One-Dimensional Collocation Method for the Korteweg-de Vries Equation

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Overview

- Physical Background of KdV
- Collocation Method
- Implementing the Method
- Simulations
- Convergence

Physical Background

KdV:

$$\partial_t \phi(x,t) + \mu \partial_x^3 \phi(x,t) + \epsilon\phi \partial_x \phi(x,t) = 0,$$

$$x \in [a,b], \quad t \geq 0$$

- KdV is commonly used in Hydrodynamics to model shallow water waves.
- Similar to Advection Equation

Other Uses of KdV

- Electrical Transmission in Lines
- Blood Pressure
- Gravity Waves in Geophysics

Collocation Method

In one dimension, a collocation method approximates a solution to an ordinary differential equation, $F(\phi) = 0$, with a function in a finite dimensional function space.

- Given our function space, let our basis functions be $\{v_1, \ldots, v_N\}$
- We approximate the solution to our differential equation with a linear combination of these basis functions:

$$\phi \approx \Phi = \sum_{i=1}^{N} \alpha_i v_i(x).$$

Collocation Method Cont.

Suppose we are approximating the solution on the interval $[a, b]$ with a uniform partition $\{x_1, \ldots, x_m\}$. We construct Φ such that:

$$F(\Phi(x_{j+1/2})) = 0 \;\forall\; j \in \{1, \ldots, M-1\}$$

where

$$x_{j+1/2} = \frac{x_{j+1} + x_j}{2}$$

We now have a system of $M - 1$ equations.
Using boundary conditions, more equations can be constructed.

Collocation Method Cont.

To implement 1-D collocation method for a PDE, we approximate the solution ϕ with

$$\phi \approx \Phi = \sum_{i=1}^{N} \alpha_i(t) v_i(x).$$

Following the same process as before, we come up with a system of differential equations, with unknown α_i, which may be solved by a Finite Difference Method.

Problem Statement

KdV:

$$\partial_t \phi(x,t) + \mu \, \partial_x^3 \phi(x,t) + \epsilon\phi \, \partial_x \phi(x,t) = 0,$$

$$x \in [a,b], \quad t \geq 0$$

With boundary conditions:

- $\phi(a,t) = 0$
- $\phi(b,t) = 0$
- $\partial_x \phi(b,t) = 0$

And an initial condition:

- $\phi(x,0) = \phi_0(x)$

Function Space

- We choose a partition of the interval $[a, b]$: $P = \{x_1, \ldots, x_N\}$.
- Let S_3 be our function space. S_3 is the set of all cubic splines on P such that $f \in S_3$, then $f \in C^2[a, b]$.
- For a given P, S_3 has a basis consisting of $N + 2$ elements: $B = \{v_1, \ldots, v_{N+2}\}$

Thus

$$\Phi = \sum_{i=1}^{N+2} \alpha_i(t) v_i(x)$$

System of ODEs

$$\partial_t \Phi(t, x_{j+1/2}) + \mu \partial_x^3 \Phi(t, x_{j+1/2}) + \epsilon \Phi \partial_x \Phi(t, x_{j+1/2}) = 0,$$

$$\forall j \in \{1, \ldots, N-1\}$$

represents our system of differential equations.
With boundary conditions:

- $\Phi(a, t) = 0$
- $\Phi(b, t) = 0$
- $\partial_x \Phi(b, t) = 0$

we have N-1 differential equations and 3 algebraic relations for the boundary conditions.

Inital Condition for System of ODEs

Let us choose the inital condition for our approximation to satisfy

$$\Phi(x_{j+1/2}, 0) = \phi_0(x_{j+1/2}), \quad \forall\, j \in \{1, \ldots, N-1\}$$

Using these equations and the boundary conditions, our initial condition is found solving a linear system.

Let $t \in [0, T]$, and choose a uniform partition of $[0, T]$: $\{t_0, \ldots, t_m\}$, $\Delta t = 1/m$.

Backward Euler

We employ Backward Euler, with a slight modification:

$$\frac{\Phi(t_{i+1}, x_{j+1/2}) - \Phi(t_i, x_{j+1/2})}{\Delta t} + \dots$$

$$\mu\, \partial_x^3\, \Phi(t_{i+1}, x_{j+1/2}) + \dots$$

$$\epsilon\Phi(t_i, x_{j+1/2})\, \partial_x\, \Phi(t_{i+1}, x_{j+1/2}) = 0,$$

where we have assumed $\Phi(t_i, x_{j+1/2}) \approx \Phi(t_{i+1}, x_{j+1/2})$.

Implementation in Summary

- Using Collocation, we reduced KdV to a system of nonlinear ODEs.
- With the aid of Backward Euler and a reasonable assumption, we reduced finding the solution at each time step to a linear system.

All that is left is to model.

Maxwellian Initial Condition

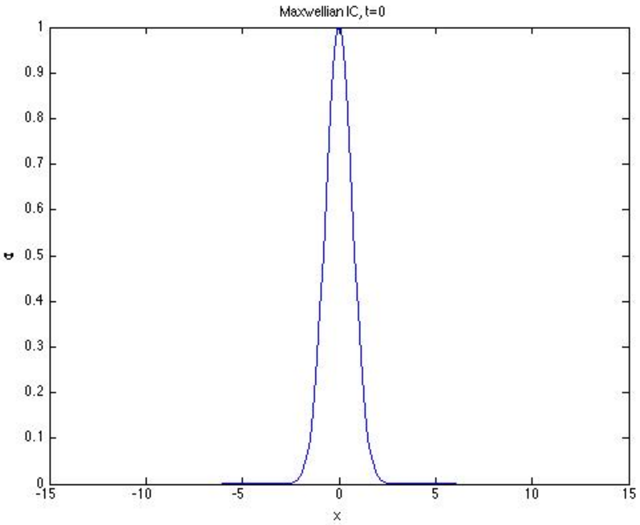
Initial Condition:

- $\phi(x, 0) = e^{-x^2}$

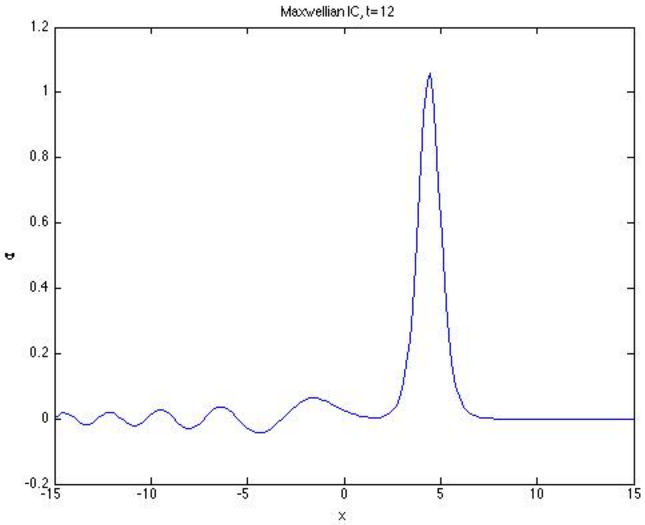
Spatial Interval and Parameters:

- $x \in [-15, 15]$
- $\Delta t = .01$
- $\Delta x = .01$
- $\epsilon = 1$
- $\mu = .05$

Maxwellian Initial Condition Cont.



Maxwellian Initial Condition



Soliton Initial Condition

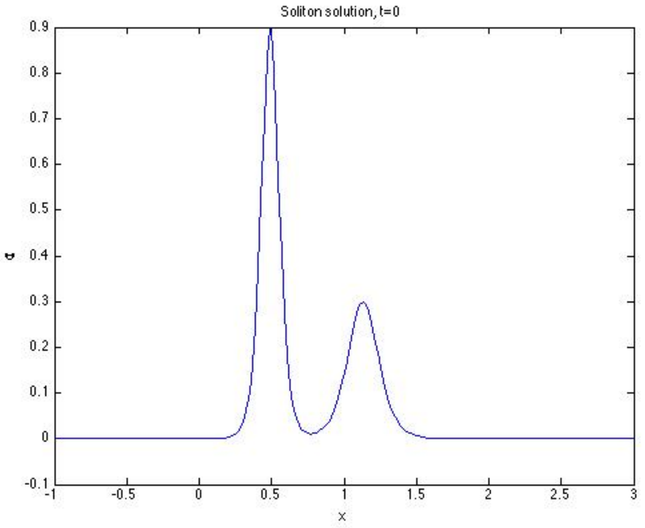
Initial Condition:

- $\phi(x,0) = 3c_1 sech^2(a_1 x - 6) + 3c_2 sech^2(a_2 x - 8)$
- $c_1 = .3$
- $c_2 = .1$
- $a_i = \sqrt{(\epsilon c_i/\mu)}/2$

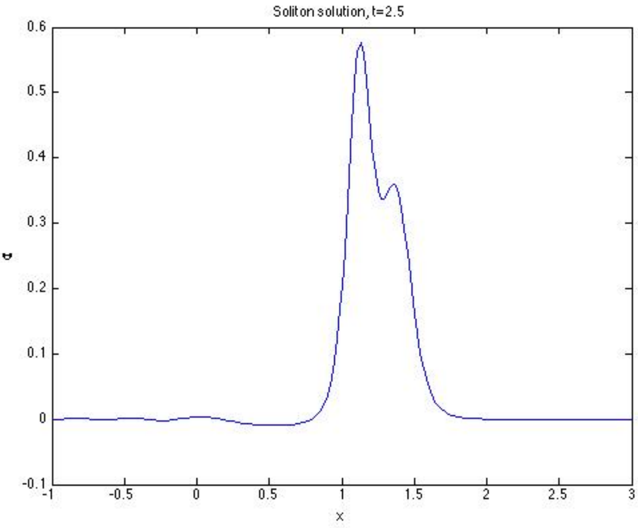
Spatial Interval and Parameters:

- $x \in [-1, 3]$
- $\Delta t = .005$
- $\Delta x = .01$
- $\epsilon = 1$
- $\mu = 0.0005$

Soliton Initial Condition Cont.



Soliton Initial Condition Cont.



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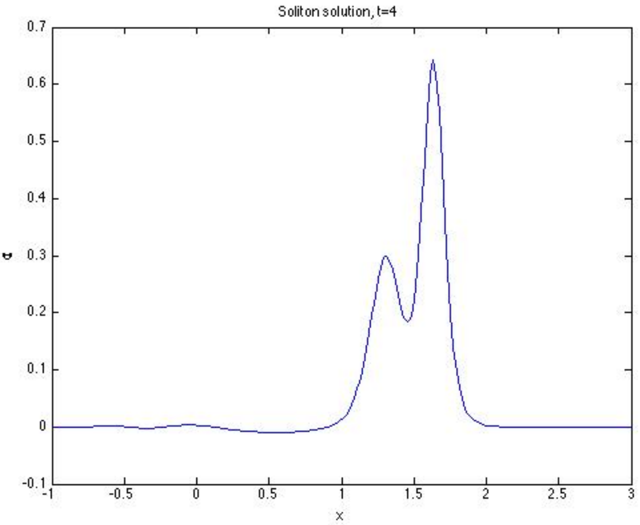
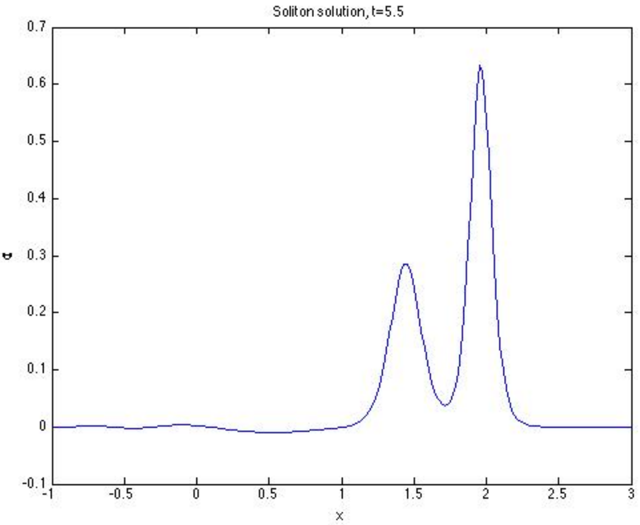


Soliton Initial Condition Cont.



Numerical Finding on Convergence

- The method exhibits $O(\Delta x^2 + \Delta t)$ convergence.

This rate was found with respect to the Soliton Initial Condition, where we know the analytic solution.

In Conclusion

The 1-D Collocation method provides a clear way to solve the KdV equation.

- Reduces nonlinear PDE to system of linear equations at each time step.