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Crystal Precipitation with Discrete Initial Data

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We consider a model of crystal precipitation, described mathematically as a conservation law with nonlinear nonlocal flux function. The initial data consist of a finite number of Dirac measures. It is proved that, as $t \to \infty$, the radii of the crystals tend to a critical radius.

1. THE CRYSTAL PRECIPITATION PROBLEM

In this paper we consider the problem: Find a function $n(x, t)$ satisfying

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(\hat{G}n) = 0 \qquad \text{for} \quad x > 0, t > 0, \tag{1.1}$$

$$n(x, 0) = n_0(x) \qquad \text{for} \quad x > 0, \tag{1.2}$$

where

$$\hat{G}(x, t) = \begin{cases} k_\gamma(c(t) - c^* e^{\Gamma/x})^\gamma & \text{if} \quad x > x^*(t) \\ -k_\delta(c^* e^{\Gamma/x} - c(t))^\delta & \text{if} \quad x < x^*(t), \end{cases} \tag{1.3}$$

$$x^*(t) = \frac{\Gamma}{\log(c(t)/c^*)}, \tag{1.4}$$

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and

$$c(t) = c_0 + \beta \int_0^\infty x^3 n_0(x)\, dx - \beta \int_0^\infty x^3 n(x, t)\, dx. \tag{1.5}$$

Here γ, δ, k_γ, k_δ, Γ, c_0, c^*, β are positive constants,

$$c_0 > c^*, \qquad \gamma \geqslant 1, \qquad \delta \geqslant 1$$

and

$$n_0(x) = \sum_{m=1}^{N} \mu_m\, \delta(x - x_m), \tag{1.6}$$

where μ_m are positive constants and

$$0 < x_1 < x_2 < \cdots < x_N < \infty$$

are given; $\delta(x)$ is the Dirac measure with unit mass at $x = 0$.

This problem was recently studied by Friedman and Ou [2] in case $n_0(x)$ is a continuous nonnegative function with compact support. The system describes a model of crystal precipitation [5] (see also [1, 3, 4]). The initial values of the form (1.6) are of particular interest for the model.

In Section 2 we establish a solution for (1.1)–(1.5). In Section 3 we prove that, as $t \to \infty$, the radii of all the crystals that have not disappeared (in finite time) converge to a critical radius ξ, which is either one of the two zeros ξ_1, ξ_2 of the transcendental equation

$$\beta \mu_N \xi^3 + c^* e^{\Gamma/\xi} = c_0 + \beta \sum_{m=1}^{N} \mu_m x_m^3 \equiv c_1. \tag{1.7}$$

It follows that for a large class of initial data,

$$c^* < \lim_{t \to \infty} c(t) < c_1;$$

this is in sharp contrast to the asymptotic behavior in case $n_0(x)$ is a continuous function where (see [2])

$$\lim_{t \to \infty} c(t) \text{ is equal to either } c^* \text{ or } c_1.$$

In Section 4 we analyze for a given c_1 the three sets of initial data (all having the same c_1) for which all the crystals disappear in finite time, converge to ξ_1 as $t \to \infty$ or converge to ξ_2 as $t \to \infty$.

2. Solution of the Crystal Precipitation Problem

In order to arrive at a natural definition of the solution to the crystal precipitation problem we approximate $n_0(x)$ by smooth initial data $n_{0j}(x)$. Let $\rho_j(x)$ be smooth functions satisfying:

$$\rho_j \geqslant 0, \qquad \rho_j(x) = 0 \quad \text{if} \quad |x| > \frac{1}{j}, \qquad \int_{-\infty}^{\infty} \rho_j(x)\,dx = 1.$$

Then we take

$$n_{0j}(x) = \sum_{m=0}^{N} \mu_m \rho_j(x - x_m). \tag{2.1}$$

By [2] there exists a solution $(n(x,t), c(t))$ of the crystal growth problem; we denote it by $(n_j(x,t), c_j(x,t))$. Then

$$c^* < c_j(t) < c_{1,j}, \tag{2.2}$$

where

$$c_{1,j} = c_0 + \beta \sum_{m=1}^{N} \mu_m \int_0^\infty x^3 \rho_j(x - x_m)\,dx \to c_1 \qquad \text{as} \quad j \to \infty.$$

Recall the relations

$$\begin{aligned} c_j(t) &= c_{1,j} - \beta \int_0^\infty x^3 n_j(x,t)\,dx, \\ \frac{dc_j(t)}{dt} &= -3\beta \int_0^\infty x^2 n_j(x,t)\,\hat{G}_j(x,t)\,dx, \end{aligned} \tag{2.3}$$

where

$$\hat{G}_j(x,t) = \begin{cases} k_\gamma(c(t) - c^* e^{\Gamma/x})^\gamma & \text{if} \quad x > x_j^*(t) \\ -k_\delta(c^* e^{\Gamma/x} - c(t))^\delta & \text{if} \quad x < x_j^*(t) \end{cases} \tag{2.4}$$

and

$$x_j^*(t) = \frac{\Gamma}{\log(c_j(t)/c)}. \tag{2.5}$$

Denote by $x_j(t) \equiv x_j(t; x)$ the solution of

$$\begin{aligned} \frac{dx_j}{dt} &= \hat{G}_j(x,t), \\ x_j(0) &= x. \end{aligned} \tag{2.6}$$

As in [2],

$$\hat{G}_j(x, t) \leqslant C, \qquad \frac{dx_j}{dt} \leqslant C$$

and thus $x_j(t) \leqslant C(t+1)$ if $0 \leqslant t \leqslant T$. Using these estimates, we deduce from (2.3) that

$$\begin{aligned} \frac{dc_j}{dt} &\geqslant -3\beta \int_{x_j^*(t)}^{c(T+1)} x^2 n_j(x, t)\, \hat{G}_j(x, t)\, dx \\ &\geqslant -C \int_{x_j^*(t)}^{\infty} n_j(x, t)\, dx \geqslant -C, \end{aligned}$$

since $\int_0^\infty n_j(x, t)\, dx \leqslant \int_0^\infty n_{0j}(x)\, dx \leqslant C$ (see [2]).

It follows that the functions $c_j(t) + Ct$ are monotone increasing and uniformly bounded for $0 \leqslant t \leqslant T$. Hence, by Helly's theorem, we can extract a subsequence $c_j(t)$ which is pointwise convergent, say to a function $c(t)$. We conclude, by diagonalization, that for some subsequence,

$$c_j(t) \to c(t) \qquad \text{for all} \quad t > 0. \tag{2.7}$$

From (2.3)–(2.7) we thus easily deduce that, for any $\varepsilon > 0$,

$$x_j(t) \to x(t)$$

uniformly in bounded t-intervals as long as $x(t) \geqslant \varepsilon$, where

$$\begin{aligned} \frac{dx}{dt} &= \hat{G}(x, t), \\ x(0) &= x, \end{aligned} \tag{2.8}$$

and $G(x, t)$ is given by (1.3).

Suppose $x(t; x_0) \geqslant \varepsilon$ for $0 \leqslant t \leqslant T$. Then $x(t; x) \geqslant \varepsilon/2$ if $0 \leqslant t \leqslant T$ and $|x - x_0| \leqslant \varepsilon'$ for ε' small enough, and $x_j(t; x) \to x(t; x)$ uniformly in $t \in [0, T]$, $x \in [x_0 - \varepsilon', x_0 + \varepsilon']$. By [2, Lemma 5.1] we get

$$\int_{x_j(t;\, x_0 + \varepsilon')}^{x_j(t;\, x_0 + \varepsilon')} n_j(x, t)\, dx = \int_{x_0 - \varepsilon'}^{x_0 + \varepsilon'} n_j(x)\, dx = \mu_0.$$

The same considerations apply to each of the points x_m; further

$$\int_{x(t;\, \bar{x})}^{x(t;\, \bar{\bar{x}})} n_j(x, t)\, dx = 0$$

if the interval $\{\bar{x} \leqslant x \leqslant \bar{\bar{x}}\}$ does contain any points x_m and j is large enough. It follows that, for any $0 < a < b < \infty$,

$$\int_a^b n_j(x, t)\,dx \to \int_a^b \sum_{m=1}^{N} \mu_m\,\delta(x(t; x_m) - x)\,dx$$

as $j \to \infty$. Thus, in the sense of weak convergence of measures, for any $t > 0$

$$n_j(x, t) \to n(x, t),$$

where, upon setting $x_m(t) = x(t; x_m)$,

$$n(x, t) = \sum_{m=1}^{N} \mu_m\,\delta(x_m(t) - x). \tag{2.9}$$

We also have

$$c(t) = c_1 - \beta \sum_{m=1}^{N} \mu_m x_m^3(t) \tag{2.10}$$

and, if $x^*(t)$ is defined as in (1.4),

$$\begin{aligned} \frac{dx_m}{dt} &= G(x_m, c(t)), \\ x_m(0) &= x_m, \end{aligned} \tag{2.11}$$

where

$$G(x_m, c(t)) = \begin{cases} k_\gamma(c_1 + \sum_{j=1}^{N} \beta\mu_j x_j^3(t) - c^* e^{\Gamma/x_m})^\gamma & \text{if } x_m > x^*(t), \\ -k_\gamma(c^* e^{\Gamma/x_m} - c_1 - \sum_{j=1}^{N} \beta\mu_j x_j^3(t))^\delta & \text{if } x_m < x^*(t). \end{cases} \tag{2.12}$$

THEOREM 2.1. *There exists a unique solution of the differential system* (2.11) *with G defined by* (2.12).

Indeed, existence was already proved. To prove uniqueness we suppose that there is another solution with $\tilde{x}_m(t)$, $\tilde{c}(t)$. From (2.12) we then deduce that, for $0 < t < T$,

$$\begin{aligned} &|G(x_m(t), c(t)) - G(\tilde{x}_m(t), \tilde{c}(t))| \\ &\quad \leqslant C(\|x - \tilde{x}\|_T + \|c - \tilde{c}\|_T), \end{aligned}$$

where $x=(x_1, ..., x_N)$, $\tilde{x}=(\tilde{x}_1, ..., \tilde{x}_N)$, $\|u\|_T=\sup_{0\leqslant t\leqslant T}|u(t)|$, and from (2.10) we get $\|c-\tilde{c}\|_T \leqslant C\|x-\tilde{x}\|_T$. Thus

$$|G(x_m(t), c(t)) - G(\tilde{x}_m(t), \tilde{c}(t))| \leqslant C\|x-\tilde{x}\|_T,$$

and using (2.11) uiqueness easily follows.

From the uniqueness part of Theorem 2.1 it follows that the limit function $n(x, t)$ is independent of the choice of the approximating sequence ρ_j, and that the full sequence n_j is convergent to n. This motivates:

DEFINITION 2.1. The function $n(x, t)$ defined (uniquely) by (2.9) (with (2.10)–(2.12)) is called the solution of the *crystal growth problem.*

3. ASYMPTOTIC BEHAVIOR

Set

$$c_1 = c_0 + \beta \sum_{m=1}^{N} \mu_m x_m^3. \tag{3.1}$$

Since the function

$$\mu_m \beta \xi^3 + c^* e^{\Gamma/\xi}$$

is convex and tends to $+\infty$ as $\xi \to 0$ and as $\xi \to \infty$, there exist at most two positive zeros ξ_1, ξ_2 of the equation

$$\mu_m \beta \xi^3 + c^* e^{\Gamma/\xi} = c_1. \tag{3.2}$$

We choose them so that $\xi_1 \leqslant \xi_2$. We shall prove:

THEOREM 3.1. (a) *For any* $m<N$, $x_m(t)\to 0$ *in finite time*; (b) *either* (i) $x_N(t)\to 0$ *in finite time or* (ii) $x_N(t)\to \xi_i$ (*for* $i=1$ *or* $i=2$) *as* $t\to\infty$.

We note that case (b)(ii) cannot occur if (3.2) has no positive solutions.

Clearly $x_j(t) < x_{j+1}(t)$ for each j as long as $x_j(t)$ remains positive. If all the $x_m(t)$ converge to zero in finite time, then the assertions (a), (b)(i) follow. Thus it remains to consider the case where, for some $k\leqslant N$, $x_k(t)>0$ for all $t>0$ whereas $x_{k-1}(t)$ converges to zero in finite time $t=t_0$. For simplicity we take $t_0=0$.

We first consider the case $k<N$.

LEMMA 3.2. *For any* $m \geqslant k+1$,

$$x_m(t) - x_{m-1}(t) \text{ is strictly increasing in } t \tag{3.3}$$

and

$$x_N(t) \leqslant \left(\frac{c_1}{\beta\mu_N}\right)^{1/3}. \tag{3.4}$$

Proof. Since $G(x, c(t))$ is strictly monotone increasing in x,

$$\frac{dx_m}{dt} = G(x_m(t), c(t)) > G(x_{m-1}(t), c(t)) = \frac{dx_{m-1}}{dt}$$

and (3.3) follows. Next, since $c(t) > 0$,

$$\beta\mu_N x_N^3(t) \leqslant \beta \sum_{m=k}^{N} \mu_m x_m^3(t) \leqslant c_1$$

and (3.4) follows.

LEMMA 3.3. *There exists t_0 such that either*

$$x_N(t) > x^*(t) \qquad \textit{for all} \quad t > t_0 \tag{3.5}$$

or

$$x_N(t) < x^*(t) \qquad \textit{for all} \quad t > t_0. \tag{3.6}$$

Proof. It suffices to show that whenever $x = x_N(t)$ and $x = x^*(t)$ intersect at some time $t = t_1$, then $dx_N(t_1)/dt = 0$ and $dx^*(t_1)/dt < 0$. The equality is obvious. To prove the inequality we use

$$\begin{aligned}\frac{dc(t)}{dt} &= -3\beta \sum_{m=k}^{N} \mu_m x_m^2(t) \frac{dx_m(t)}{dt} \\ &= -3\beta \sum_{m=k}^{N} \mu_m x_m^2(t)\, G(x_m(t), c(t)) > 0 \qquad \text{at} \quad t = t_1\end{aligned}$$

since $G(x_N(t_1), c(t_1)) = 0$ whereas $G(x_m(t_1), c)t_1)) < 0$ if $k \leqslant m < N$. Hence, from (1.4), $dx^*(t_1)/dt < 0$.

Consider first the case (3.6). Then

$$\frac{dx_N}{dt} = G(x_N(t), c(t)) < 0 \tag{3.7}$$

and similarly $dx_m/dt < 0$ if $k \leqslant m < N$. From (2.10) we then deduce that $dc/dt > 0$. Hence $c(\infty) \equiv \lim_{t\to\infty} c(t)$ exists. From (3.7) we also deduce that $x_N(\infty) \equiv \lim_{t\to\infty} x_N(t)$ exists and

$$G(x_N(\infty), c(\infty)) = 0. \tag{3.8}$$

From Lemma 3.2 we see that, for some $\eta_0 > 0$,

$$x_m(t) \leqslant x_N(\infty) - \eta_0 \qquad (k \leqslant m < N) \tag{3.9}$$

for all t large enough and therefore, by (3.8),

$$\frac{dx_m}{dt} = G(x_m(t), c(t)) < -\eta < 0 \tag{3.10}$$

for all such t's. It follows that $x_m(t)$ must converge to zero in finite time, which is a contradiction.

It remains to consider the case (3.5). In this case $dx_N/dt > 0$ for all $t > 0$, and $x_N(\infty) = \lim_{t \to \infty} x_N(t)$ exists and is finite (by (3.6)). Proceeding as in [2, Sect. 5] we can prove that $x^*(\infty) = \lim_{t \to \infty} x^*(t)$ exists and $x^*(\infty) = x_N(\infty)$. But then $c(\infty) \equiv \lim_{t \to \infty} c(t)$ exists and therefore (3.8) must hold. Proceeding to argue as in (3.9), (3.10) we again derive a contradiction.

We have thus proved that the case $k < N$ cannot occur. Thus all the $x_m(t)$ with $m < N$ must tend to zero in finite time. If $x = x_N(t)$ intersects $x = x^*(t)$ for some $t = t_0$ then $x_N(t) \equiv x_N(t_0)$ is a solution of the differential equation (2.11) with $c(t) \equiv c(t_0)$ for $t > t_0$. It follows that $\xi = \lim_{t \to \infty} x_N(t)$ satisfies (3.8) with $x^*(\infty) = \xi$; consequently ξ is a solution of (3.2), and (b)(ii) follows.

It remains to consider the case where $x = x_N(t)$ does not intersect $x = x^*(t)$. In this case either (3.5) or (3.6) hold. In both cases we can argue as before and deduce that $x_N(\infty)$ and $x^*(\infty)$ exist and are equal. In view of (3.8), their common value must satisfy (3.2), and this completes the proof of Theorem 3.1.

In Section 4 we shall show that the case (b)(ii) occurs for a large class of initial data $x_1, ..., x_N$.

COROLLARY 3.4. *In case* (b)(ii) *occurs,*

$$c^* < c(\infty) < c_1. \tag{3.11}$$

This is in sharp contrast to the situation when $n_0(x)$ is a continuous function; in fact, in that case, as shown in [2], $c(\infty)$ must be equal either to c_1 or to c^*.

4. Asymptotic Behavior (Continued)

We define

$$c_1' = \min_{x>0} \{c^* c^{\Gamma/x} + \beta \mu_N x^3\}, \tag{4.1}$$

$$c_1'' = \min_{x>0} \left\{c^* e^{\Gamma/x} + \beta \left(\sum_{m=1}^{N} \mu_m \right) x^3 \right\} \tag{4.2}$$

and notice that $c_1'' > c_1'$ if $N = 1$.

If $c_1 > c_1'$ then the equation

$$c^* e^{\Gamma/x} - (c_1 - \beta \mu_N x^3) = 0 \tag{4.3}$$

has two roots which we have labeled ξ_1 and ξ_2. For $c_1 < c_1'$ this equation has no roots. If $c_1 > c_1''$ then the equation

$$c^* e^{\Gamma/x} - \left(c_1 - \beta \left(\sum_{m=1}^{N} \mu_m \right) x^3 \right) = 0 \tag{4.4}$$

has two roots which we shall denote by η_1 and η_2, $\eta_1 < \eta_2$. For $c_1 < c_1''$ this equation has no roots.

As easily seen from (4.3),

$$\frac{d\xi_1}{d\mu_N} > 0, \qquad \frac{d\xi_2}{d\mu_N} < 0$$

and therefore

$$\xi_1 \leqslant \eta_1 < \eta_2 \leqslant \xi_2 \qquad \text{for} \quad c_1 > c'', \tag{4.5}$$

with strict inequalities if $N > 1$.

We define

$$G(x, c) = \begin{cases} k_\gamma (c - c^* e^{\Gamma/x})^\gamma & \text{if} \quad x > x^*(t) \\ -k_\delta (c^* e^{\Gamma/x} - c)^\delta & \text{if} \quad x < x^*(t), \end{cases} \tag{4.6}$$

where $x = x^*(t)$ is defined by $c = c^* e^{\Gamma/x}$. We would like to analyze the asymptotic behavior of the solution $(x_1(t), ..., x_N(t))$ given initial data $(x_1, ..., x_N)$ in the set

$$S = \left\{ (x_1, ..., x_N);\, \beta \sum_{m=1}^{N} \mu_m x_m^3 = c_1 - c_0 \right\};$$

each such data yields the same concentration c_1, and thus, the roots ξ_1, ξ_2, η_1, η_2 are independent of the particular point in S.

Introduce

$$S_0 = \{(x_1, ..., x_N) \in S;\, x_N(t) \to 0 \text{ in finite time}\},$$
$$S_j = \{(x_1, ..., x_N) \in S;\, x_N(t) \to \xi_j \text{ if } t \to \infty\}$$

for $j = 1, 2$. Then, by Theorem 3.1, $S = S_0 \cup S_1 \cup S_2$.

Consider first the case $N = 1$ and assume that $c_1 > c_1'$. Then

$$G(x_N, c) < 0 \qquad \text{if} \quad x_N < \xi_1 \text{ or } x_N > \xi_2$$

and

$$G(x_N, c) > 0 \qquad \text{if} \quad \xi_1 < x_N < \xi_2.$$

Hence $x_N(t) \to 0$ in finite time if $x_N(0) < \xi_1$, $x_N(t) \to \xi_2$ if $x_N(0) > \xi_1$, and $x_N(t) \equiv \xi_1$ if $x_N(0) = \xi_1$.

For $N > 1$ the situation is much more difficult to analyze. We shall give some partial results.

If $c_1 < c_1'$ then Theorem 3.1 implies that $S = S_0$. We also have, for fixed $c_0 > 0$,

$$S = S_0 \qquad \text{if} \quad c_1 - c_0 \qquad \text{is sufficiently small.} \tag{4.7}$$

Indeed, $x_N(0)$ is then sufficiently small and, as easily seen, $x_N(t) \downarrow 0$ in finite time.

THEOREM 4.1. *If $c_1 > c_1''$ and*

$$c_1 - c_0 < \beta \left(\sum_{m=1}^{N} \mu_m \right) \eta_1^3 \tag{4.8}$$

then $S = S_2$.

Proof. Set

$$\bar{c} = c_1 - \beta \left(\sum_{m=1}^{N} \mu_m \right) \eta_1^3, \qquad \bar{c} < c_0. \tag{4.9}$$

Since $c_1 - c_0 = \beta \sum \mu_m x_m^3(0)$, it follows from (4.8) that $x_N(0) > \eta_1$. We claim that

$$x_N(t) > \eta_1 \qquad \text{for all} \quad t. \tag{4.10}$$

Indeed otherwise consider the smallest t such that $x_N(t) = \eta_1$; then $x_N'(t) \leqslant 0$. Also

$$c(t) = c_1 - \beta \sum_{m=1}^{N} \mu_m x_m^3(t) > \bar{c} \qquad \text{by (4.9)}$$

(some of the $x_m(t)$ may vanish). Since $G(\eta_1, \bar{c}) = 0$ (by the definitions of η_1 and $\bar{c}$) and G is strictly monotone increasing in c,

$$0 < G(\eta_1, c(t)) = G(x_N(t), c(t))$$

and thus $dx_N(t)/dt > 0$, a contradiction.

The above proof of (4.10) also shows that $x_N(t) \geqslant \eta_1 + \varepsilon$ for some sufficiently small $\varepsilon > 0$. It follows that $\lim_{t \to \infty} x_N(t) > \eta_1$ and, in view of (4.5) and Theorem 3.1, $\lim x_N(t) = \xi_2$, i.e., $(x_1, ..., x_N) \in S_2$ for any $(x_1, ..., x_N)$ in S.

THEOREM 4.2. *The sets S_0 and S_2 are open subsets of S.*

It follows that S_1 is closed in S; further if S_0 and S_2 are both nonempty then any continuous curve in S which connects a point in S_0 to a point in S_2 must intersect S_1. Thus S_1 "separates" S_0 from S_2.

Proof. To prove that S_0 is open, let $(x_1^0, ..., x_N^0) \in S_0$ and let $(\tilde{x}_1, ..., \tilde{x}_N) \in S$ with $\sum |x_i^0 - \tilde{x}_i| < \delta$, δ small. Denote the corresponding solutions by $(x_1^0(t), ..., x_N^0(t))$ and $(\tilde{x}_i(t), ..., \tilde{x}_N(t))$. Then, for some $t_0 > 0$, $x_N^0(t) > 0$ for $0 \leqslant t < t_0$ and $x_N(t_0) = 0$. Further, if $\tilde{x}_N(t) > 0$ for $t \leqslant t_0$, then

$$\tilde{x}_N(t_0) < \sigma(\delta),$$

where $\sigma(\delta) \to 0$ if $\delta \to 0$ (by continuous dependence of solutions on the initial data). If $\sigma(\delta)$ is small enough then

$$d\tilde{x}_N(t)/dt < 0 \qquad \text{for} \quad t_0 < t < t_1,$$

for some t_1, and $\tilde{x}_N(t_1) = 0$. Thus $(\tilde{x}_1, ..., \tilde{x}_N) \in S_0$ and, consequently, S_0 is open.

We next prove that S_2 is open. Let $(x_1^0, ..., x_N^0) \in S_2$ and $(\tilde{x}_1, ..., \tilde{x}_N) \in S$ with $\sum |x_i^0 - \tilde{x}_i| < \delta$, δ small. Choose T large enough such that

$$x_{n-1}^0(t) = 0, \qquad \tilde{x}_{n-1}(t) = 0 \qquad \text{if} \quad t \geqslant T.$$

Denote the concentrations $c(t)$ corresponding the solutions $(x_1^0(t), ..., x_N^0(t))$ and $(\tilde{x}_1(t), ..., \tilde{x}_N(t))$, respectively, by $c_0(t)$ and $\tilde{c}(t)$. Consider the functions

$$\begin{aligned} I^0(t) &= c^0(t) - c^* e^{\Gamma/x_N^0(t)} \\ &= \left(c_0 + \sum \beta\mu_m (x_m^0)^3\right) - \mu\beta(x_N^0(t))^3 - c^* e^{\Gamma/x_N^0(t)} \\ \tilde{I}(t) &= \tilde{c}(t) - c^* e^{\Gamma/\tilde{x}_N(t)} \\ &= \left(c_0 + \sum \beta\mu_m \tilde{x}_m^3\right) - \mu\beta(\tilde{x}_N(t))^3 - c^* e^{\Gamma/\tilde{x}_N(t)}. \end{aligned}$$

Since $\lim x_N(t) = \xi_2$, we may assume that

$$\frac{\xi_1 + \xi_2}{2} < x_N(t) \qquad \forall t > T;$$

hence $I^0(t) > 0$ and $dx_N^0(t)/dt > 0$ for $t > T$ (as in case $N = 1$) as long as $x_N^0(t) < \xi_2$. Similarly, as long as

$$\frac{\xi_1 + \xi_2}{2} < \tilde{x}_N(t) < \xi_2$$

we have $\tilde{I}(t) > 0$ and $d\tilde{x}_N(t)/dt > 0$. We can choose δ small enough so that $\tilde{x}_N(T) > (\xi_1 + \xi_2)/2$ (by continuous dependence of solutions on the initial data) and, then, by the previous observation, $\tilde{x}_N(t)$ cannot decrease to $(\xi_1 + \xi_2)/2$ for any $t > T$. It follows, by Theorem 3.1, that $\lim \tilde{x}_N(t) = \xi_2$ and thus $(\tilde{x}_1, ..., \tilde{x}_N) \in S_2$.

THEOREM 4.3. *If* $S_1 \neq \varnothing$ *then* $S_1 \cap \bar{S}_0 \neq \varnothing$ *and* $S_1 \cap \bar{S}_2 \neq 0$; *consequently* $\operatorname{int} S_1 = \varnothing$.

Proof. Let $(x_1^0, ..., x_N^0) \in S_1$ and set

$$B_\delta = \left\{ (\tilde{x}_1, ..., \tilde{x}_N) \in S; \sum |\tilde{x}_i - x_i^0|^2 < \delta^2 \right\}, \qquad \delta > 0.$$

Choose T such that $x_{N-1}^0(T) = 0$. For any $\varepsilon > 0$ there is $\delta > 0$ such that for each solution with initial data in B_δ

$$\tilde{x}_{N-1}(t) = 0 \qquad \text{for all} \quad t \geqslant T + \varepsilon.$$

We have, by assumption,

$$x_N^0(t) \to \xi_1 \qquad \text{as} \quad t \to \infty. \tag{4.11}$$

We claim that actually

$$x_N^0(t) = \xi_1 \qquad \text{for all} \quad t \geqslant T. \tag{4.12}$$

Indeed, if $x_N^0(T_0) > \xi_1$ for some $T_0 > T$ then, as in the proof of Theorem 4.2,

$$I^0(t) > 0 \qquad \text{and} \qquad \frac{dx_N^0(t)}{dt} > 0 \qquad \text{for} \quad t > T_0$$

and thus (4.11) cannot occur. Similarly $x_N^0(T_0) < \xi_1$ implies

$$I^0(t) < 0 \qquad \text{and} \qquad \frac{dx_N^0(t)}{dt} < 0 \qquad \text{if} \quad t \geqslant T_0$$

and (4.11) cannot occur.

From the diffeomorphism

$$(\tilde{x}_1, ..., \tilde{x}_N) \to (\tilde{x}_1(t), ..., \tilde{x}_N(t))$$

(for t fixed) it follows that for any $0 < \delta' < \delta$ there exists a point $(\tilde{x}_1, ..., \tilde{x}_N)$ in B_δ, with

$$\xi_1 < \tilde{x}_N(T+\varepsilon) < \xi_2.$$

Arguing as above we find that

$$\tilde{x}_N(t) \geqslant \tilde{x}_N(T+\varepsilon) > \xi_1 \qquad \text{for all} \quad t \in T+\varepsilon;$$

consequently $(\tilde{x}_1, ..., \tilde{x}_N) \in S_2$. Thus $\bar{S}_2 \cap S_1 \neq \varnothing$. Similarly one can prove that $\bar{S}_1 \cap S_1 \neq \varnothing$.

From the above proof we get:

COROLLARY 4.4. *If* $(x_1, ..., x_N) \in S_1$ *then* $x_N(t) \to \xi_1$ *is finite time.*

Set

$$c(x) = c_1 - \beta \sum_{m=1}^{N} \mu_m x_m^3,$$

$$\tilde{S} = \{(x_1, ..., x_N); x_j > 0 \; \forall j, c_* < c(x) \leqslant c_1\}$$

and consider the dynamical system

$$\frac{dx_j}{dt} = G(x_j, c(x)) \qquad (x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N) \tag{4.13}$$

with initial values $(x_1(0), ..., x_N(0))$ in $\tilde{S}$. Then the flow remains in $\tilde{S}$ (since $G(x_j, c^*) > 0$). The roots ξ_i, η_i can be defined as before (they depend only on c_1), and we also define:

$$\tilde{S}_0 = \{\text{all } (x_1, ..., x_N) \in \tilde{S} \text{such that } x_N(t) \to 0 \text{ in finite time}\},$$

$$\tilde{S}_j = \{\text{all } (x_1, ..., x_N) \in \tilde{S} \text{ such that } x_N(t) \to \xi_i \text{ as } t \to \infty\}.$$

For each $c_0 \in (0, c_1 - c^*)$ the sets

$$S^{c_0} = \tilde{S} \cap \{c(x) = c_1 - c_0\}, \qquad S_j^{c_0} = \tilde{S}_j \cap \{c(x) = c_1 - c_0\}$$

coincide with the sets S, S_j defined above, for the same c_0. The previous analysis can easily be extended to the present sets; in particular,

$$\tilde{S}_0 \text{ and } \tilde{S}_2 \text{ are open subsets of } \tilde{S}, \tag{4.14}$$

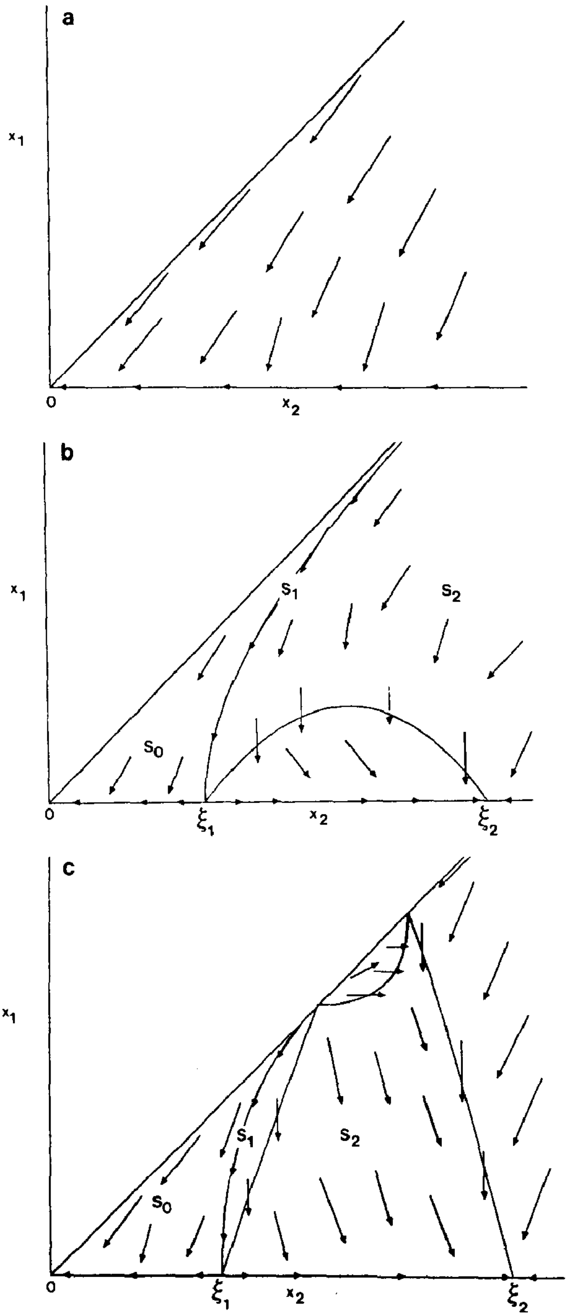

FIG. 1. Phase diagrams for the dynamical system for $N = 2$. (a) $c_1 < c_1'$, (b) $c_1' < c_1 < c_1''$, and (c) $c_1'' < c_1$.

$$\operatorname{int} \tilde{S}_1 \neq \varnothing \qquad \text{if} \quad \tilde{S}_1 \neq \varnothing, \tag{4.15}$$

and

$$\tilde{S}_0 = [0, -\xi_1), \tilde{S}_1 = \{\xi_1\}, \quad S_2 = \left(\xi_1, \left(\frac{c_1 - c^*}{\beta\mu_N}\right)^{1/3}\right] \quad \text{if} \quad N = 1. \tag{4.16}$$

The system (4.13) is autonomous and this suggests that some dynamical system approach may be useful for analyzing the set $\tilde{S}_1$. It can be shown that $G(c(x), x_j) < 0$ on the set $\tilde{S}_1$, for all j.

Phase diagrams of the dynamical system are in Fig. 1 for $N = 2$.

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