Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH 500: Linear Vector Spaces

Assignment #1 - Review of Linear Algebra
Due Thursday, September 2, 2021

1. (10 points) Recall that for $A \in \mathbb{R}^{p\times p}$, the **trace** of A is defined by

$$\operatorname{tr}(A) = \sum_{k=1}^{p} A_{kk}.$$

Prove that if $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{q\times p}$, then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

2. (10 points) A square matrix A is called **skew-symmetric** if $A^T = -A$.

(a) Show that if A is any square matrix, then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

(b) Let A be a square matrix satisfying $A = A_1 + A_2$ where A_1 is symmetric and A_2 is skew-symmetric. Find representations for both A_1 and A_2 in terms of A and A^T.

3. (10 points) Let $A, B \in \mathbb{C}^{p\times p}$ be Hermitian matrices. Prove that AB is Hermitian if and only if A and B commute (i.e. $AB = BA$).

4. (10 points) Let $A, B,$ and $A + B$ be nonsingular matrices. Show that $A^{-1} + B^{-1}$ is nonsingular.
Hint: Compute a formula for $(A^{-1} + B^{-1})^{-1}$ first.

5. (10 points) Let $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{q\times r}$ be given. Prove

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A).$$

Hint: Show $\operatorname{Col}(AB)$ is a subspace of $\operatorname{Col}(A)$.

6. (10 points) Assume $A \in \mathbb{C}^{p\times q}$ satisfies $\operatorname{rank}(A) = q$. Show that $A^H A$ is nonsingular.