MATH 500: Linear Vector Spaces **SOLUTIONS**
Midterm Exam - Fall 2021

The exam will last from Thursday, October 21 at 9am to Tuesday, October 26 at 2pm. You may use our class notes, but NOT HW, HW solutions, or any other source, including internet or computational sources. Additionally, **you may NOT collaborate with others**. Show your work for each question. Throughout, let $p \in \mathbb{N}$ be given.

1. (20 points) Let $\mathcal{V}$ be a vector space, with nested subspaces $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}$ satisfying $\dim(\mathcal{V}_0) = \dim(\mathcal{V}_1) < \infty$. Prove (using proof by contradiction) that $\mathcal{V}_0 = \mathcal{V}_1$.

Solution: We prove the result by contradiction. First, assume that $\mathcal{V}_1 \setminus \mathcal{V}_0 \neq \emptyset$ and thus let $u \in \mathcal{V}_1$ with $u \notin \mathcal{V}_0$ be given. Since $\mathcal{V}_0$ is a subspace with finite dimension, let $B_0 = \{v_1, ..., v_n\}$ be a basis for $\mathcal{V}_0$, and notice that $\mathcal{V}_1$ must also have a basis B_1 consisting of n elements. Because $u \notin \mathcal{V}_0 = \text{span}(B_0)$, we note that u cannot be written as a linear combination of the vectors in B_0. Next, define the set $\tilde{B}_0 = B_0 \cup \{u\} \subseteq \mathcal{V}_1$ and consider $\alpha_1, ..., \alpha_{n+1} \in K$ such that

$$\sum_{k=1}^{n} \alpha_k v_k + \alpha_{n+1} u = 0. \tag{1}$$

If $\alpha_{n+1} \neq 0$, then we may divide by this number within (1) and write

$$u = \sum_{k=1}^{n} -\frac{\alpha_k}{\alpha_{n+1}} v_k$$

which implies that $u \in \text{span}(B_0)$. Of course, this contradicts the fact that $u \notin \mathcal{V}_0 = \text{span}(B_0)$.

If, instead, $\alpha_{n+1} = 0$, then we see that

$$\sum_{k=1}^{n} \alpha_k v_k = 0$$

and by the linear independence of the basis B_0, this implies $\alpha_1 = ... = \alpha_n = 0$. Therefore, $\tilde{B}_0$ is a linearly independent subset of $\mathcal{V}_1$. However, $\tilde{B}_0$ contains $n+1$ elements, and since B_1 is a basis (and thus a spanning set) for $\mathcal{V}_1$ with n elements, a theorem from class implies $|\tilde{B}_0| \leq |B_1|$. Thus, we conclude $n + 1 \leq n$, providing a contradiction.

In either case, we find a contradiction to our original assumption, namely that $\mathcal{V}_1 \setminus \mathcal{V}_0 \neq \emptyset$, and we finally conclude that $\mathcal{V}_1 \setminus \mathcal{V}_0 = \emptyset$, which implies $\mathcal{V}_1 = \mathcal{V}_0$.

2. (25 points) Let $A \in \mathbb{R}^{p\times p}$ be given and define

$$\operatorname{Nul}\left(A^T\right) = \{x \in \mathbb{R}^p : A^T x = 0\}, \qquad \operatorname{Col}(A) = \{Ax \in \mathbb{R}^p : x \in \mathbb{R}^p\}.$$

(a) Prove that every element of $\operatorname{Nul}\left(A^T\right)$ is orthogonal to every element of $\operatorname{Col}(A)$ with respect to the standard inner product on $\mathbb{R}^p$ (i.e., the dot product).

(b) Assume further that A is nonsingular and show that the function $\|\cdot\|_A : \mathbb{R}^p \to [0,\infty)$ defined for every $x \in \mathbb{R}^p$ by

$$\|x\|_A = \|Ax\|_2$$

is a norm.

Hint: You may use the fact that $\|\cdot\|_2$ is a norm on $\mathbb{R}^p$.

(c) If we endowed $\mathbb{R}^p$ with the norm $\|\cdot\|_A$, is the resulting normed vector space complete? Justify your answer with a sentence or two.

Solution:

(a) Let $u \in \operatorname{Nul}(A^T)$ and $v \in \operatorname{Col}(A)$ be given. Note that both of these vectors are in $\mathbb{R}^p$. Since $v \in \operatorname{Col}(A)$, there is $x \in \mathbb{R}^p$ such that $v = Ax$, and since $u \in \operatorname{Nul}(A^T)$, we see that $A^T u = 0$. Thus, taking the inner product of these elements yields

$$\langle u, v\rangle_{\mathbb{R}^p} = u^T v = u^T Ax = (A^T u)^T x = 0^T x = 0.$$

Therefore, u and v are orthogonal with respect to this inner product.

(b) We merely show the properties of a norm and use the fact that $\|\cdot\|_2$ is a norm throughout. Let $x \in \mathbb{R}^p$ be given. First, we find

$$\|0\|_A = \|A0\|_2 = \|0\|_2 = 0.$$

Additionally, if $x \neq 0$, then because A is nonsingular, by the IMT we find that $Ax \neq 0$. Therefore,

$$\|x\|_A = \|Ax\|_2 > 0.$$

Next, let $\alpha \in \mathbb{R}$ be given. Then,

$$\|\alpha x\|_A = \|A(\alpha x)\|_2 = |\alpha|\|Ax\|_2 = |\alpha|\|x\|_A.$$

Finally, the triangle inequality follows from the triangle inequality of $\|\cdot\|_2$; namely for any $x, y \in \mathbb{R}^p$

$$\|x+y\|_A = \|A(x+y)\|_2 = \|Ax + Ay\|_2 \leq \|Ax\|_2 + \|Ay\|_2 = \|x\|_A + \|y\|_A.$$

(c) Due to a theorem from class, every finite-dimensional vector space is necessarily complete. Hence, $\mathbb{R}^p$ with any norm must be complete.

3. (12 points) Let $\mathcal{V}$ be a Hilbert space. Prove that if $u, v \in \mathcal{V}$ with $u \perp v$ then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

This is known as the Pythagorean Theorem.

Solution: Let $u, v \in \mathcal{V}$ be given with $u \perp v$ so that $\langle u, v\rangle = 0$. Then, we find

$$\langle v, u\rangle = \overline{\langle u, v\rangle} = \overline{0} = 0.$$

Thus, we merely compute

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2 = \|u\|^2 + \|v\|^2$$

and the proof is complete.

4. (20 points) Let $n \in \mathbb{N}$ be given and assume $\mathcal{S} = \{v_1, ..., v_n\} \subset \mathcal{V}$ is orthogonal with $0 \notin \mathcal{S}$. Show that $\mathcal{S}$ is linearly independent.

Solution: Let $\alpha_1, .., \alpha_n \in \mathbb{K}$ be given such that

$$\sum_{j=1}^{n} \alpha_j v_j = 0.$$

Then, since $\mathcal{S}$ is orthogonal, we have for any $k = 1, ..., n$

$$\begin{aligned} 0 &= \langle v_k, 0 \rangle \\ &= \left\langle v_k, \sum_{j=1}^{n} \alpha_j v_j \right\rangle \\ &= \sum_{j=1}^{n} \alpha_j \langle v_k, v_j \rangle \\ &= \alpha_k \langle v_k, v_k \rangle \\ &= \alpha_k \|v_k\|^2. \end{aligned}$$

Since $0 \notin \mathcal{S}$, we see that $v_k \neq 0$ and thus $\|v_k\|^2 > 0$. Hence, $\alpha_k = 0$ and since k was arbitrary, this holds for every $k = 1, ..., n$. Because $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only solution, this then implies that $\mathcal{S}$ must be linearly independent.

5. (23 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Compute a normalized QR Factorization of A

(b) Let $M = \mathrm{Col}(A)$ and compute $M^{\perp}$.

Solution: (a) Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad v_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

and then define $u_1 = v_1$. Using GS orthogonalization, we compute

$$\alpha_{12} = \frac{u_1 \cdot v_2}{\|u_1\|_2^2} = \frac{1}{2}$$

and thus

$$u_2 = v_2 - \alpha_{12} u_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Next,

$$\alpha_{13} = \frac{u_1 \cdot v_3}{\|u_1\|_2^2} = \frac{3}{2},$$

$$\alpha_{23} = \frac{u_2 \cdot v_3}{\|u_2\|_2^2} = -1,$$

and thus

$$u_3 = v_3 - \alpha_{13} u_1 - \alpha_{23} u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we discard u_3 and compute

$$\|u_1\|_2 = \sqrt{2} \qquad \text{and} \qquad \|u_2\|_2 = \sqrt{\frac{3}{2}} = \frac{3}{\sqrt{6}}$$

and form the resulting normalized QR factorization as

$$Q = \begin{bmatrix} \frac{u_1}{\|u_1\|_2} & \frac{u_2}{\|u_2\|_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

and

$$R = \begin{bmatrix} \|u_1\|_2 & \alpha_{12}\|u_1\|_2 & \alpha_{13}\|u_1\|_2 \\ 0 & \|u_2\|_2 & \alpha_{23}\|u_2\|_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} \end{bmatrix}.$$

(b) To compute the orthogonal complement of this space, we could merely take the cross product of u_1 and u_2 and then take the span of the resulting vector. Another possibility is to use Problem 2 part (a) and note that $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$. Alternatively, it's enough to find all $x \in \mathbb{R}^3$ that are orthogonal to the orthogonal basis for $\mathrm{Col}(A)$ that we just constructed, i.e. $x \perp u_1$ and $x \perp u_2$. Hence, we find x_1, x_2, x_3 such that

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 + 2x_3 &= 0. \end{aligned}$$

Adding the two equations gives

$$x_3 = -x_1$$

and combining this with the first equation

$$x_2 = -x_1$$

gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\mathrm{Col}(A)^\perp = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$