Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH 500: Linear Vector Spaces

Assignment #4: Inner Products & Orthogonality
Due Thursday, October 14, 2021

1. (10 points) Let $p, q \in \mathbb{N}$ be given.

(a) Let A be a real, symmetric $p \times p$ matrix satisfying $x^T Ax > 0$ for every $x \in \mathbb{R}^p \setminus \{0\}$. Prove that the function
$$\langle x, y \rangle_A := x^T Ay$$
defined for any $x, y \in \mathbb{R}^p$ is an inner product on $\mathbb{R}^p$.

(b) Prove that the function
$$\|A\|_F := \sqrt{\mathrm{tr}(A^T A)}$$
defined for any $A \in \mathbb{R}^{p\times q}$ is a norm on $\mathbb{R}^{p\times q}$.
Hint: Use a formula for $\|A\|_F$ in terms of the columns or entries of A.

2. (8 points) Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathcal{V}$ with
$$\|v_n\| \to \|v\| \quad \text{and} \quad \langle v_n, v\rangle \to \|v\|^2.$$
Show that v_n converges to v in $\mathcal{V}$.

3. (12 points) Let $\mathcal{V}$ be a vector space over $\mathbb{K}$, endowed within an inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and induced norm $\| \cdot \|_{\mathcal{V}}$. Show that if $S = \{v_1, ..., v_q\} \subset \mathcal{V}$ with $q \in \mathbb{N}$ is orthonormal, then for every $v \in \mathcal{V}$ we have
$$\sum_{k=1}^{q} |\alpha_k|^2 \le \|v\|_{\mathcal{V}}^2 \quad \text{where} \quad \alpha_k = \langle v_k, v\rangle_{\mathcal{V}}.$$
Hint: Consider $w = \sum_{k=1}^{q} \alpha_k v_k$ and $\langle v, w\rangle_{\mathcal{V}}$.

4. (10 points) Consider the vector space $\mathcal{V} = L^2(-1, 1)$ defined over $\mathbb{R}$ and endowed with the inner product
$$\langle f, g\rangle_2 := \int_{-1}^{1} f(x)g(x)\, dx$$
for every $f, g \in \mathcal{V}$. Let $S = \{1, x, x^2\} \subset \mathcal{V}$.

(a) Show that S is not an orthogonal set with respect to this inner product.

(b) Construct an orthonormal basis for $\text{span}(S)$.

<u>*Remark:*</u> The basis you are constructing is known as the *orthonormal Legendre Polynomials*, and if this process is continued for all $n \in \mathbb{N}$, they form an orthonormal basis for $L^2(-1, 1)$.

5. (10 points) Let A be a real $p \times q$ matrix satisfying $A = QR$, where Q is a real $p \times k$ matrix with orthonormal columns and R is a real $k \times q$ matrix with $\text{rank}(R) = k$.

(a) Show that R has a right inverse; that is, show that there exists a real $q \times k$ matrix X such that $RX = I_k$.

(b) Show that the columns of Q form an orthonormal basis for $\text{Col}(A)$.