Differential Game Theory & Pedestrian Traffic

Andrew Glaws

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Outline

Control Theory

Standard problem in control theory:

- state variable, $\boldsymbol{x} \in \mathbb{R}^m$
- initial set, $\boldsymbol{X}_0$, and target set, $\boldsymbol{X}_1$
- control variable, $\boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^n$
  - $\mathcal{U} =$ set of all admissible controls
- system dynamics

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \boldsymbol{X}_0 \end{cases}$$

- cost functional

$$J(\boldsymbol{u}) = \psi(\boldsymbol{x}(t_1)) + \int_{t_0}^{t_1} L(s, \boldsymbol{x}, \boldsymbol{u})\, ds$$

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Goal: Find the optimal control $\boldsymbol{u}^* \in \mathcal{U}$ which minimizes the cost functional

$$J(\boldsymbol{u}^*) \leq J(\boldsymbol{u}) \qquad \text{for all } \boldsymbol{u} \in \mathcal{U}$$

Differential Game Theory

Definition (Differential Game)

A control theory problem with multiple controls, each operated by a different player who is attempting to minimize their own cost functional, is called a differential game.

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Two-Player Game

System dynamics:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2)$$

Cost functionals:

$$J_1(\boldsymbol{u}_1) = \psi_1(\boldsymbol{x}(t_1)) + \int_{t_0}^{t_1} L_1(s, \boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2)\, ds$$

$$J_2(\boldsymbol{u}_2) = \psi_2(\boldsymbol{x}(t_1)) + \int_{t_0}^{t_1} L_2(s, \boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2)\, ds$$

Differential Game Theory

Differential games appear in a variety of applications:

- Flight and rocket/missile control
- Political science
- Economics and business management
- Military strategy
- Traffic

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and take a variety of forms:

- Cooperative vs. non-cooperative games
- Open-loop vs. closed-loop games
- Sequential-move vs. simultaneous-move games
- Zero-sum vs. nonzero-sum games
- Pursuit-evasion games

The Hamiltonian

Definition (Hamiltonian)

Define $\boldsymbol{\lambda} \in \mathbb{R}^m$ where $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$ to be the adjoint system. Then, the Hamiltonian is given by

$$H\left(t, \boldsymbol{x}(t), \boldsymbol{\lambda}(t), \boldsymbol{u}(t)\right) = L(t, \boldsymbol{x}, \boldsymbol{u}) + \left[\boldsymbol{\lambda}(t)\right]^T \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}).$$

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The state and adjoint dynamics are given by

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} \qquad \text{and} \qquad \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}}$$

with $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$ and $\boldsymbol{x}(t_1) \in \boldsymbol{X}_1$.

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Definition (Transversality Condition)

Let $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ be the tangent spaces to $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$, respectively. Then,

$$\boldsymbol{\lambda}^*(t_0) \perp \boldsymbol{T}_0 \qquad \text{and} \qquad \boldsymbol{\lambda}^*(t_1) \perp \boldsymbol{T}_1.$$

Pontryagin Maximum Principle

Theorem (Pontryagin Maximum Principle)

Let $\boldsymbol{u}^* \in \mathcal{U}$ *be the optimal control with corresponding optimal state* $\boldsymbol{x}^*$ *and optimal adjoint* $\boldsymbol{\lambda}^*$. *Then,* $\boldsymbol{u}^*$ *minimizes the Hamiltonian.*

$$H\left(t, \boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{u}^*\right) \leq H\left(t, \boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{u}\right) \qquad \text{for any } \boldsymbol{u} \in \mathcal{U}$$

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Thus,

$$\left.\frac{\partial H}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}=\boldsymbol{u}^*} = \boldsymbol{0}$$

1D Example

Consider the one-dimensional domain $\Omega = (0,1)$ with a single pedestrian.

$$\boldsymbol{x}(t) = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix},$$

$$\boldsymbol{X}_0 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \qquad \boldsymbol{X}_1 = \left\{ \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\}, \text{ where } \alpha \in \mathbb{R},$$

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Notice that t_1 is not specified.

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The Hamiltonian is given by

$$H\left(t, \boldsymbol{x}(t), \boldsymbol{\lambda}(t), u(t)\right) = \frac{1}{2}\left([u(t)]^2 + 1\right) + \lambda_1(t)v(t) + \lambda_2(t)u(t)$$

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System of ODEs:

$$\begin{cases} \dot{r}(t) = v(t) & r(0) = 0 \\ \dot{v}(t) = u(t) & v(0) = 0 \\ \dot{\lambda}_1(t) = 0 & r(t_1) = 1 \\ \dot{\lambda}_2(t) = -\lambda_1(t) & \lambda_2(t_1) = 0 \end{cases}$$

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Maximum Principle:

$$\frac{\partial H}{\partial u}\left(t, \boldsymbol{x}^*(t), \boldsymbol{\lambda}^*(t), u^*(t)\right) = u^*(t) + \lambda_2^*(t) = 0$$

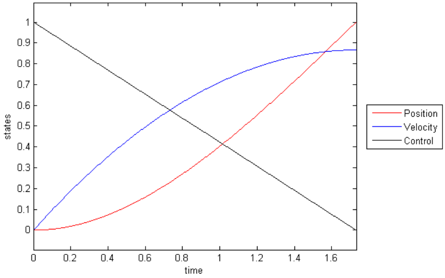
$$\Rightarrow u^*(t) = -\lambda_2^*(t)$$

1D Example

Solution:

$$\begin{cases} r^*(t) = \frac{1}{6\sqrt{3}}\left(\sqrt{3}-t\right)^3 + \frac{\sqrt{3}}{2}t - \frac{1}{2} \\ v^*(t) = -\frac{1}{2\sqrt{3}}\left(\sqrt{3}-t\right)^2 + \frac{\sqrt{3}}{2} \\ \lambda_1^*(t) = -\frac{1}{\sqrt{3}} \\ \lambda_2^*(t) = -\frac{1}{\sqrt{3}}\left(\sqrt{3}-t\right) \end{cases}$$

$$\Rightarrow u^*(t) = -\lambda_2^*(t) = \frac{1}{\sqrt{3}}\left(\sqrt{3}-t\right)$$



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Assumptions:

1. Pedestrians continuously observe the current state and update their plan

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Assumptions:

1. Pedestrians continuously observe the current state and update their plan
2. Pedestrians have perfect knowledge of the current state
3. Pedestrians have limited prediction capabilities
4. Pedestrians work harder to avoid groups than individuals
5. Pedestrians' cost functional is made up of 3 components:
    - straying from the optimal velocity
    - discomfort from walking too near to other pedestrians
    - accelerating/decelerating/turning

Pedestrian Traffic Model

System state:

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{r}_1(t) \\ \vdots \\ \boldsymbol{r}_N(t) \\ \boldsymbol{v}_1(t) \\ \vdots \\ \boldsymbol{v}_N(t) \end{bmatrix}$$

System dynamics:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_N \\ \boldsymbol{u}_1 + \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{u}_N + \boldsymbol{w}_N \end{bmatrix}$$

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$\Rightarrow \boldsymbol{u}_p =$ controllable portion of pedestrian p's acceleration

$\Rightarrow \boldsymbol{w}_p =$ uncontrollable portion of pedestrian p's acceleration

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$$\boldsymbol{w}_p = \boldsymbol{w}_{\text{bound}} + \sum_{q \neq p} \Big[k_{p,1} \left(\left(R_p^* + R_q^* \right) - R_{p,q} \right)^+ \Big) \boldsymbol{n}_{pq} + k_{p,2} \left(\left(\boldsymbol{v}_q - \boldsymbol{v}_p \right)^T \boldsymbol{n}_{pq}^{\perp} \right) \left(\left(R_p^* + R_q^* \right) - R_{p,q} \right)^+ \boldsymbol{n}_{pq}^{\perp} \Big]$$

Pedestrian Traffic Model

Cost functional for pedestrian p:

$$J_p(\boldsymbol{u}_p) = \int_{t_0}^{\infty} e^{-\eta_p s} \left[c_{p,1} L_{p,1} + c_{p,2} L_{p,2} + c_{p,3} L_{p,3} \right] ds$$

where

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where

- $L_{p,1} = \frac{1}{2}\left(\boldsymbol{v}_p^* - \boldsymbol{v}_p\right)^T\left(\boldsymbol{v}_p^* - \boldsymbol{v}_p\right)$
  $\Rightarrow$ cost of drifting from optimal velocity

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- $L_{p,2} = \sum_{q \neq p} e^{-D_{pq}/R_p^0}$
  $\Rightarrow$ cost of discomfort from walking too near to other pedestrians

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- $L_{p,2} = \sum_{q \neq p} e^{-D_{pq}/R_p^0}$
  $\Rightarrow$ cost of discomfort from walking too near to other pedestrians
- $L_{p,3} = \theta_p\left(\boldsymbol{u}_p^T\boldsymbol{e}_p\right)^2 + (1-\theta_p)\left(\boldsymbol{u}_p^T\boldsymbol{e}_p^{\perp}\right)^2$
  $\Rightarrow$ cost of accelerating/decelerating/turning

Pedestrian Traffic Model

Hamiltonian:

$$H\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) = e^{-\eta_p t}\left[c_{p,1} L_{p,1} + c_{p,2} L_{p,2} + c_{p,3} L_{p,3}\right] + \boldsymbol{\lambda}^T \boldsymbol{f}$$

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Applying the maximum principle yields:

$$\boldsymbol{u}_p^* = -\frac{1}{c_{p,3}} e^{-\eta_p t} M_p \boldsymbol{\lambda}_{\boldsymbol{v}_p}$$

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Applying the maximum principle yields:

$$\boldsymbol{u}_p^* = -\frac{1}{c_{p,3}} e^{-\eta_p t} M_p \boldsymbol{\lambda}_{\boldsymbol{v}_p}$$

and solving the adjoint dynamics for $\boldsymbol{v}_p$ results in

$$\boldsymbol{u}_p^* = M_p\left[I - \frac{1}{\eta_p}\left(\frac{\partial \boldsymbol{v}_p^*}{\partial \boldsymbol{r}_p}\right)^T\right]\left(\frac{\boldsymbol{v}_p^* - \boldsymbol{v}_p}{\tau_p}\right) - A_p^0 M_p\left[\frac{\partial L_{p,2}}{\partial \boldsymbol{r}_p} + \eta_p \frac{\partial L_{p,2}}{\partial \boldsymbol{v}_p}\right]$$

where

$$M_p = \frac{1}{2}\left[\theta_p \boldsymbol{e}_p \boldsymbol{e}_p^T + (1-\theta_p)\,\boldsymbol{e}_p^{\perp}\left(\boldsymbol{e}_p^{\perp}\right)^T\right]^{-1},\ \tau_p = \frac{\eta_p c_{p,3}}{c_{p,1}},\ A_p^0 = \frac{c_{p,2}}{\eta_p^2 c_{p,3}}$$

2D Examples

Let's go to the tape....

Model Parameters

The model contains several (seemingly arbitrary) parameters:

- R_p^* - physical radius of pedestrians
- R_p^0 - spatial discount factor
- η_p - temporal discount factor
- $\left.\begin{array}{l} k_{p,1} \\ k_{p,2} \end{array}\right\}$ weighting factors for inter-pedestrian interactions
- $\left.\begin{array}{l} c_{p,1} \\ c_{p,2} \\ c_{p,3} \end{array}\right\}$ weighting factors for costs functions
- $\left.\begin{array}{l} c_p^+ \\ c_p^- \end{array}\right\}$ weighting factors for longitudinal/latitudinal discomfort
- θ_p - relative weight of longitudinal vs. latitudinal acceleration

Future Work

- Eliminate simplifying assumptions in model
  - Introduce cooperation between pedestrians
  - Expand anisotropy property of pedestrians
  - Limit pedestrians' knowledge of the current state
- Study of parameter values
  - Macroscopic data vs. microscopic data
  - Relationship between parameters and outputs within model
- Examine how variations in environment influence behavior

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