Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH 500: Linear Vector Spaces

Assignment #2 - PageRank & Vector Spaces
Due Thursday, September 16, 2021

For problems which require computational simulation, please print and submit both your code and results (e.g., pictures).

1. (15 points) Consider a graph with vertices

$$V = \{1, 2, 3, 4, 5, 6\}$$

and edges

$$E = \{1 \to 2, 1 \to 3, 1 \to 4, 2 \to 1, 2 \to 3, 3 \to 4, 4 \to 1, 4 \to 3, 4 \to 5, 4 \to 6, 5 \to 6, 6 \to 5\}.$$

(a) Draw the directed graph for this vertex set V and edge set E.

(b) Determine the adjacency matrix of the graph.

(c) The quantity $N_j = \sum_{i=1}^{n} a_{i,j}$ is called the degree of the vertex j. Compute the degree of each vertex of the graph.

(d) From this, compute the matrix W whose entries are given by

$$w_{i,j} = \begin{cases} 1/N_j & \text{if } i \in F_j \\ 0 & \text{otherwise.} \end{cases}$$

Then, use MATLAB to solve the equation $W\pi = \pi$ where $\pi_i \geq 0$ and $\sum_{i=1}^{6} \pi_i = 1$. Next, compute W^2, W^3, W^{10}, W^{30}. What do you observe?

(e) Now let $\alpha = 0.85$ and let $G = \alpha W + (1-\alpha)S$, where S is the matrix with all of its entries equal to $1/6$. Solve the equation $G\pi = \pi$, where $\pi_i \geq 0$ and $\sum_{i=1}^{6} \pi_i = 1$. Compute G^2, G^3, G^{10}, G^{30}. What do you observe?

(f) Delete the vertices 5 and 6, as well as, any related edges and compute the resulting steady-state vector π for the associated Google matrix G.

2. (10 points) Assume that $P, Q \in \mathbb{R}^{p\times p}$ are stochastic matrices.

(a) Let $0 \leq \alpha \leq 1$ be given, and prove that $R = \alpha P + (1-\alpha)Q$ is a stochastic matrix.

(b) Prove that P^n is a stochastic matrix for every $n \in \mathbb{N}$.
Hint: Prove that the product of stochastic matrices is stochastic and then use induction.

3. (10 points) Let V be a given set and $f : \mathbb{R} \to V$ be a one-to-one and onto function. For every $u, v \in V$ and $\alpha \in \mathbb{R}$, define the sum and scalar product operations on V by

$$u \oplus v = f\left(f^{-1}(u) + f^{-1}(v)\right) \qquad \text{and} \qquad \alpha \odot v = f\left(\alpha f^{-1}(v)\right).$$

Prove that V (defined over $\mathbb{R}$) is a vector space.

4. (15 points) Let $\mathbb{C}^p(\mathbb{R})$ be the vector space of complex-valued p-tuples defined over the field of real numbers, and $\mathbb{C}^p(\mathbb{C})$ be the vector space of complex-valued p-tuples defined over the field of complex numbers. Further, let

$$T_n = \left\{ \sum_{k=1}^{n} a_k \sin(k\pi x) : a_k \in \mathbb{R} \text{ for every } k = 1, ..., n \right\}$$

be the vector space of sinusoidal trigonometric polynomials of degree at most n and

$$C(0,1) = \{f : (0,1) \to \mathbb{R} \text{ such that } f \text{ is continuous}\}$$

be the vector space of continuous functions on $(0,1)$, both of which are defined over $\mathbb{R}$. For each of the following pairs of sets A and B, determine whether or not A is a subspace of B and justify your answer by providing either a proof or a counterexample.

(a) $A = \mathbb{R}^p$ and $B = \mathbb{C}^p(\mathbb{R})$.

(b) $A = \mathbb{R}^p$ and $B = \mathbb{C}^p(\mathbb{C})$

(c) $A = T_n$ for a given $n \in \mathbb{N}$ and $B = C(0,1)$.

(d) $A = \bigcup_{n \in \mathbb{N}} T_n$ and $B = C(0,1)$. **(Extra Credit)**