Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH484: Capstone - Mathematical and Computational Modeling

Assignment #4, Spring 2015
SOLUTIONS

1. Friedman & Littman, p.39, Problem **2.7.2**
Let $U > 0$ be given. For the equation

$$\partial_t c + U \partial_x c = 0,$$

investigate the **stability** of the FTFS (forward time, forward space) scheme

$$\frac{c_j^{n+1} - c_j^n}{\Delta t} + U \frac{c_{j+1}^n - c_j^n}{\Delta x} = 0.$$

Throughout this assignment, we will define and make use of the constant

$$\sigma := \frac{U \Delta t}{\Delta x}.$$

As in class, we begin by using the discrete solution

$$c_j^n = \alpha^n e^{i\beta j}.$$

Multiplying through by Δt and using this solution within the FTFS scheme yields

$$\alpha_j^{n+1} e^{i\beta j} - \alpha_j^n e^{i\beta j} = -U \frac{\Delta t}{\Delta x} \left(\alpha_j^n e^{i\beta(j+1)} - \alpha_j^n e^{i\beta j} \right).$$

Dividing by $c_j^n = \alpha_j^n e^{i\beta j} \neq 0$, this becomes

$$\alpha = 1 - \sigma(e^{i\beta} - 1)$$

which simplifies to

$$\alpha = 1 - \sigma(\cos(\beta) - 1) - i\sigma \sin(\beta)$$

using Euler's formula. Computing the square modulus of this complex number and putting the 1 on the left side gives us

$$\begin{aligned} |\alpha|^2 - 1 &= -2\sigma(\cos(\beta) - 1) + \sigma^2(\cos(\beta) - 1)^2 + \sigma^2 \sin^2(\beta) \\ &= 2\sigma(1 - \cos(\beta)) + \sigma^2(1 - \cos(\beta))^2 + \sigma^2 \sin^2(\beta). \end{aligned}$$

Now, if $\cos(\beta) = 1$ then this becomes $|\alpha| = 1$, which is fine. However, since $\cos(\beta) \leq 1$, the only other option is $1 - \cos(\beta) > 0$, and in this case the right side of the equation above is strictly positive (as long as $\sigma > 0$). Hence, in this case $|\alpha| > 1$ regardless of the value of σ, and there is no condition which guarantees that $|\alpha| \leq 1$ for all $\beta \in \mathbb{R}$. Therefore, the method is **unconditionally unstable**.

2. Friedman & Littman, p.40, Problem **2.7.5**
For the same equation, investigate the **stability** of the Lax-Wendroff scheme

$$c_j^{n+1} = c_j^n - \frac{U}{2}\frac{\Delta t}{\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right) + \frac{U^2}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left(c_{j+1}^n - 2c_j^n + c_{j-1}^n\right).$$

Proceeding as in the first problem, the scheme reduces to

$$\alpha_j^{n+1}e^{i\beta j} = \alpha_j^n e^{i\beta j} - \frac{1}{2}\sigma\left(\alpha_j^n e^{i\beta(j+1)} - \alpha_j^n e^{i\beta(j-1)}\right) + \frac{1}{2}\sigma^2\left(\alpha_j^n e^{i\beta(j+1)} - 2\alpha_j^n e^{i\beta j} + \alpha_j^n e^{i\beta(j-1)}\right).$$

As before, dividing by $c_j^n \neq 0$, this becomes

$$\alpha = 1 - \frac{1}{2}\sigma(e^{i\beta} - e^{-i\beta}) + \frac{1}{2}\sigma^2(e^{i\beta} - 2 + e^{-i\beta})$$

Using Euler's formula and the symmetry of $\sin(-\beta)$ and $\cos(-\beta)$, this becomes

$$\alpha = 1 - i\sigma\sin(\beta) + \sigma^2(\cos(\beta) - 1) \tag{1}$$

Computing the modulus and subtracting one from both sides, we finally find

$$\begin{aligned} |\alpha|^2 - 1 &= \sigma^2\sin^2(\beta) + 2\sigma^2(\cos(\beta) - 1) + \sigma^4(\cos(\beta) - 1)^2 \\ &= \sigma^2(1 - \cos^2(\beta)) - 2\sigma^2(1 - \cos(\beta)) + \sigma^4(1 - \cos(\beta))^2 \end{aligned}$$

If $\cos(\beta) = 1$ then this becomes $|\alpha| = 1$, which is fine. The only other option is $1 - \cos(\beta) > 0$, and in this case imposing $|\alpha|^2 - 1 \leq 0$ is equivalent to

$$\sigma^2(1 - \cos^2(\beta)) - 2\sigma^2(1 - \cos(\beta)) + \sigma^4(1 - \cos(\beta))^2 \leq 0$$

Since $\sigma^2(1 - \cos(\beta)) > 0$, we can divide by it and the condition becomes

$$(1 + \cos(\beta)) - 2 + \sigma^2(1 - \cos(\beta)) \leq 0$$

or after simplifying

$$(\sigma^2 - 1)(1 - \cos(\beta)) \leq 0.$$

Hence, it must be the case that $|\sigma| \leq 1$ in order for the scheme to be stable.

3. Friedman & Littman, p.40, Problem **2.7.6**
Use the Lax-Wendroff scheme for Problem 2.4.1 and compare your results with those of the FTBS scheme we previously used. As in the directions for the first problem of HW#3, use

$dt = 0.001$ and $dx = 0.1$. For the spatial interval use $[-2, 200]$, fix the y axis in your plots to $[0, 5]$, and create a 5×1 matrix of plots on the same figure.

The maximum concentration values appear to overestimate the true value, rather than underestimate it. These values are $6.3917, 6.2192, 5.8773, 5.3328$, and 4.7589 for $U = 1, 5, 10, 20, 40$, respectively. Below is the associated code.

MATLAB Code

```
clear; clc;

dx = 1e-1;
dt = 1e-3;

U = [1, 5, 10, 20, 40];

a = -2;
b = 200;

T = 4;
x = a:dx:b;
n = T/dt;

I = find(abs(x) < 1);
c(1,:) = zeros(1,length(x));
c(1,I) = 5;

for vel = 1:5
    sig = U(vel)*(dt/dx);
    sig2 = sig^2;
    for i = 1:n
        for j = 2:length(x)-1
            c(i+1,j) = c(i,j) - (sig/2)*(c(i,j+1) - c(i,j-1)) + ...
            (sig2/2)*(c(i,j+1) -2*c(i,j) + c(i,j-1));
        end

    end
    subplot(5,1, vel);
    plot(x, c(n+1, :)), title(['Problem 3, U = ', num2str(U(vel))]),
    axis([-2,200,0,7]), xlabel('x'), ylabel('c(t,x)')

    %Maximum concentration at t=4
    M = max(c(end, :))
end
```

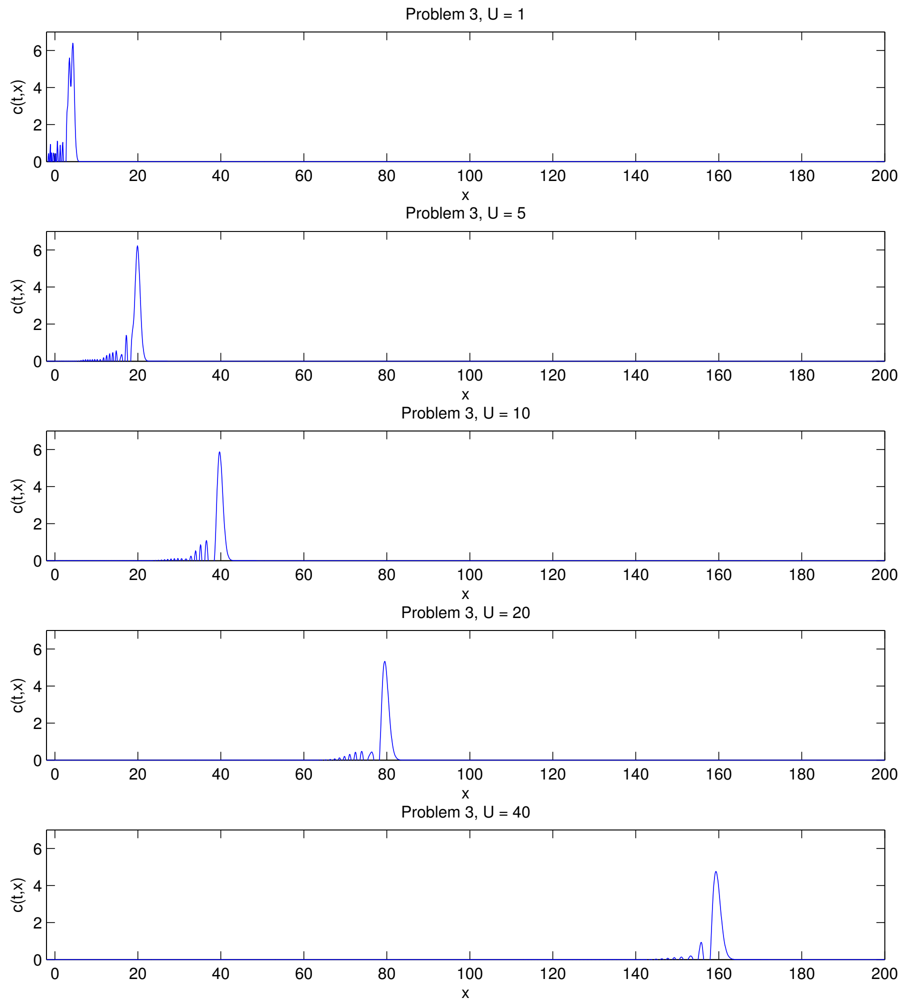


Figure 1: Graphs for Problem 3: $U = 1, 5, 10, 20, 40$.

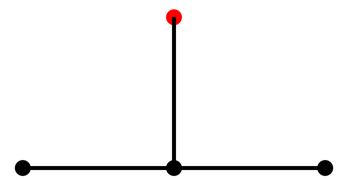
4. Draw the stencil for the Lax-Wendroff scheme above. Then, show that this numerical method is **consistent** with the advection equation

$$\partial_t c + U \partial_x c = 0$$

and use this to demonstrate that it is first-order accurate in time and second-order accurate in space, i.e.

$$|E_j^n| \leq O\left(\Delta t + (\Delta x)^2\right).$$

The stencil is the same as a central difference method would be, i.e.



Next, we derive the wave speed of solutions to the equation. As in class, we consider solutions of the form

$$c(t,x) = e^{vt}e^{i\kappa x}$$

and if this is to solve the given advection equation, we see that

$$v = -iU\kappa.$$

So, computing the first part of the consistency definition by Taylor expansion, we find

$$e^{v\Delta t} = e^{-iU\kappa\Delta t} = 1 - iU\kappa\Delta t + O((\Delta t)^2) \tag{2}$$

Next, we use the second problem to derive a representation for α. More specifically, beginning with (1) and using $\beta = \kappa\Delta x$ along with the Taylor expansions

$$\sin(x) = x + O(x^3) \qquad \cos(x) = 1 + \frac{1}{2}x^2 + O(x^4)$$

provides the representation

$$\begin{aligned}
\alpha &= 1 - i\sigma\sin(\beta) + \sigma^2(\cos(\beta) - 1)\\
&= 1 - i\sigma\sin(\kappa\Delta x) + \sigma^2(\cos(\kappa\Delta x) - 1)\\
&= 1 - i\sigma\left(\kappa\Delta x + O((\kappa\Delta x)^3)\right) + \sigma^2\left(\frac{1}{2}(\kappa\Delta x)^2 + O(\kappa\Delta x)^4\right)\\
&= 1 - iU\left(\kappa\Delta t + \Delta t O((\Delta x)^2)\right) + U^2(\Delta t)^2\left(\frac{1}{2}\kappa^2 + O((\Delta x)^2)\right)\\
&= 1 - iU\kappa\Delta t + \Delta t O((\Delta x)^2) + O((\Delta t)^2).
\end{aligned}$$

Thus, using (2) with this expression for α in the definition for consistency, we see that the first portions cancel and

$$\begin{aligned}
\left|\frac{e^{v\Delta t} - \alpha}{\Delta t}\right| &= \frac{1}{\Delta t}\left(\Delta t O((\Delta x)^2) + O((\Delta t)^2)\right)\\
&= O\left(\Delta t + (\Delta x)^2\right).
\end{aligned}$$

Therefore, the quantity on the left tends to 0 as $\Delta t, \Delta x \downarrow 0$. Moreover, using the inequalities from class, this implies

$$|E_k^n| \le Ct\left|\frac{e^{v\Delta t} - \alpha}{\Delta t}\right| = O\left(\Delta t + (\Delta x)^2\right)$$

which further implies that the scheme is first-order accurate in t and second-order in x.

Additionally, it can be shown that the second-order Δt terms cancel as well! This happens because the $O((\Delta t)^2)$ term in (2) is actually

$$\frac{1}{2}(iU\kappa\Delta t)^2 = -U^2(\Delta t)^2\frac{1}{2}\kappa^2$$

which cancels with this same term in the representation of α above (see the second-to-last line).