A Finite Difference Method for Modeling Seismic Wave Propagation

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Capstone Final Project

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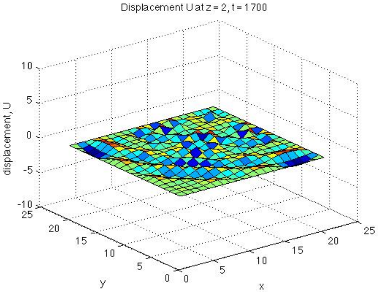
Why model seismic waves?

- Information about the material properties and structure of the earth
- Loss estimation
- Future earthquake scenarios



- FDMs were first use in the 1960's
- Equation of Motion:

$$\rho \ddot{u}_i - \sigma_{ij,j} - f_i = 0$$

ρ: material density, $\ddot{u}_i$: particle acceleration, $\sigma_{ij,j}$: stress tensor, f: forcing function

- Constitutive Relation: relationship between the stress and the strain of a medium

$$\sigma_{ij} = \kappa \epsilon_{kk} \delta_{ij} + 2\mu(\epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij})$$

κ, μ, λ: bulk, shear, and elastic moduli, $(\lambda = \kappa - \frac{2}{3}\mu)$, σ: stress, ϵ:strain

- Displacement:

$$\rho \ddot{u}_i = [(\kappa - \frac{2}{3}\mu)u_{k,k}]_{,i} + (\mu u_{i,j})_{,j} + (\mu u_{j,i})_{,j} + f_i$$

- Let u, v, w be the x, y, z components of the displacement. For $\phi \in \{u, v, w\}$, let $\Phi \in \{U, V, W\}$ be the corresponding approximation

$$\phi(x_i, y_j, z_k, t_m) \approx \Phi^m_{i,j,k}$$

with uniform spatial step h and time step Δt,

$$x_i = ih, y_j = jh, z_k = kh, t_m = m\Delta t$$

- Central difference approximations
$\frac{d^2\phi}{dx^2} = \frac{\phi(x-h)-2\phi(x)+\phi(x+h)}{h^2}$, $\frac{d\phi}{dx} = \frac{\phi(x+h)-\phi(x-h)}{2h}$
- Must account for mixed and non-mixed derivatives
Define auxiliary function $\phi = \mu u_{,x}$. Then

$$\phi_{,x}\big|_{i,j,k} = \frac{\phi_{i+1/2,j,k} + \phi_{i-1/2,j,k}}{h}$$

$$\phi_{,z}\big|_{i,j,k} = \frac{\phi_{i,j,k+1/2} - \phi_{i,j,k-1/2}}{h}$$

We need to approximate $\phi_{i+1/2,j,k}$, $\phi_{i-1/2,j,k}$, $\phi_{i,j,k+1/2}$ and $\phi_{i,j,k-1/2}$

- Non-mixed:

$$\phi_{,x}|_{i,j,k} = \frac{\phi_{i+1/2,j,k} + \phi_{i-1/2,j,k}}{h}$$

$$\phi_{i+1/2,j,k} = \frac{1}{h}\mu^{x}_{i+1/2,j,k}(U_{i+1,j,k} - U_{i,j,k})$$

$$\phi_{i-1/2,j,k} = \frac{1}{h}\mu^{x}_{i-1/2,j,k}(U_{i,j,k} - U_{i-1,j,k})$$

- Mixed:

$$\phi_{,z}|_{i,j,k} = \frac{\phi_{i,j,k+1/2} - \phi_{i,j,k-1/2}}{h}$$

$$\phi_{i,j,k+1/2} = \frac{1}{4h}\mu_{i,j,k+1/2}(U_{i+1,j,k+1} - U_{i-1,j,k+1} + U_{i+1,j,k-1} - U_{i-1,j,k-1})$$

$$
\begin{aligned}
U^{m+1}_{i,j,k} = \frac{1}{h^2}[\lambda^x_{i+1,j,k}(U^m_{i+1,j,k} - U^m_{i,j,k}) - \lambda^x_{i-1,j,k}(U^m_{i,j,k} - U^m_{i-1,j,k}) \\
+ 2(\mu^x_{i+1,j,k}(U^m_{i+1,j,k} - U^m_{i,j,k}) - \mu^x_{i-1,j,k}(U^m_{i,j,k} - U^m_{i-1,j,k})) \\
+ \mu^y_{i,j+1,k}(U^m_{i,j+1,k} - U^m_{i,j,k}) - \mu^y_{i,j-1,k}(U^m_{i,j,k} - U^m_{i,j-1,k}) \\
+ \mu^z_{i,j,k+1}(U^m_{i,j,k+1} - U^m_{i,j,k}) - \mu^z_{i,j,k-1}(U^m_{i,j,k} - U^m_{i,j,k-1}) \\
+ \frac{1}{4}[\lambda^y_{i,j+1,k}(V^m_{i,j+1,k} + V^m_{i+1,j+1,k} - V^m_{i,j-1,k} - V^m_{i+1,j-1,k}) \\
- \lambda^y_{i,j-1,k}(V^m_{i-1,j+1,k} + V^m_{i,j+1,k} - V^m_{i-1,j-1,k} - V^m_{i,j-1,k}) \\
+ \lambda^z_{i,j,k+1}(W^m_{i,j,k+1} + W^m_{i+1,j,k+1} - W^m_{i,j,k-1} - W^m_{i+1,j,k-1}) \\
- \lambda^z_{i,j,k-1}(W^m_{i-1,j,k+1} + W^m_{i,j,k+1} - W^m_{i-1,j,k-1} - W^m_{i,j,k-1}) \\
+ \mu^y_{i+1,j,k}(V^m_{i+1,j,k} + V^m_{i+1,j+1,k} - V^m_{i-1,j,k} - V^m_{i-1,j-1,k}) \\
- \mu^y_{i-1,j,k}(V^m_{i+1,j-1,k} + V^m_{i+1,j,k} - V^m_{i+1,j-1,k} - V^m_{i-1,j,k}) \\
+ \mu^z_{i+1,j,k}(W^m_{i+1,j,k} + W^m_{i+1,j,k+1} - W^m_{i-1,j,k} - W^m_{i-1,j,k-1}) \\
- \mu^z_{i-1,j,k}(W^m_{i-1,j,k+1} + W^m_{i,j,k+1} - W^m_{i-1,j,k-1} - W^m_{i,j,k-1})]] + F^{x,m}_{i,j,k}
\end{aligned}
$$

$$L_{\gamma,\gamma}(a,\phi) = \frac{1}{h^2}[a^{\gamma+}(\phi_+^m - \phi^m) - a^{\gamma-}(\phi^m - \phi_-^m)],\ a^{\gamma+} = \frac{1}{h}[\int_{\gamma_n}^{\gamma_{n+1}} \frac{1}{a} d\gamma]^{-1}$$

$\gamma \in \{x, y, z\}$, $a \in \{\mu, \lambda\}$ $\phi \in \{U, V, W\}$

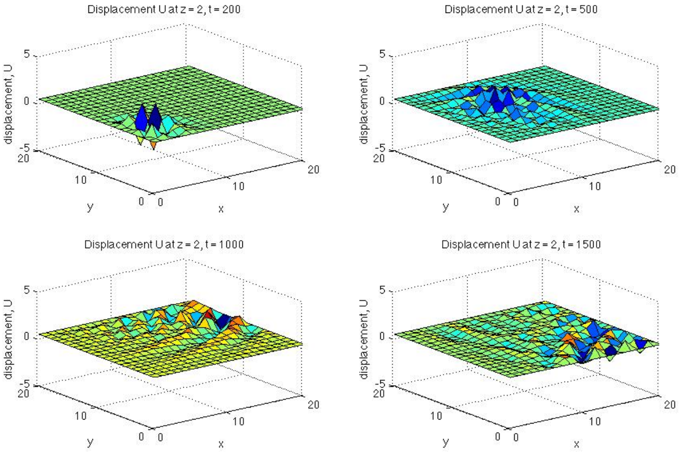
$$L_{\gamma,\eta}(a,\phi) = \frac{1}{4h^2}[a^{\eta+}(\phi_{2+}^m + \phi_{3+}^m - \phi_{2-}^m - \phi_{3-}^m) - a^{\eta-}(\phi_{1+}^m + \phi_{2+}^m - \phi_{1-}^m - \phi_{2-}^m)]$$

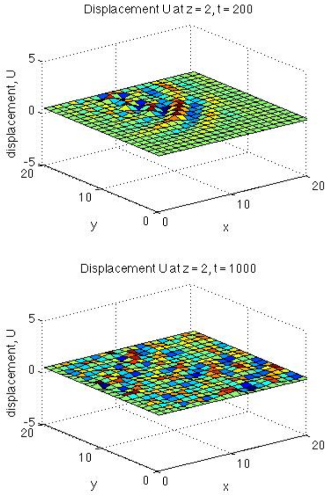
$$U_{i,j,k}^{m+1} = 2U_{i,j,k}^m - U_{i,j,k}^{m-1} + \frac{\Delta t^2}{\rho_{i,j,k}}[L_{xx}(\lambda, U) + 2L_{xx}(\mu, U) + L_{yy}(\mu, U) + L_{zz}(\mu, U) L_{yx}(\lambda, V) + L_{zx}(\lambda, W) + L_{xy}(\mu, V) + L_{xz}(\mu, W)] + F_{i,j,k}^{x,m}$$

Stability condition for a second order central difference formula on the general equation of motion in an isotropic, homogeneous medium:

$$\Delta t^2 \leq \frac{h^2}{2\alpha^2(1 + 2\beta^2/\alpha^2)}$$

where $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$ is the P-wave velocity and $\beta = \sqrt{\frac{\mu}{\rho}}$ is the S-wave velocity.


Displacement U at z = 2, t = 200
Displacement U at z = 2, t = 500
Displacement U at z = 2, t = 1000
Displacement U at z = 2, t = 1500
displacement, U
x
y
5
0
-5
0
10
20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Displacement U at z = 2, t = 1000
Displacement U at z = 2, t = 1500
displacement, U
5
0
-5
20
10
0
y
x

Questions

Moczo,P., Kristek, J., Galis, M., Pazak, P., Balazovjech, M. The finite difference and finite element modeling of seismic wave propagation and earthquake motion. *Acta Physica Slovaca*, 57(2):177-406, April 2007.
Bormann, P., Engdahl, B., Kind, R. Seismic Wave Propagation and Earth Models. In: Bormann, P. (Ed.) *New Manual of Seismological Observatory Practice (NMSOP)* Postdam: Deutsches Geo-Forschungs Zentrum GFZ, 2009.