Department of Applied Mathematics and Statistics
COLORADO SCHOOL OF MINES
MATH 500: Linear Vector Spaces

Assignment #7: Diagonalizable & Unitary Operators
Due Tuesday, November 23, 2021

1. (12 points) Let $T : \mathcal{V} \to \mathcal{V}$ be linear and $k \in \mathbb{N}$. Show that if $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T and $v_1, \ldots, v_k$ are any associated eigenvectors, then the set $S = \{v_1, \ldots, v_k\}$ is linearly independent.

2. (12 points) Recall that $A \in \mathbb{C}^{p\times p}$ is diagonalizable if a basis for $\mathbb{C}^p$ can be formed from a collection of the eigenvectors of A. Prove that $A \in \mathbb{C}^{p\times p}$ is diagonalizable if and only if there exists a nonsingular $P \in \mathbb{C}^{p\times p}$ and diagonal $\Lambda \in \mathbb{C}^{p\times p}$ such that

$$A = P\Lambda P^{-1}.$$

3. (8 points) Let $P \in \mathbb{R}^{p\times p}$ be given. Prove that P is orthogonal if and only if its columns are orthonormal with respect to the standard inner product (and associated norm) on $\mathbb{R}^p$.

4. (8 points) Assume $A \in \mathbb{C}^{p\times p}$ is a unitary matrix, and let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ represent the standard inner product and its induced norm on $\mathbb{C}^p$, respectively. Additionally, let $\|\cdot\|_2$ represent the associated operator (matrix) norm. Prove the following results without invoking analogous theorems presented in class about unitary transformations:

(a) For all $x, y \in \mathbb{C}^p$
$$\langle Ax, Ay\rangle = \langle x, y\rangle$$

(b) For all $x \in \mathbb{C}^p$
$$\|Ax\| = \|x\|$$

(c) $\|A\|_2 = 1$

Extra Credit. (5 points) Let $L^2(\mathbb{R}; \mathbb{C})$ represent the set of all complex-valued, square-integrable functions defined on $\mathbb{R}$. The Fourier transform $\mathscr{F} : L^2(\mathbb{R}; \mathbb{C}) \to L^2(\mathbb{R}; \mathbb{C})$ is defined by

$$\mathscr{F}[u](\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} u(x)dx.$$

Furthermore, the inverse Fourier transform is

$$\mathscr{F}^{-1}[v](x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} v(\xi) d\xi.$$

Show that $\mathscr{F}$ is a unitary operator with respect to the inner product

$$\langle u, v \rangle = \int \overline{u(x)} v(x) \, dx.$$