In Problems 1.10.2–1.10.4 you will be asked to verify Theorems 1.8.3–1.8.5 numerically. To facilitate this we will choose the constants g and d in (1.11) to be one ($g = d = 1$), thus yielding the simpler equation

$$\text{(E)} \qquad \frac{dx}{dt} = k(c_1 - \mu x^3 - c^* e^{\Gamma/x}),$$

where we have taken $k_g = k_d = k$. To make the numbers somewhat more manageable we convert to new units: We use microns (μm) for length and picograms (pg) for mass:

$$1 \text{ micron} = 10^{-6} \text{ meters},$$

$$1 \text{ picogram} = 10^{-12} \text{ grams} = 10^{-15} kg,$$

$$1 \frac{\text{gram}}{cm^3} = 1 \frac{pg}{\text{microns}^3}.$$

We keep the time units as seconds.

In each of the following problems you will first have to find the two roots ξ_1 and ξ_2 of the equation

$$(1.19) \qquad \mu x^3 + c^* e^{\Gamma/x} = c_1 \qquad (= c_0 + \mu(x^*)^3)$$

using Newton's method. Since ξ_1 and ξ_2 may be very close together, care will have to be taken to obtain *both* roots. You should choose your initial guesses in Newton's method with care. Try a rough graph of the left-hand side of (1.19) if necessary.

PROBLEM 1.10.2. Solve (E) up to $t = 0.5$. Take

$$\begin{aligned}
\Gamma &= 4 \times 10^{-3} \mu m, \\
\mu &= 10^{-3}, \\
c^* &= 7.52 \times 10^{-7} pg/(\mu m)^3, \\
c_0 &= 1.05 c^*, \\
x^* &= x(0) = .05 \mu m, \\
k &= 5 \times 10^7, \\
\text{so that} \quad k\mu &= 5 \times 10^4 \quad \text{and} \quad kc^* = 37.6.
\end{aligned}$$

In this case kc_1 should turn out to be 45.73, and $\xi_1 = 2.16736 \times 10^{-2}$, $\xi_2 = 4.53479 \times 10^{-2}$. (Check this out.) Thus $x^* > \xi_2 > \xi_1$.

PROBLEM 1.10.3. Solve (E) up to $t = 0.5$. Take $x^* = .0975$ and all the other constants as in Problem 1.10.2.

In this case kc_1 should turn out to be 85.8229 while the roots ξ_1, ξ_2 should be

$$\xi_1 = 4.84721 \times 10^{-3},$$

$$\xi_2 = 9.771616 \times 10^{-2}.$$

Thus $\xi_1 < x^* < \xi_2$ here. Check this out.

PROBLEM 1.10.4. Solve (E) until $t = 0.16$. Take Γ, k, c^* as before but $x^* = .08$ and $k\mu = 5 \times 10^2$. In this case kc_1 should turn out to be 39.736, while $\xi_1 = 8.3510 \times 10^{-2}$ and $\xi_2 = 1.197 \times 10^{-1}$. Note that $x^* < \xi_1 < \xi_2$.

The graphs for the solutions of Problems 1.10.2–1.10.4 are given in Fig. 1.4.

1.11. The Runge–Kutta method (BASIC)

While the Euler method could easily be motivated, it very often does not give the accuracy needed for scientific investigation. It is for that reason that many other methods are introduced—to give greater accuracy. A whole series of related methods going by the name of "Runge–Kutta" (dating from about the turn of the century) have the advantage that they are quite accurate and, at the same time, not too difficult to carry out. We are going to describe the most popular of these, leaving the motivation for later.

Again we look at the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

We will prescribe the approximate values of $y(t)$ by giving them only at the points t_k.

Suppose we know $y(t_k)$ and we want to determine an approximation y_{k+1} to $y(t_k + h)$. The idea of this method is to compute the value of $f(x, y)$ at several clearly chosen points near the solution curve in the interval $(t_k, t_k + h)$ and to combine these values so as to get a good value for $y_{k+1} - y_k$. We set

$$y_{k+1} = y_k + \tfrac{1}{6}\,(K_1 + 2K_2 + 2K_3 + K_4),$$

where

$$\begin{aligned} K_1 &= hf(t_k, y_k), \\ K_2 &= hf(t_k + \tfrac{1}{2}\,h,\ y_k + \tfrac{1}{2}K_1)\ , \\ K_3 &= hf(t_k + \tfrac{1}{2}\,h,\ y_k + \tfrac{1}{2}K_2)\ , \\ K_4 &= hf(t_k + h,\ y_k + K_3). \end{aligned}$$

The simplest choice for t_k is given by $t_k = t_0 + kh$, while t_0 is the point at which the initial condition is given: $y(t_0) = y_0$.