} \leq o^{n,\ell}$$

to mean  $\forall \epsilon > 0 \exists N$  such that  $n, \ell > N \Rightarrow \sigma^{n,\ell} \leq \epsilon$ . Recall that

$$\sup_{t \in [0, T]} \|(E^n B^n) - (E^\ell, B^\ell)\|_{H^1(\mathbb{R})} + \sup_{t \in [0, T]} \iint v_0^a (f^{n+1} - f^{\ell+1})^2 dv dx \leq o^{n,\ell}.$$

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\*Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado 80401, ([pankavic@mines.edu](mailto:pankavic@mines.edu))

†Department of Mathematics Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, ([js5m@andrew.cmu.edu](mailto:js5m@andrew.cmu.edu))

By (4.14) of [1] we have

$$\begin{aligned} |K^n - K^\ell| &\leq v_0 |(E^n, B^n) - (E^\ell, B^\ell)| + v_0^{1+\epsilon/2} |B^\ell| 2^{-n\epsilon/2} \\ &\leq v_0^{1+\epsilon/2} o^{n,\ell} \end{aligned}$$

where we assume  $\ell > n$ .

Recall that  $\alpha = a + 2 + \epsilon$  and note that

$$\begin{aligned} &\iint |f^{\ell+1} (K^n - K^\ell) \cdot v_0^a \nabla_v (f^{n+1} - f^{\ell+1})| \, dv dx \\ &\leq o^{n,\ell} \iint |f^{\ell+1}| v_0^{a+1+\epsilon/2} |\nabla_v (f^{n+1} - f^{\ell+1})| \, dv dx \\ &\leq o^{n,\ell} \sqrt{\iint v_0^\alpha (f^{\ell+1})^2 \, dv dx} \sqrt{\iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx} \\ &\leq o^{n,\ell} \sqrt{\iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\iint |f^{\ell+1} (K^n - K^\ell) \cdot (f^{n+1} - f^{\ell+1}) \nabla_v v_0^a| \, dv dx \\ &\leq o^{n,\ell} \sqrt{\iint v_0^a (f^{n+1} - f^{\ell+1})^2 \, dv dx} \leq o^{n,\ell}. \end{aligned}$$

So, by Equation (2.1) we find

$$\begin{aligned} &\frac{d}{dt} \iint v_0^a (f^{n+1} - f^{\ell+1})^2 \, dv dx \\ &\leq - \iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx + o^{n,\ell} \\ &\quad + \left[ o^{n,\ell} \sqrt{\iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx} - \iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx \right] \\ &\leq o^{n,\ell} - \iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx. \end{aligned}$$

It follows that

$$\int_0^T \iint v_0^a |\nabla_v (f^{n+1} - f^{\ell+1})|^2 \, dv dx d\tau \leq o^{n,\ell}. \tag{2.2}$$

In a similar manner we may show that

$$\int_0^T \iint v_0^\alpha |\nabla_v f^{n+1}|^2 \, dv dx d\tau \leq C. \tag{2.3}$$

Note that the exponent of  $v_0$  in the inequality (2.2) is  $a$ , but in the inequality (2.3) it is  $\alpha$ .

Next, we derive  $L^p$  bounds on  $\mathcal{G}$ . Considering  $0 \leq \tau \leq t \leq T$ , letting  $p \geq 1$ ,  $b, \theta \geq 0$ , and using the change of variables

$$u = \frac{v-w}{\sqrt{t-\tau}}, \quad z = \frac{x-y - \frac{t-\tau}{2}(v_1+w_1)}{(t-\tau)^{3/2}}$$

we have

$$\begin{aligned} & \iint w_0^{-b\theta} \mathcal{G}^p dw dy \\ &= C \iint \left( \sqrt{1+|v-\sqrt{t-\tau}u|^2} \right)^{-b\theta} \left[ (t-\tau)^{-5/2} e^{-|u|^2/4} e^{-3z^2} \right]^p (t-\tau)^{5/2} dz du \\ &= C(t-\tau)^{\frac{5}{2}(1-p)} \left( \int_{|u| < \frac{1}{2}(t-\tau)^{-1/2}|v|} \left( \sqrt{1+|v-\sqrt{t-\tau}u|^2} \right)^{-b\theta} e^{-p|u|^2/4} du \right. \\ & \quad \left. + \int_{|u| > \frac{1}{2}(t-\tau)^{-1/2}|v|} \left( \sqrt{1+|v-\sqrt{t-\tau}u|^2} \right)^{-b\theta} e^{-p|u|^2/4} du \right) \\ &\leq C(t-\tau)^{\frac{5}{2}(1-p)} \left[ \left( \sqrt{1+\left(\frac{1}{2}|v|\right)^2} \right)^{-b\theta} \int e^{-|u|^2/4} du + e^{-\frac{1}{8}\left(\frac{|v|}{2\sqrt{t-\tau}}\right)^2} \int e^{-|u|^2/8} du \right] \\ &\leq C(t-\tau)^{\frac{5}{2}(1-p)} \left[ \left( \sqrt{1+|v|^2} \right)^{-b\theta} + e^{-C|v|^2} \right] \\ &\leq C(t-\tau)^{\frac{5}{2}(1-p)} v_0^{-b\theta}. \end{aligned}$$

So for  $1 \leq p < 7/5$  we have

$$\int_0^t \iint w_0^{-b\theta} \mathcal{G}^p dw dy d\tau \leq C v_0^{-b\theta}. \tag{2.4}$$

Here, constants may depend on  $p, b$ , and  $\theta$ . Later, specific choices of  $p, b$ , and  $\theta$  are used and this dependence is removed.

Next, we bound  $|\nabla_w \mathcal{G}|$  and  $|\nabla_v \mathcal{G}|$ . Note that

$$\begin{aligned} & \iint w_0^{-b\theta} \left( \frac{|v-w|}{t-\tau} \mathcal{G} \right)^p dw dy \\ &= C \iint \left( \sqrt{1+|v-\sqrt{t-\tau}u|^2} \right)^{-b\theta} \left[ \frac{|u|}{\sqrt{t-\tau}} (t-\tau)^{-5/2} e^{-|u|^2/4} e^{-3z^2} \right]^p (t-\tau)^{5/2} dz du \\ &= C(t-\tau)^{\frac{5}{2}-3p} \int \left( \sqrt{1+|v-\sqrt{t-\tau}u|^2} \right)^{-b\theta} \left[ |u| e^{-|u|^2/4} \right]^p du \\ &\leq C(t-\tau)^{\frac{5}{2}-3p} v_0^{-b\theta} \end{aligned}$$

and similarly,

$$\iint w_0^{-b\theta} \left( \frac{|x-y - \frac{t-\tau}{2}(v_1+w_1)|}{(t-\tau)^2} \mathcal{G} \right)^p dw dy \leq C(t-\tau)^{\frac{5}{2}-3p} v_0^{-b\theta}.$$

Hence,

$$\iint w_0^{-b\theta} (|\nabla_w \mathcal{G}|^p + |\nabla_v \mathcal{G}|^p) dw dy \leq C(t-\tau)^{\frac{5}{2}-3p} v_0^{-b\theta}$$

and for  $1 \leq p < 7/6$

$$\int_0^t \iint w_0^{-b\theta} (|\nabla_w \mathcal{G}|^p + |\nabla_v \mathcal{G}|^p) dw dy d\tau \leq C v_0^{-b\theta}. \tag{2.5}$$

In a very similar manner it may be shown that

$$\int_0^T \iint v_0^{-b\theta} \mathcal{G}^p dv dx dt \leq C w_0^{-b\theta} \tag{2.6}$$

for  $p < 7/5$  and

$$\int_0^T \iint v_0^{-b\theta} (|\nabla_v \mathcal{G}|^p + |\nabla_w \mathcal{G}|^p) dv dx dt \leq C w_0^{-b\theta}$$

for  $p < 7/6$ .

Next we derive two inequalities that will be used repeatedly. Let  $p, q \in [1, \infty)$  and  $r \in [1, \infty]$  satisfy

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}.$$

We will first consider the case  $r \neq \infty$ , but what follows may be easily adapted to the case  $r = \infty$ . Let  $\theta \geq 0$  and define

$$b = \left(\frac{1}{p} - \frac{1}{r}\right)^{-1} \quad c = \left(\frac{1}{q} - \frac{1}{r}\right)^{-1}$$

and note that

$$\frac{1}{r} + \frac{1}{b} + \frac{1}{c} = 1.$$

By Hölder’s inequality and using the inequality (2.4), we have for  $p < 7/5$  and any  $h(\tau, y, w) \geq 0$

$$\begin{aligned} \int_0^t \iint \mathcal{G} h dw dy d\tau &= \int_0^t \iint \left[ \mathcal{G}^{p/r} (w_0^\theta h)^{q/r} \right] \left[ \mathcal{G}^{p(\frac{1}{p} - \frac{1}{r})} w_0^{-\theta} \right] \left[ w_0^\theta h \right]^{q(\frac{1}{q} - \frac{1}{r})} dw dy d\tau \\ &\leq \left( \int_0^t \iint \mathcal{G}^p (w_0^\theta h)^q dw dy d\tau \right)^{\frac{1}{r}} \left( \int_0^t \iint \mathcal{G}^p w_0^{-b\theta} dw dy d\tau \right)^{\frac{1}{b}} \left( \int_0^t \iint (w_0^\theta h)^q dw dy d\tau \right)^{\frac{1}{c}} \\ &\leq C v_0^{-\theta} \left( \int_0^t \iint \mathcal{G}^p (w_0^\theta h)^q dw dy d\tau \right)^{1/r} \left( \int_0^t \iint (w_0^\theta h)^q dw dy d\tau \right)^{1/c}. \end{aligned}$$

Hence, using the inequality (2.6) we have, for  $p < 7/5$

$$\begin{aligned}
 & \left[ \int_0^T \iint \left( v_0^\theta \int_0^t \iint \mathcal{G} h d w d y d \tau \right)^r d v d x d t \right]^{1/r} \\
 & \leq C \left[ \int_0^T \iint \left( \int_0^t \iint \mathcal{G}^p (w_0^\theta h)^q d w d y d \tau \right) \right. \\
 & \quad \left. \left( \int_0^t \iint (w_0^\theta h)^q d w d y d \tau \right)^{r/c} d v d x d t \right]^{1/r} \\
 & \leq C \left[ \left( \int_0^T \iint (w_0^\theta h)^q d w d y d \tau \right)^{r/c} \right. \\
 & \quad \left. \int_0^T \iint \left( \int_0^T \iint \mathcal{G}^p d v d x d t \right) (w_0^\theta h)^q d w d y d \tau \right]^{1/r} \\
 & \leq C \left[ \left( \int_0^T \iint (w_0^\theta h)^q d w d y d \tau \right)^{1+r/c} \right]^{1/r} \\
 & = C \left[ \int_0^T \iint (w_0^\theta h)^q d w d y d \tau \right]^{1/q}.
 \end{aligned} \tag{2.7}$$

Similarly, using the inequality (2.5) we find for  $p < 7/6$

$$\begin{aligned}
 & \left[ \int_0^T \iint \left( v_0^\theta \int_0^t \iint [|\nabla_w \mathcal{G}| + |\nabla_v \mathcal{G}|] h d w d y d \tau \right)^r d v d x d t \right]^{1/r} \\
 & \leq C \left( \int_0^T \iint (w_0^\theta h)^q d w d y d \tau \right)^{1/q}.
 \end{aligned} \tag{2.8}$$

Now we will show that  $f^n$  converges in  $L^\infty$ . We have

$$f^{n+1} - f^{\ell+1} = - \int_0^t \iint \mathcal{G} (K^n \cdot \nabla_v f^{n+1} - K^\ell \cdot \nabla_v f^{\ell+1})(\tau, y, w) d w d y d \tau. \tag{2.9}$$

Using (4.14) of [1] and taking  $\ell > n$

$$\begin{aligned}
 & |K^n \cdot \nabla_v f^{n+1} - K^\ell \cdot \nabla_v f^{\ell+1}| \\
 & \leq |K^n - K^\ell| |\nabla_v f^{n+1}| + |K^\ell| |\nabla_v (f^{n+1} - f^{\ell+1})| \\
 & \leq |(E^n, B^n) - (E^\ell, B^\ell)| w_0 |\nabla_v f^{n+1}| \\
 & \quad + C w_0^{1+\epsilon/2} 2^{-n\epsilon/2} |\nabla_v f^{n+1}| + C w_0 |\nabla_v (f^{n+1} - f^{\ell+1})| \\
 & \leq o^{n,\ell} w_0^{1+\epsilon/2} |\nabla_v f^{n+1}| + C w_0 |\nabla_v (f^{n+1} - f^{\ell+1})|.
 \end{aligned} \tag{2.10}$$

Next, we will apply the inequality (2.7) with  $p < 7/5$  and  $q = 2$ . Note that by taking  $p$  close to  $7/5$ , we may make  $r = \left(\frac{1}{p} + \frac{1}{q} - 1\right)^{-1}$  close to

$$\left[ \frac{5}{7} + \frac{1}{2} - 1 \right]^{-1} = 14/3.$$

Thus, applying the inequality (2.7) with  $p < 7/5$ ,  $q = 2$ ,  $\theta = \frac{\alpha}{2} - 1$ , and

$$h = w_0 |\nabla_v (f^{n+1} - f^{\ell+1})|$$

while using the inequality (2.2) yields

$$\begin{aligned} & \left[ \int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1} \int_0^t \iint \mathcal{G} w_0 |\nabla_v (f^{n+1} - f^{\ell+1})| dw dy d\tau \right)^r dv dx dt \right]^{1/r} \\ & \leq C \left[ \int_0^T \iint \left( w_0^{\frac{\alpha}{2}-1} w_0 |\nabla_v (f^{n+1} - f^{\ell+1})| \right)^2 dw dy d\tau \right]^{\frac{1}{2}} \leq o^{n,\ell} \end{aligned}$$

for  $r < 14/3$ . Applying the inequality (2.7) with  $p < 7/5$ ,  $q = 2$ ,  $\theta = \frac{\alpha}{2} - 1 - \epsilon/2$ , and

$$h = w_0^{1+\epsilon/2} |\nabla_v f^{n+1}|$$

and then using the inequality (2.3) yields

$$\begin{aligned} & \left[ \int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1-\epsilon/2} \int_0^t \iint \mathcal{G} w_0^{1+\epsilon/2} |\nabla_v f^{n+1}| dw dy d\tau \right)^r dv dx dt \right]^{1/r} \\ & \leq C \left[ \int_0^T \iint \left( w_0^{\frac{\alpha}{2}-1-\epsilon/2} w_0^{1+\epsilon/2} |\nabla_v f^{n+1}| \right)^2 dw dy d\tau \right]^{\frac{1}{2}} \leq C \end{aligned}$$

for  $r < 14/3$ . Since  $\frac{\alpha}{2} - 1 - \frac{\epsilon}{2} > \frac{\alpha}{2} - 1$ , Equation (2.9) and inequality (2.10) now yield for  $r < 14/3$

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1} |f^{n+1} - f^{\ell+1}| \right)^r dv dx dt \leq o^{n,\ell}. \tag{2.11}$$

Similarly, using the inequality (2.3) we may show that for  $r < 14/3$

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1} |f^{n+1}| \right)^r dv dx dt \leq C. \tag{2.12}$$

To use the inequality (2.11) we integrate by parts in Equation (2.9) to obtain

$$f^{n+1} - f^{\ell+1} = \int_0^t \iint \nabla_w \mathcal{G} \cdot (K^n f^{n+1} - K^\ell f^{\ell+1}) dw dy d\tau \tag{2.13}$$

and using (4.14) of [1]

$$\begin{aligned} & |K^n f^{n+1} - K^\ell f^{\ell+1}| \\ & \leq |(E^n, B^n) - (E^\ell, B^\ell)| w_0 |f^{n+1}| \\ & \quad + C w_0^{1+\epsilon/2} 2^{-n\epsilon/2} |f^{n+1}| + C w_0 |f^{n+1} - f^{\ell+1}| \\ & \leq o^{n,\ell} w_0^{1+\epsilon/2} |f^{n+1}| + C w_0 |f^{n+1} - f^{\ell+1}|. \end{aligned} \tag{2.14}$$

Next, we will apply the inequality (2.8), but now with  $p < 7/6$  and  $q < 14/3$ . Note that we may take  $p$  close to  $7/6$  and  $q$  close to  $14/3$  to make  $r$  close to

$$\left( \frac{6}{7} + \frac{3}{14} - 1 \right)^{-1} = 14.$$

Thus, applying inequality (2.8) with  $p < 7/6$ ,  $q < 14/3$ ,  $\theta = \frac{\alpha}{2} - 2$ , and

$$h = w_0 |f^{n+1} - f^{\ell+1}|$$

and then using the inequality (2.11) yields

$$\begin{aligned} & \left[ \int_0^T \iint \left( v_0^{\frac{\alpha}{2}-2} \int_0^t \iint |\nabla_w \mathcal{G}| w_0 |f^{n+1} - f^{\ell+1}| dw dy d\tau \right)^r dv dx dt \right]^{1/r} \\ & \leq C \left[ \int_0^T \iint \left( w_0^{\frac{\alpha}{2}-2} |f^{n+1} - f^{\ell+1}| \right)^q dw dy d\tau \right]^{1/q} \leq o^{n\ell} \end{aligned}$$

for  $r < 14$ . Applying inequality (2.8) again with  $p < 7/6$ ,  $q < 14/3$ ,  $\theta = \frac{\alpha}{2} - 2 - \frac{\epsilon}{2}$  and

$$h = w_0^{1+\epsilon/2} |f^{n+1}|$$

yields

$$\begin{aligned} & \left[ \int_0^T \iint \left( v_0^{\frac{\alpha}{2}-2-\epsilon/2} \int_0^t \iint |\nabla_w \mathcal{G}| w_0^{1+\epsilon/2} |f^{n+1}| dw dy d\tau \right)^r dv dx dt \right]^{1/r} \\ & \leq \left[ \int_0^T \iint \left( w_0^{\frac{\alpha}{2}-1} |f^{n+1}| \right)^q dw dy d\tau \right]^{1/q} \leq C \end{aligned}$$

for  $r < 14$  by using the inequality (2.12). With these two estimates, (2.13) and (2.14) yield

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-2} |f^{n+1} - f^{\ell+1}| \right)^r dv dx dt \leq o^{n,\ell}$$

for  $r < 14$ . Similarly,

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-2} |f^{n+1}| \right)^r dv dx dt \leq C.$$

Finally, we can apply estimate (2.8) with  $p = \frac{7}{6.4}$ ,  $q = \frac{14}{1.2}$ , and  $r = \infty$ . Proceeding as above we obtain

$$\left\| v_0^{\frac{\alpha}{2}-3} (f^{n+1} - f^{\ell+1}) \right\|_{L^\infty} \leq o^{n,\ell}.$$

Defining

$$f = \lim_{n \rightarrow \infty} f^n$$

it follows that

$$\left\| v_0^{\frac{\alpha}{2}-3} (f^n - f) \right\|_{L^\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Next, we bound  $\nabla_v f^n$  in  $L^\infty$ . We have

$$f^{n+1} = H - \int_0^t \iint \mathcal{G} K^n \cdot \nabla_v f^{n+1} dw dy d\tau$$

so

$$\begin{aligned}
 |\nabla_v(f^{n+1} - H)| &= \left| \int_0^t \iint \nabla_v \mathcal{G} K^n \cdot \nabla_v f^{n+1} dw dy d\tau \right| \\
 &\leq C \int_0^t \iint |\nabla_w \mathcal{G}| w_0 |\nabla_v f^{n+1}| dw dy d\tau.
 \end{aligned}
 \tag{2.15}$$

Applying estimate (2.8) with  $p < 7/6$ ,  $q = 2$ ,  $\theta = \frac{\alpha}{2} - 1$ , and  $h = w_0 |\nabla_v f^{n+1}|$ , and then using (2.3) yields

$$\begin{aligned}
 &\left[ \int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1} \int_0^t \iint |\nabla_w \mathcal{G}| w_0 |\nabla_v f^{n+1}| dw dy d\tau \right)^r dv dx dt \right]^{1/r} \\
 &\leq \left[ \int_0^T \iint \left( w_0^{\frac{\alpha}{2}-1} w_0 |\nabla_v f^{n+1}| \right)^2 dw dy d\tau \right]^{1/2} \leq C.
 \end{aligned}$$

Note that taking  $p$  close to  $7/6$  yields  $r$  close to

$$\left( \frac{6}{7} + \frac{1}{2} - 1 \right)^{-1} = 14/5.$$

Hence, using estimate (2.15) we find for  $r < 14/5$

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-1} |\nabla_v f^{n+1}| \right)^r dv dx dt \leq C.$$

We then apply estimate (2.8) three more times. In each application we take  $h = w_0 |\nabla_v f^{n+1}|$ . First, using  $p < \frac{7}{6}$ ,  $q < 14/5$ , and  $\theta = \frac{\alpha}{2} - 2$  yields

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-2} |\nabla_v f^{n+1}| \right)^r dv dx dt \leq C$$

for  $r < 14/3$ . Using  $p < 7/6$ ,  $q < 14/3$ , and  $\theta = \frac{\alpha}{2} - 3$  yields

$$\int_0^T \iint \left( v_0^{\frac{\alpha}{2}-3} |\nabla_v f^{n+1}| \right)^r dv dx dt \leq C$$

for  $r < 14$ . Using  $p = \frac{7}{6.4}$ ,  $q = \frac{14}{1.2}$ ,  $r = \infty$ , and  $\theta = \frac{\alpha}{2} - 4$  yields

$$\left\| v_0^{\frac{\alpha}{2}-4} \nabla_v f^{n+1} \right\|_{L^\infty} \leq C.
 \tag{2.16}$$

Recall that  $a > 8$  so  $\frac{\alpha}{2} - 4 = \frac{a+2+\epsilon}{2} - 4 > 1$ . Now  $\forall h \in \mathbb{R}^2$

$$|f^{n+1}(t, x, v+h) - f^{n+1}(t, x, v)| \leq C|h|,$$

and so

$$|f(t, x, v+h) - f(t, x, v)| \leq C|h|.$$

Finally, we show that  $f$  is Hölder continuous in  $x$ . Let  $h > 0$  and

$$e(t, x, v) = f^{n+1}(t, x+h, v) - f^{n+1}(t, x, v),$$



then

$$\partial_t e + v_1 \partial_x e + K^n \cdot \nabla_v e - \Delta_v e = -(K^n(t, x+h, v) - K^n(t, x, v)) \cdot \nabla_v f^{n+1}(t, x+h, v).$$

Note that

$$\begin{aligned} |K^n(t, x+h, v) - K^n(t, x, v)| &\leq v_0 (|E^n(t, x+h) - E^n(t, x)| + |B^n(t, x+h) - B^n(t, x)|) \\ &\leq v_0 h^{1/2} \left( \sqrt{\int (\partial_x E)^2 dx} + \sqrt{\int (\partial_x B)^2 dx} \right) \\ &\leq C v_0 h^{1/2}. \end{aligned}$$

Thus, by the bound (2.16) we find

$$|\partial_t e + v_1 \partial_x e + K^n \cdot \nabla_v e - \Delta_v e| \leq C v_0 h^{1/2} C v_0^{4-\alpha/2} \leq C h^{1/2}.$$

By Lemma 2.1 of [1]

$$|e| \leq C h^{1/2}.$$

It follows that  $f$  is Hölder continuous with exponent  $1/2$  in  $x$ .

Now, Theorem II.1 of [2] shows that  $f$  possesses the regularity stated in Theorem 1.1 of [1], and this completes the argument.

#### REFERENCES

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