MATH 500: Linear Vector Spaces **NAME:**
Midterm Exam - Fall 2021

The exam will last from Thursday, October 21 at 9am to Tuesday, October 26 at 2pm. You may use our class notes, but NOT HW, HW solutions, or any other source, including internet or computational sources. Additionally, **you may NOT collaborate with others**. Show your work for each question. Throughout, let $p \in \mathbb{N}$ be given.

1. (20 points) Let $\mathcal{V}$ be a vector space, with nested subspaces $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}$ satisfying $\dim(\mathcal{V}_0) = \dim(\mathcal{V}_1) < \infty$. Prove (using proof by contradiction) that $\mathcal{V}_0 = \mathcal{V}_1$.

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2. (25 points) Let $A \in \mathbb{R}^{p \times p}$ be given and define

$$\text{Nul}\left(A^T\right) = \{x \in \mathbb{R}^p : A^T x = 0\}, \qquad \text{Col}(A) = \{Ax \in \mathbb{R}^p : x \in \mathbb{R}^p\}.$$

   (a) Prove that every element of $\text{Nul}\left(A^T\right)$ is orthogonal to every element of $\text{Col}(A)$ with respect to the standard inner product on $\mathbb{R}^p$ (i.e., the dot product).

   (b) Assume further that A is nonsingular and show that the function $\|\cdot\|_A : \mathbb{R}^p \to [0, \infty)$ defined for every $x \in \mathbb{R}^p$ by

$$\|x\|_A = \|Ax\|_2$$

   is a norm.

   Hint: You may use the fact that $\|\cdot\|_2$ is a norm on $\mathbb{R}^p$.

   (c) If we endowed $\mathbb{R}^p$ with the norm $\|\cdot\|_A$, is the resulting normed vector space complete? Justify your answer with a sentence or two.

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3. (12 points) Let $\mathcal{V}$ be a Hilbert space. Prove that if $u, v \in \mathcal{V}$ with $u \perp v$ then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

   This is known as the Pythagorean Theorem.

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4. (20 points) Let $n \in \mathbb{N}$ be given and assume $\mathcal{S} = \{v_1, ..., v_n\} \subset \mathcal{V}$ is orthogonal with $0 \notin \mathcal{S}$. Show that $\mathcal{S}$ is linearly independent.

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5. (23 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

   (a) Compute a normalized QR Factorization of A

   (b) Let $M = \text{Col}(A)$ and compute $M^{\perp}$.