Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH 500: Linear Vector Spaces

Assignment #5: Linear Operators
Due Thursday, November 4, 2021

1. (10 points) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces with $\dim(\mathcal{V}) < \infty$, and let $T : \mathcal{V} \to \mathcal{W}$ be a linear operator. Assume that $\{w_1, ..., w_k\}$ is a basis for $\mathrm{R}(T)$ and let $B_1 = \{v_1, .., v_k\}$ where $T(v_j) = w_j$ for all $j = 1, ..., k$. Prove that if $B_2 = \{u_1, ..., u_n\}$ is a basis for $\mathrm{Ker}(T)$, then $B := B_1 \cup B_2$ is basis for $\mathcal{V}$.

2. (8 points) Let $p, q \in \mathbb{N}$ and $A \in \mathbb{C}^{p\times q}$ be given. Prove that

$$\mathrm{rank}(A^H A) = \mathrm{rank}(A).$$

Hint: First show that $\mathrm{Nul}(A^H A) = \mathrm{Nul}(A)$.

3. (10 points) Let $\mathcal{V}$ be a Hilbert space and assume that $T : \mathcal{V} \to \mathcal{V}$ is a bounded linear operator. Show that

$$\|T^*\| = \|T\|.$$

4. (12 points) Construct a normalized QR Factorization of A and use it to solve the least squares problem $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

5. (10 points) Let $p, q \in \mathbb{N}$ and recall for $A \in \mathbb{R}^{p\times q}$, the norm $\|A\|_1$ is defined by

$$\|A\|_1 = \max_{x\in\mathbb{R}^q\setminus\{0\}} \frac{\|Ax\|_1}{\|x\|_1}.$$

Show that $\|A\|_1 = \max\limits_{1\le j\le q} \sum\limits_{i=1}^{p} |a_{ij}|$.

Extra Credit. (10 points) Let $\mathcal{V}$ and $\mathcal{W}$ be normed spaces and assume $T : \mathcal{V} \to \mathcal{W}$ is a linear operator. We say that T is **continuous** on $\mathcal{V}$ if for every $v_0 \in \mathcal{V}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|v - v_0\|_{\mathcal{V}} < \delta \quad \text{implies} \quad \|T(v) - T(v_0)\|_{\mathcal{W}} < \epsilon.$$

Prove that T is continuous if and only if T is bounded.