# An Undergraduate's Guide to the Hartman-Grobman and Poincaré-Bendixon Theorems

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MATH181HM: Dynamical Systems
Spring 2008

## 1 Introduction

The Hartman-Grobman and Poincaré-Bendixon Theorems are two of the most powerful tools used in dynamical systems. The Hartman-Grobman theorem allows us to represent the local phase portrait about certain types of equilibria in a nonlinear system by a similar, simpler linear system that we can find by computing the system's Jacobian matrix at the equilibrium point. The Poincaré-Bendixon theorem gives us a way to find periodic solutions on 2D surfaces. One way in which we can use this theorem is by finding an annulus-shaped region (2D donut shape) such that the vectors on both edges point into the region.

This document is a guide to the proofs of these two powerful theorems. These proofs are not generally covered in dynamical systems courses at the undergraduate level. Many such courses do not require previous knowledge of topics such as mathematical analysis and topology. This guide is intended to be a self-contained explanation of the proofs of these theorems in the sense that it should be comprehensible to those who have a basic understanding of set theory, calculus, linear algebra and differential equations and who are currently studying dynamical systems.

# 2 The Hartman-Grobman Theorem

## 2.1 Why does linearization at fixed points tell us about behavior around the fixed point?

If we have a n-dimensional linear system of differential equations ($\dot{\vec{x}} = Ax$) with a single fixed point at the origin we can observe several types of behaviors, such as saddle points, spirals, cycles, stars and nodes, which are well-understood. We classify these cases based on the eigenvalues of the matrix A used to classify the system. With a nonlinear system the behavior of the system is more difficult to analyze. Fortunately, we are not left completely in the dark. We can find the Jacobian matrix, or "total derivative", J, corresponding to the system and evaluate it at a fixed point to obtain a linear system with a characteristic coefficient matrix. The Hartman-Grobman theorem tells us that, at least in a neighborhood of the fixed point, if J's eigenvalues all have nonzero real part then we can get a qualitative idea of the behavior of solutions in the nonlinear system. Such qualitative characteristics we can glean include whether solution trajectories approach or move away from the equilibrium point over time, and whether the solutions spiral or if the equilibrium point acts as a node.

## 2.2 Definitions

**Definition 2.1** *Homeomorphism*

*A function $h : X \to Y$ is a homeomorphism between X and Y if it is a continuous bijection (1-1 and onto function) with a continuous inverse (denoted $h^{-1}$). The existence of homeomorphisms tell us that X and Y have analogous structures. This is because $h$ and $h^{-1}$, when applied to the entire space (X or Y, respectively), may be thought of as continuously pushing the points around such that each point retains all of its original neighbors. Topologists sometimes explain this concept as stretching and bending without tearing.*

**Definition 2.2** *Topological Conjugacy*

*Given two maps, $f : X \to X$ and $g : Y \to Y$, the map $h : X \to Y$ is a topological semi-conjugacy if it is continuous, onto and $h \circ f = g \circ h$, where $\circ$ denotes function composition (sometimes written $h(f(\vec{x})) = g(h(\vec{x}))$ where $\vec{x}$ is a point in X. Furthermore, h is a topological conjugacy if it is a* *homeomorphism between X and Y (i.e. h is also 1-1 and has a continuous inverse). We then say that X and Y are homeomorphic.*

**Definition 2.3** *Hyperbolic Fixed Point*

*A hyperbolic fixed point for a system of differential equations a point at which the eigenvalues of the Jacobian for the system evaluated at that point all have nonzero real part.*

**Definition 2.4** *Cauchy Sequence*

*For the purposes of this document I will provide a non-technical definition. A Cauchy sequence of functions is a series of functions* $x_k = x_1, x_2 \ldots$ *such that the functions become more and more similar as* $k \to \infty$.

**Definition 2.5** *flow*

*Let* $\dot{\vec{x}} = F(\vec{x})$ *be a system of differential equations and* $\vec{x}_0$ *be an initial condition for* $F(\vec{x})$. *Provided that the solutions to the differential equation exist and are unique (the conditions of which are given in the existence and uniqueness theorem. See, for example, Strogatz (1995), pg. 149), then* $\phi(t; \vec{x}_0)$, *the flow of* $F(\vec{x})$, *gives the spatial solution of* $F(\vec{x})$ *given the initial condition over time. An important result of flows is that changing initial conditions in phase space will change flows in a continuous fashion because we have a continuous vector field in* $\mathbb{R}^n$.

**Definition 2.6** *orbit/trajectory*

*The set of all points in a flow* $\phi(t; \vec{x}_0)$ *for the set of differential equations* $\dot{\vec{x}} = F(\vec{x})$ *is called the "orbit" or "trajectory" of* $F(\vec{x})$ *with initial condition* $\vec{x}_0$. *We write the orbit as* $\phi(\vec{x}_0)$. *When we consider only* $t \geq 0$, *we say we consider the "forward orbit" or "forward trajectory."*

## 2.3 Theorem and Proof

**Theorem 2.7** *The Hartman-Grobman Theorem*

*Let* $\vec{x} \in \mathbb{R}^n$. *Consider the nonlinear system* $\dot{\vec{x}} = f(\vec{x})$ *with the flow* $\phi_t$ *and the linear system* $\dot{\vec{x}} = A\vec{x}$, *where A is the Jacobian* $Df(\vec{x}^*)$ *of f and* $\vec{x}^*$ *is a hyperbolic fixed point. Assume that we have appropriately translated* $\vec{x}^*$ *to origin, i.e.* $\vec{x}^* = \vec{0}$.

*Let f be* $C^1$ *on some* $E \subset \mathbb{R}^n$ *with* $\vec{0} \in E$. *Let* $I_0 \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^n$ *such that U, V and* $I_0$ *each contain the origin. Then* $\exists$ *a homeomorphism*

*$H : U \to V$ such that, $\forall$ initial points $\vec{x}_0 \in U$ and all $t \in I_0$,*

$$H \circ \phi_t(\vec{x}_0) = e^{At} H(\vec{x}_0)$$

*Thus the flow of the nonlinear system is homeomorphic to the flow, $e^{At}$, of the linear system given by the fundamental theorem for linear systems.*

**Proof**

This theorem essentially states that the nonlinear system $\dot{\vec{x}} = f(\vec{x})$ is locally homeomorphic the linear system $\dot{\vec{x}} = A\vec{x}$. For the proof, we begin by writing A as the matrix

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where P and Q are partitions or "sub-matrices" of A such that the real part of the eigenvalues of P are negative and the real part of the eigenvalues of Q have positive real part. Finding such a matrix A may require finding a new basis for our linear system using techniques of linear algebra. For more information see section 1.8 on Jordan forms of matrices in Perko (1991).

Consider the solution $\vec{x}(t, \vec{x}_0) \in \mathbb{R}^n$ given by

$$\vec{x}(t, \vec{x}_0) = \phi_t(\vec{x}) = \begin{pmatrix} \vec{y}(t, \vec{y}_0, \vec{z}_0) \\ \vec{z}(t, \vec{y}_0, \vec{z}_0) \end{pmatrix}$$

with $\vec{x}_0 \in \mathbb{R}^n$ given by

$$\vec{x}_0 = \begin{pmatrix} \vec{y}_0 \\ \vec{z}_0 \end{pmatrix}$$

and $\vec{y}_0 \in E^S$ (the stable subspace of A), $\vec{z}_0 \in E^U$ (the unstable subspace of A). The stable and unstable subspaces of A are given by the spans of the negative and positive eigenvectors of A, respectively. Let

$$\begin{aligned} \tilde{Y}(\vec{y}_0, \vec{z}_0) =& \vec{y}(1, \vec{y}_0, \vec{z}_0) - e^P \vec{y}_0, \\ \tilde{Z}(\vec{y}_0, \vec{z}_0) =& \vec{z}(1, \vec{y}_0, \vec{z}_0) - e^Q \vec{z}_0. \end{aligned}$$

$\tilde{\mathrm{Y}}$ and $\tilde{\mathrm{Z}}$ are functions of the trajectory with initial condition $\vec{x}_0$ evaluated at $t = 1$. Then if $\vec{x}_0 = \vec{0}$, it follows that $\vec{y}_0 = \vec{z}_0 = \vec{0}$ so we have $\tilde{\mathrm{Y}}(\vec{0}) =$

$\tilde{Z}(\vec{0}) = 0$ and thus $D\tilde{Y}(\vec{0}) = D\tilde{Z}(\vec{0}) = \vec{0}$ since $\vec{x}_0$ is located at the fixed point $\vec{0}$. Since f is $C^1$ on $E$, $\tilde{Y}$ and $\tilde{Z}$ are also $C^1$ on $E$. Since we know that the $D\tilde{Y}$ and $D\tilde{Z}$ are zero at the origin and $\tilde{Y}$ and $\tilde{Z}$ are continuously differentiable, we can define a region about the origin such that $|\vec{y}_0|^2 + |\vec{z}_0|^2 \leq s_0^2$ for some sufficiently small $s_0 \epsilon \mathbb{R}$, where the norms of $D\tilde{Y}$ and $D\tilde{Z}$ are each smaller than some real number $a$:

$$||D\tilde{Y}(\vec{y}_0, \vec{z}_0)|| \leq a$$
$$||D\tilde{Z}(\vec{y}_0, \vec{z}_0)|| \leq a.$$

We now use the mean value theorem: Let Y and Z be smooth functions such that if $|\vec{y}_0|^2 + |\vec{y}_0|^2 \geq s_0^2$, then $Y = Z = 0$, whereas if $|\vec{y}_0|^2 + |\vec{y}_0|^2 \leq (\frac{s_0}{2})^2$, $Y = \tilde{Y}$ and $Z = \tilde{Z}$. Then the mean value theorem yields

$$|Y| \leq a\sqrt{|\vec{y}_0|^2 + |\vec{z}_0|^2} \leq a(|\vec{y}_0| + |\vec{z}_0|),$$
$$|Z| \leq a\sqrt{|\vec{y}_0|^2 + |\vec{z}_0|^2} \leq a(|\vec{y}_0| + |\vec{z}_0|).$$

Let $B = e^P$ and $C = e^Q$. Given proper normalization (see Hartman (1964)) we have $b = ||B|| < 1$ and $c = ||C^{-1}|| < 1$. We now prove that there is a homeomorphism H from U to V such that $H \circ T = L \circ H$ by the method of successive approximations. Define the transformations L, T and H as follows:

$$L(\vec{y}, \vec{z}) = \begin{pmatrix} B\vec{y} \\ C\vec{z} \end{pmatrix} = e^A\vec{x}, \tag{2.1}$$

$$T(\vec{y}, \vec{z}) = \begin{pmatrix} B\vec{y} + Y(\vec{y}.\vec{z}) \\ C\vec{z} + Z(\vec{y}, \vec{z}) \end{pmatrix}, \tag{2.2}$$

$$H(\vec{x}) = \begin{pmatrix} \Phi(\vec{y}, \vec{z}) \\ \Psi(\vec{y}, \vec{z}) \end{pmatrix}. \tag{2.3}$$

From (2.1)-(2.3) and our desired relation $H \circ T = L \circ H$, we have that

$$B\Phi = \Phi(B\vec{y} + Y(\vec{y}, \vec{z}), C\vec{z} + Z(\vec{y}, \vec{z}))$$
$$C\Psi = \Psi(B\vec{y} + Y(\vec{y}, \vec{z}), C\vec{z} + Z(\vec{y}, \vec{z}))$$

Successive approximations for (2.3) are given recursively by

$$\Psi_0 = \vec{z}, \tag{2.4}$$
$$\Psi_{k+1} = C^{-1}\Psi_k(B\vec{y} + Y(\vec{y}, \vec{z}), C\vec{z} + Z(\vec{y}, \vec{z})), k \epsilon \mathbb{N}_0. \tag{2.5}$$

This means that we can get closer and closer to the function $\Phi$ by following the recursion relation defined by (2.4)-(2.5). By induction it follows that all of the $\Psi_k$ are continuous because the flow $\phi_t$ is continuous and therefore it follows that $\Psi_0$ is continuous. $C^{-1}$ is continuous so $\Psi_1$ is continuous, and by induction $\Psi_k$ is continuous $\forall\ k \epsilon \mathbb{N}_0$. It also follows that $\Psi_k(\vec{y}, \vec{z}) = \vec{z}$ for $|\vec{y}| + |\vec{z}| \geq 2s_0$, [Perko,1991].

It can be shown by induction [Perko,1991] that

$$|\Psi_j(\vec{y}, \vec{z}) - \Psi_{j-1}(\vec{y}, \vec{z})| \leq Mr^j(|\vec{y}| + |\vec{z}|)^\sigma$$

Where $j = 1, 2, ...$ and $r = c[2\max(a, b, c)]^\sigma$, $c < 1$, and $\sigma\ \epsilon\ (0, 1)$ such that $r < 1$. This yields the result that $\Psi_k(\vec{y}, \vec{z})$ is a Cauchy sequence of continuous functions. These functions converge uniformally as $k \rightarrow \infty$, and we can call the limiting function $\Psi(\vec{y}, \vec{z})$. It As for the $\Psi_k$, it is true that $\Psi(\vec{y}, \vec{z}) = \vec{z}$ for $|\vec{y}| + |\vec{z}| \geq 2s_0$.

The case is similar for $B\Phi = \Phi(B\vec{y} + Y(\vec{y}, \vec{z}), C\vec{z} + Z(\vec{y}, \vec{z}))$, which can be written as $B^{-1}\Phi(\vec{y}, \vec{z}) = \Phi(B^{-1}\vec{y} + Y_1(\vec{y}, \vec{z}), C^{-1}\vec{z} + Z_1(\vec{y}, \vec{z}))$, where $T^{-1}$ defines $Y_1$ and $Z_1$ as follows

$$T^{-1}(\vec{y}, \vec{z}) = \begin{pmatrix} B^{-1}\vec{y} + Y_1(\vec{y}, \vec{z}) \\ C^{-1}\vec{z} + Z_1(\vec{y}, \vec{z}) \end{pmatrix}.$$

Then we can solve for $\Phi$ in a manner excatly as we solved for $\Psi$ above using $\Phi_0 = \vec{y}$. Once we have carried out the calculations to find $\Psi$ and $\Phi$ we obtain the homeomorphism $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$H = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \tag{2.6}$$

∎

# 3 The Poincaré-Bendixon Theorem

## 3.1 How do we know if we have a periodic orbit?

Often when analyzing a two-dimensional dynamical system we can classify the behavior at all of the equilibrium points, but it is still unclear what happens in between them. Numerically solving a system and plotting solutions in the phase plane may make us suspect the existence of closed orbits in a particular region. The Poincaré-Bendixon Theorem tells us that if we can show that an orbit with an initial condition in a region is contained in that region for all future time then there must be a closed orbit or a fixed point in the region. Since fixed points are relatively easy to find by simultaneously solving the differential equations that make up the system, we should know whether a fixed point is in the region, and thus whether a closed orbit is in the region. Strogatz shows a useful technique in which one can construct a "trapping region" for trajectories and then use the Poincaré-Bendixon Theorem to show the existence of closed orbits (see Figure 1below).

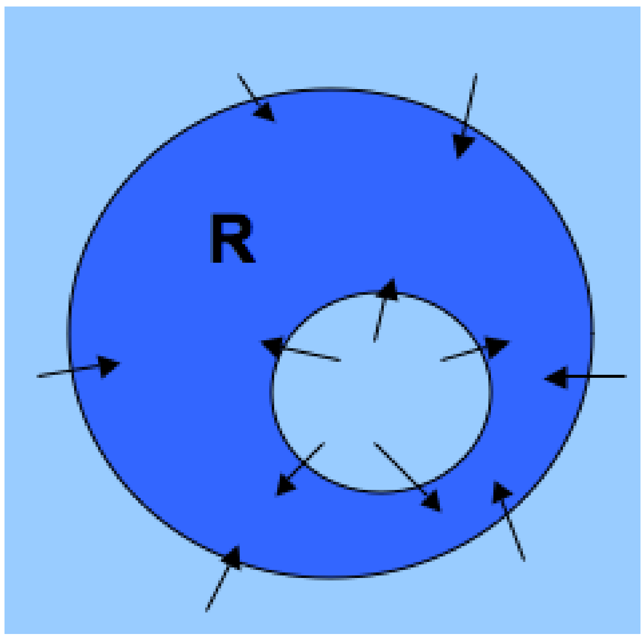


Figure 1: Trapping region

## 3.2 Definitions

**Definition 3.1** *metric, metric space*

*Given a set M and a function d, the ordered pair (M,d) is a metric space and d is a metric provided that* $\forall$ *x,y,z* $\epsilon$ *M the following are true:*

*a)* $d(x, y) = d(y, x)$,

*b)* $0 \leq d(x, y) < \infty$,

*c)* $d(x, y) = 0$ *if and only if* $x = y$, *and*

*d)* $d(x,z) \leq d(x,y) + d(y,z)$.
*The metric d provides us with a concept of distance between any two points in the set M. For this theorem we work in the set* $\mathbb{R}^2$, *in which the most common metric to use for two points* $(x_1,y_1)$ *and* $(x_2,y_2)$ *is the euclidian distance* $d = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$.

**Definition 3.2** *bounded set*
*Let (M,d) be a metric space. Let* $B(\alpha, C) = \{x \,\epsilon\, \mathbb{R}^n \mid \|x - \alpha\| \leq C\}$ *(i.e. B is a ball of radius C centered at* $\alpha$*). M is bounded if and only if* $\exists$ *a real-valued constant C such that* $M \subset B(0, C)$.

**Definition 3.3** *positively invariant set*
*Let* $\phi(t; \vec{x}_0)$ *be the flow for the set of differential equations* $\dot{\vec{x}} = F(\vec{x})$ *defined on* $\mathbb{R}^n$*. If, for* $S \subset \mathbb{R}^n$ *and* $\phi(t; \vec{x}_0) \,\epsilon\, S$ *for any point* $\vec{x}_0 \,\epsilon\, S$, $t \geq 0$*, then S is positively invariant. In other words, if the forward orbits of all initial conditions in S are subsets of S, then S is positively invariant.*

**Definition 3.4** $\omega$*-limit point,* $\omega$*-limit set*
*Let* $\phi(t; \vec{x}_0)$ *be the flow for the set of differential equations* $\dot{\vec{x}} = F(\vec{x})$ *defined on* $\mathbb{R}^n$ *with initial condition* $\vec{x}_0$. $\vec{z}$ *is called an* $\omega$*-limit point of* $\vec{x}_0$ *if* $\exists$ *an infinite sequence of times* $t_0, t_1, ..., t_n, t_{n+1}, ...$ *such that* $\phi(t_n; \vec{x}_0)$ *converges to* $\vec{z}$*. The* $\omega$*-limit set of* $\vec{x}_0$*, denoted* $\omega(\vec{x}_0)$*, is the set of all* $\omega$*-limit points of* $\vec{x}_0$.

## 3.3 Theorem and Proof

**Theorem 3.5** *The Poincaré-Bendixon Theorem*
*Let* $\dot{\vec{x}} = F(\vec{x})$ *be a system of differential equations defined on* $\mathbb{R}^2$.

*We assume:*
*i)* $F(\vec{x})$ *is defined* $\forall\ \vec{x} \,\epsilon\, \mathbb{R}^2$*, and*
*ii) A forward orbit* $\phi(\vec{q}) = \{\phi(t; \vec{x}_0) \mid t \geq 0\}$*, with initial condition* $\vec{x}(t_0) = \vec{x}_0$ *at* $t = t_0$*, is bounded.*

*Then either:*
*a)* $\omega(\vec{x}_0)$ *contains a fixed point, or,*
*b)* $\omega(\vec{x}_0)$ *is a periodic orbit.*

**Proof**

First we define some points that we use in the proof and examine their important properties. Let $\vec{x}_0$ be an initial value of the flow $\phi(\vec{x}_0)$ in a closed, bounded, and positively invariant subset of $\mathbb{R}^2$. We know that $\phi(\vec{x}_0)$ is bounded and the forward orbit is defined for the infinite set of times $t \geq 0$, so the orbit must pass increasingly close to at least one point infinitely many times and thus $\omega(\vec{x}_0)$ is nonempty.

If we let $\vec{q}$ be a point in $\omega(\vec{x}_0)$, then $\phi(\vec{q})$ is a subset of $\omega(\vec{x}_0)$ due to the continuity of flows. Then $\omega(q)$ is bounded since $\phi(\vec{q})$ is bounded. Let $\vec{z}$ be a point in $\omega(\vec{q})$. We know that $\vec{z}$ is nonempty since $\phi(\vec{q})$ is bounded and thus $\omega(\vec{q})$ is nonempty.

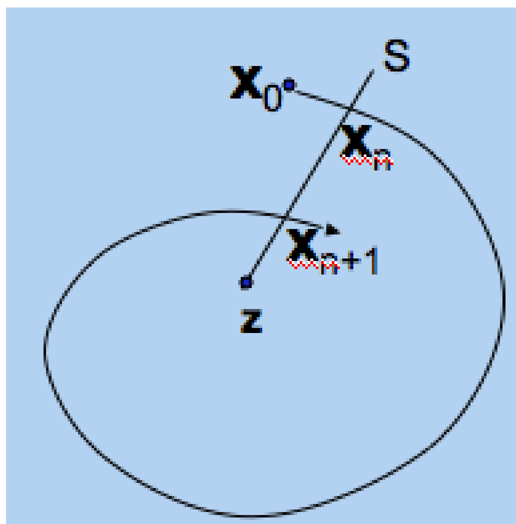


Figure 2: The orbit begining at $x_0$ may cross S as shown. Notice that the intersections between S and the orbit occur closer to $\vec{z}$ as time passes. Here $\vec{z}$ is on the interior of $\Gamma$. The next intersection would occur between $\vec{x_{n+1}}$ and $\vec{z}$.

We can construct a line segment S through $\vec{x}$ such that all of the orbits that intersect S pass through S (are one one side of S immediately before being in S and are on the other side of S immediately afterwards). This condition implies that no trajectory that intersects S is tangent to S, and hence, since our vector field's flow is continuous, all of the orbits crossing S must do so in the same direction. This can be done because we can make S sufficiently small such that the continuity of the vector field ensures that all

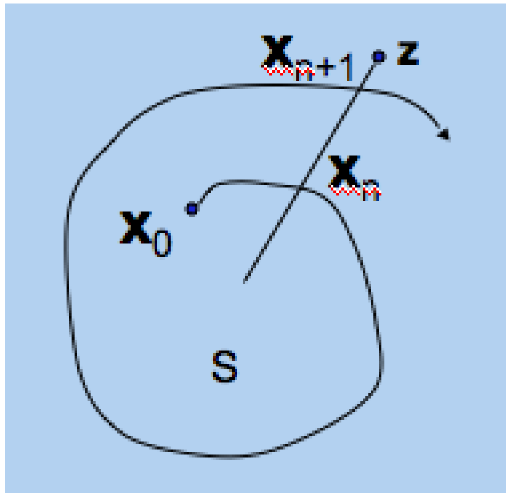


Figure 3: Another example of an orbit begining at $x_0$ crossing S over time. The intersections between S and the orbit still occur closer to $\vec{z}$ as time passes, but $\vec{z}$ is outside the region enclosed by $\Gamma$

trajectories crossing S do so in the same direction.

Since $\phi(\vec{x}_0)$ and $\phi(\vec{q})$ both come near $\vec{z}$ infinitely many times, they must repeatedly intersect S. Thus there is a sequence of times $t_i = t_1, t_2, ..., t_n, ...$ such that $\vec{x}_n = \phi(t_n, \vec{x}_0)$ is a point at which $\phi(\vec{x}_0)$ intersects S.

We can define the section of S between $\vec{x}_n$ and $\vec{x}_{n+1}$ as $S'_n$. We can also define $\{\phi(t; \vec{q}) \mid t_n \leq t \leq t_{n+1}\}$ to be the piece of $\phi(\vec{x}_0)$ between the same points $\vec{x}_n$ and $\vec{x}_{n+1}$. It is then possible to construct a closed curve, $\Gamma$, by taking the union $S'_n$ and $\{\phi(t; \vec{q}) \mid t_n \leq t \leq t_{n+1}\}$.

The trajectory at $\phi(t_{n+1}, \vec{x}_0)$ must either enter the interior or the exterior of of $\Gamma$. We then know that all of the trajectories along $S'_n$ also enter the interior of $\Gamma$. Likewise if the trajectory at $\phi(t_{n+1}, \vec{x}_0)$ enters the exterior of $\Gamma$, then all other trajectories crossing with initial points on $S'_n$ also enter the exterior of $\Gamma$. Thus, because of flow continuity, if $\phi(t_{n+1}, \vec{x}_0)$ enters the interior of $\Gamma$, then $\phi(\vec{x}_0)$ is in the interior of $\Gamma \; \forall \, t > t_n$. Hence the intersections of $\phi(\vec{x}_0)$ with $S'_n$ occur monotonically along $S'_n$, occurring closer to $\vec{z}$ along $S'_n$ as t increases. Thus the intersections converge to the single point $\vec{z}$.

Similarly for $\vec{q}$, there is a sequence of time $s_i = s_0, s_1, ..., s_n, ...$ with $s_k \leq s_{k+1} \forall \, k = 1, 2, 3...$ such that $\phi(s_n, \vec{q})$ intersects $S'_n$ and accumulates on $\vec{z}$. The points $\phi(s_n, \vec{q})$ are in the intersection of $\omega(\vec{x}_0)$ and S since $\phi(\vec{q})$ is a subset of

$\omega(\vec{x}_0)$. This intersection is the single point $\vec{z}$, thus the points $\phi(s_n, \vec{q})$ are all the same.

Thus we have a series of times at which $\phi(\vec{q})$ intersects S. This may mean that $\phi(\vec{q})$ always intersects S and thus $\vec{z}$ is a fixed point, or $\phi(\vec{q})$ intersects S at an infinite number of discrete times and thus $\phi(\vec{q})$ is a periodic orbit. In the later case $\omega\vec{x}_0$ contains a periodic orbit. In the former case, $\vec{z} = \omega(\vec{x}_0)$ by flow continuity. ∎

# 4 Conclusions and Future Work

The Hartman-Grobman and Poincaré-Bendixon theorems provide us with powerful methods by which we can better understand nonlinear dynamical systems. Despite the theorems' intuitive appeal, the proofs of these theorems can be subtle. Personally I had much more success with the Poincaré-Bendixon theorem's proof because my learning style is very visual. However, I struggled with the Hartman-Grobman theorem, and feel as though I only made minor progress in making it more understandable than Perko's representation, which was the primary presentation of the proof upon which I was attempting to improve. I feel as if I made some progress in understanding the foundational concepts involved in the proof, but much more could be done given more time. I understand the intent of the proof and what it attempts to show, but I have come across many problems in understanding the analysis. It is, however, useful to point out the problems so that they may be fixed. For example, there are obvious missing steps implementation of the mean value theorem.

Both of these proofs currently rely heavily on abstract thinking. Since the Poincaré-Bendixon proof is set in 2D space and involves concepts that can be visualized, I think that the proof would benefit from a more thorough visual interpretation to compliment the abstract concepts. Some particular demonstrations could use diagrams and animation to show how the intersections of the orbit and S converge upon z, how the bounded orbit must come close to at least one point infinitely many times, and how the continuity of the flow ensures that $\phi(\vec{q})$ is bounded because $\omega(\vec{x}_0)$ is bounded. I believe that this would be the most productive avenue of future work on this proof, and would help more types of learners to understand and appreciate the proof. This kind of approach may make pure mathematics more accessible to people for whom the abstract analysis doesn't come as easily.

As for the Hartman-Grobman proof, I have struggled to come up with an organizational structure to the proof that would be more helpful to students (myself included), with little avail. However, I do believe that the way to make this proof easier to understand lies, at least in part, in restructuring it. Since this proof relies upon many other proofs, it would be helpful to compile them and present a well-structured document. Such a document would be interesting because it would involve many concepts from analysis and topology, but, rather than having the goal of teaching such subjects, the aim would be to understand a theorem that is commonly used in applied dynamical systems work. This document may include a section on Jordan canonical forms and matrix calculus as well is the ideaas from analysis and topology (successive approximations, norms, etc.).

# 5 Annotated Bibliography

J. Guckenheimer & P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.


One of the classic books on the subject of dynamical systems.

B. Hasselblatt & A. Katok, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, New York, 1995.


This book was intended to be a self-contained introduction to the theory of dynamical systems. As such it is is helpful because of its treatment of the topological and analytical ideas that relate to dynamical systems. A proof for the Poincaré-Bendixon theorem is given where the 2-dimensional manifold is assumed to be the surface of the sphere $S^2$. This is one example of how this book is more topologically-oriented than the others, and perhaps more appropriate to those interested in topology as it relates to dynamical systems than to a general undergraduate audience. However, despite these topological differences, the proof is similar to Robinson's. The Hartman-Grobman theorem and the subject of linearization do not appear in the index, which seems to be a hole in a text with such ambitious aims.

L. Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 1991.


This text provides the simpler proof of the Hartman-Grobman theorem upon which this document's treatment of the proof was based. It is simple in

comparison to the text by Robinson, which gives a more detailed proof. It assumes that we have already translated the equilibrium point about which we are analyzing phase space to the origin, which simplifies the problem greatly. The Poincaré-Bendixon theorem is also presented and its proof is similar to the one in this document.

C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics and Chaos* CRC Press, Inc., Boca Raton, 1995.

Robinson's text is very detailed and requires more knowledge of topology and analysis than the others. However, it is, for the most part, self contained and the definitions of most topological and analytical concepts can be found within the book. The proof given for the Hartman-Grobman theorem is more complicated than Perko's text. However, the text gives proofs of the global theorem, the local theorem, and the theorem as applied to flows. The Poincaré-Bendixon theorem is also covered and proved using a similar method to the approach used in this document.

R. C. Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete*, Pearson Education, Upper Saddle River, NJ, 2004.

Like Strogatz's text, this book is an "introduction" to the field and is more accessible than some of the other texts, however, it is more advanced and was the primary text used for the proof of the Poincaré-Bendixon theorem presented in this document. The text is related to Robinson's other text, and a proof for the Poincaré-Bendixon is presented in each, however, where his other text uses a Lemma-based approach, this text uses a more chronological approach to the proof and illustrates the idea of the closed curve $\Gamma$ with a picture.

S.H. Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering* Perseus Books Publishing, Cambridge, MA, 1995.

Strogatz's book is easily accessible at the undergraduate level, and is a great place to begin learning about dynamical systems. Both the he Hartman-Grobman and Poincaré-Bendixon theorems are presented, but neither is proved. However, the presentation is much more intuitive than the other books presented here.