**Department of Applied Mathematics and Statistics**
**COLORADO SCHOOL OF MINES**
MATH 500: Linear Vector Spaces

## Assignment #3 - Vector Spaces, Subspaces, and Norms
## Due Thursday, September 30, 2021

**1**. (10 points) Let $V$ be a vector space with $v_1, ..., v_n \in V$ for some $n \in \mathbb{N}$ and denote $S = \{v_1, .., v_n\}$.

(a) Prove that if $S$ is linearly dependent and $v_{n+1}, ..., v_m \in V$ for some $m \in \mathbb{N}$ with $m > n$, then the set $S \bigcup \{v_{n+1}, ..., v_m\}$ is also linearly dependent.

(b) Prove that if $S$ is linearly independent then any non-empty subset of $S$ is also linearly independent.

*Hint:* Proof by contradiction may be useful

**2**. (8 points) Let $u \in \mathbb{R}^p \setminus \{0\}$ and $v \in \mathbb{R}^q \setminus \{0\}$ be given and define $A = uv^T \in \mathbb{R}^{p \times q}$. Show that $\{u\}$ is a basis for

$$\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^q\}$$

and determine the rank of $A$.

**3**. (10 points) Define the vector space $\ell^\infty(\mathbb{R})$ to be the set of all bounded infinite sequences of real numbers $\{a_n\}_{n=1}^\infty$. Prove that $\dim(\ell^\infty(\mathbb{R})) = \infty$.
*Hint:* Proof by contradiction may be useful

**4**. (10 points) We say that two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ defined on the same vector space $V$ are **equivalent** if there exists $C_1, C_2 > 0$ such that

$$C_1\|v\|_A \leq \|v\|_B \leq C_2\|v\|_A$$

for every $v \in V$. Prove (via inequalities) that the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ defined on $\mathbb{R}^p$ are equivalent.

**5**. (12 points) A subset $A$ of a vector space $\mathcal{V}$ is **convex** if for any $u, v \in A$, the line segment

$$H = \{\alpha u + (1 - \alpha)v : 0 \leq \alpha \leq 1\}$$

is a subset of $A$.

(a) Show that for any normed space $\mathcal{V}$, the closed unit ball

$$B(0,1) = \{v \in \mathcal{V} : \|v\| \leq 1\}$$

is convex.

(b) Show that the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi(v) = \left(\sqrt{|v_1|} + \sqrt{|v_2|}\right)^2$$

is not a norm on $\mathbb{R}^2$.

(c) Sketch the curve $\phi(v) = 1$. You may use MATLAB® to do this.