MATH 500: Linear Vector Spaces **NAME:**
Final Exam - Fall 2021

The exam will be available from Tuesday, December 7 at 9am to Wednesday, December 15 at 4pm. You may use our notes, but NOT homework, homework solutions, or any other source. Additionally, **you may NOT collaborate with others**. Show all work for each question. Be precise in your proofs and cite any theorems from our notes whenever you use them. Let $p, q \in \mathbb{N}$ be given.

1. (25 points) Let $P \in \mathbb{R}^{p\times p}$ be given, and define $T : \mathbb{R}^{p\times q} \to \mathbb{R}^{p\times q}$ by
$$T(A) = PA$$
for every $A \in \mathbb{R}^{p\times q}$.
   (a) Prove that T is linear.
   (b) Recall the Frobenius inner product on $\mathbb{R}^{p\times q}$ defined by
$$\langle A, B\rangle_F = \mathrm{tr}(A^T B).$$
   Compute T^* under $\langle\cdot,\cdot\rangle_F$ and justify your answer.
   (c) Determine conditions on P which guarantee that T is normal (with respect to $\langle\cdot,\cdot\rangle_F$) and justify your answer.
   (d) Assuming P is unitary, compute the operator norm $\|T\|_F$. Justify your answer.

2. (20 points) Let $\mathcal{V}$ be a Hilbert space and $T : \mathcal{V} \to \mathcal{V}$ be a bounded, normal linear operator with $\mathrm{R}(T)$ closed.
   (a) Explain why T^* exists.
   (b) Prove that $\mathrm{Ker}(T^*) = \mathrm{Ker}(T)$.
   (c) Use part (b) to prove that $\mathrm{R}(T^*) = \mathrm{R}(T)$.

3. (15 points) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.
   (a) Determine whether A is (i) unitary, (ii) Hermitian, (iii) normal. Justify your answers.
   (b) Is A unitarily diagonalizable? Justify your answer.
   (c) Is A orthogonally diagonalizable? Justify your answer.

4. (25 points) Let $A \in \mathbb{R}^{2\times 100}$ be defined by
$$\begin{cases} A_{1k} = k, & \text{for } k = 1, ..., 100 \\ A_{2k} = 2k, & \text{for } k = 1, ..., 100. \end{cases}$$

(a) How many real scalars are needed to store A?

(b) Compute an SVD of A. It may be helpful to use the outer product notation for this decomposition.

(c) How many real scalars are needed to store the SVD of A?

5. (15 points) Let $A \in \mathbb{R}^{q \times q}$ be given with eigenvalue $\lambda \in \mathbb{C}$. Show that

$$|\lambda| \leq \|A\|_p$$

for every $p \in [1, \infty]$.