Joint Rank and Positive Semidefinite Constrained Optimization for Projection Matrix

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Abstract—Sparse signals can be sensed with a reduced number of projections and then reconstructed if compressive sensing is employed. Traditionally, the projection matrix is chosen as a random matrix, but a projection sensing matrix that is optimally designed for a certain class of signals can further improve the reconstruction accuracy. This paper considers the problem of designing the projection matrix $\Phi$ for a compressive sensing system in which the dictionary $\Psi$ is assumed to be given. A novel algorithm based on joint rank and positive semidefinite constrained optimization for optimal projection matrix searching is proposed. Simulation results reveal that the signal recovery performance of sensing matrix obtained by proposed algorithm surpasses that of other standard sensing matrix designs.

Index Terms—Compressed sensing, MSE, Equiangular Tight Frame.

I. INTRODUCTION

Compressive sensing (CS) states that signals which have a sparse representation in an certain dictionary can be exactly recovered with a much smaller number of measurements than traditional considered [1] - [3]. Over the last few years researchers have derived some new compressive sensing theory for a variety of structured sensing matrices [4]. In the same time, some progress has been made in making statements on the pursuit performance for optimized dictionaries and sensing matrices [5] - [6].

In one hand, we can represent the measuring process mathematically as

$z = \Phi y$ \hspace{1cm} (1)

where $z \in \mathbb{R}^M$ is the measurement vector, $y \in \mathbb{R}^N$ is a target signal vector and $\Phi \in \mathbb{R}^{M \times N}$, called projection matrix or sensing matrix, represents a dimensionality reduction, i.e., it maps $\mathbb{R}^N$ into $\mathbb{R}^M$, where $M$ is typically much smaller than $N$.

In the other hand, a given signal $y \in \mathbb{R}^N$ can often be expressed sparsely represented by a certain dictionary $\Psi \in \mathbb{R}^{N \times L}$.

$y = \Psi s$ \hspace{1cm} (2)

where $s$ is the representation vector. A signal is said to be sparse if there are only a few non-zeros in its the representation vector, or compressible if most of the coefficients of $s$ are very close to zero. Thus these tiny coefficients can be discarded without much loss of information of the signal.

By combing(1) and (2), $z$ can be rewritten as

$z = \Phi \Psi s \triangleq As$ \hspace{1cm} (3)

where the matrix $A \in \mathbb{R}^{M \times L}$ is equivalent dictionary of the CS system. As here $M << L$, $A$ is over-complete. Thus for given measurement vector $z$ and equivalent dictionary $A$, the coefficients vector $s$ tends to be not unique. In next section, we will present a theorem to attack this problem, that is the essence of CS.

The choice of the projection matrix $\Phi$ is governed by the principle of the mutual coherence, denoted as $\mu(A)$, represents the worst-case coherence between any two columns (atoms) of $A$ and is one of the most fundamental quantities associated with CS theory. "good" projection matrix $\Phi$ can dramatically lower the mutual coherence of $A$. In our work, we only consider the case in which the dictionary is given. Random projection matrix was shown to be a good choice, since random vectors are incoherent with any fixed basis with high probability. Even though, with appropriate optimization of the projection matrix based on above principle can drastically improve the performance of CS system, such as in the reconstruct error rate measure [6]-[8], [12].

This paper is dedicated in optimising the projection matrix in a more rigorous mathematical way. We investigate the problem of projection matrix design for sensing signals which are sparse in a given dictionary and a novel algorithm based on joint rank and positive semidefinite constrained optimization is proposed for optimal projection matrix searching. Experiments are given which show that the sensing matrix obtained using our proposed algorithm outperforms others in signal reconstruction accuracy.

The paper is arranged as follows. In Section II, we define the criteria that is used for measuring the coherence of a matrix. In Section III, some related work on the sensing matrix optimization problem is provided, and an iterative algorithm based on joint rank and positive semidefinite constrained optimization for optimal projection matrix searching is derived to find an optimal sensing matrix that minimizes the coherence of the equivalent dictionary. Simulations are also presented in Section IV to show the effectiveness of our proposed method.
in improving signal reconstruction accuracy. Some concluding remarks are given in Section V to end this paper.

II. PRELIMINARIES

The mutual coherence of the matrix $A$ is defined as

$$\mu(A) \triangleq \max_{1 \leq i \neq j \leq L} \frac{|A_i^T A_j|}{\|A_i\| \|A_j\|}$$

which measures the maximum linear dependency possibly achieved by any two columns of matrix $A$. Coherence is a blunt instrument since it only reflects the most extreme correlation in the matrix. Nevertheless, it is easy to calculate and it captures well the behavior of uniform matrices. Thus it can be used as a criteria of the CS system.

The $(i,j)$th element of the Gram matrix of $A$ is defined as

$$g_{ij} \triangleq A_i^T A_j$$

and

$$S_c \triangleq \text{diag}(g_{11}^{-1/2} \cdots g_{kk}^{-1/2} \cdots g_{LL}^{-1/2})$$

Thus the Gram matrix of $\tilde{A} \triangleq A S_c$, denoted as $\tilde{G} = \{\tilde{g}_{ij}\}$, is normalized such that $\tilde{g}_{kk} = 1, \forall k$. Obviously,

$$\mu(A) = \max_{i \neq j} |\tilde{g}_{ij}|$$

As shown in Theorem 1, the mutual coherence provides a lower bound for the perfect recovery of sparse signals.

**Theorem 1:** Consider an overcomplete dictionary $A$ with mutual coherence $\mu(A)$ and a signal $y = As$. If condition (5) is true:

$$\|s\|_0 < \frac{1}{2} \left[ 1 + \frac{1}{\mu(A)} \right]$$

then the following holds:

1) $s$ is the sparsest decomposition of $y$ in $A$, i.e. it is the solution of the optimization problem

$$\min_s \|s\|_0 \quad s.t. \quad z = As$$

which can be attacked effectively by greedy algorithms such as the matching pursuit (MP) and the orthogonal MP (OMP), which iteratively select locally optimal basis vectors.

2) $s$ is recoverable using BP technique, i.e. it is the solution of the optimization problem

$$\min_s \|s\|_1 \quad s.t. \quad z = As$$

Theorem 1 shows that having a smaller coherence of the equivalent dictionary $A$ is desirable, as it increases the set of recoverable signals. As such, an optimal projection matrix $\Phi$ is one that minimizes the mutual coherence of the equivalent $A = \Phi \Psi$, i.e. the largest off-diagonal element of the equivalent Gram matrix $G = A^T A$. Thus the optimization problem can be stated in terms of minimizing the mutual coherence, or, in general, the largest off-diagonal elements of $G$.

III. SENSING MATRIX OPTIMIZATION

In this section, we first provide some related work on the sensing matrix optimization problem, and then an iterative algorithm based on joint rank and positive semidefinite constrained optimization for optimal projection matrix searching is derived to find an optimal sensing matrix that minimizes the coherence of the equivalent dictionary.

A. Related work

This subsection contains a brief survey of the important results in optimization of the projection matrix.

Elad proposed the first work of the optimal design of sensing matrix $\Phi$ in [6]. It is due to the fact that (5) is just a worst-case bound and can not reflect the average signal recovery performance that, instead of $\mu(A)$, an averaged mutual coherence, denoted as $\mu_t(A)$, was dealt with in [6].

$$\mu_t(A) \triangleq \frac{\sum_{(i,j) \in S_t} |\tilde{g}_{ij}|}{N_t}$$

where $S_t = \{ (i,j) : |\tilde{g}_{ij}| \geq t \}$ with $0 \leq t < 1$ a given number and $N_t$ is the number of elements in the index set $S_t$. The reason for minimizing the $t$-averaged value instead of the single largest off-diagonal value is that the latter is a pessimistic bound: even under more relaxed conditions than condition (5), in practice all signals can still be adequately recovered, at the expense of a small fraction of unrecoverable signals. For this reason, it is argued that the $t$-averaged mutual coherence is a better measure for the average behaviour of the equivalent dictionary $A$.

However, if we have a close insight to Elad’s algorithm, we could find there exist some optimization drawbacks.

**Objective:** To optimize $\Phi$

**Input:** Parameters to be set:

- $\Psi \in \mathbb{R}^{N \times L}$ : the dictionary
- $\Phi \in \mathbb{R}^{M \times N}$ : the projection
- $iter$ : number of iteration

**Initialization:** With $\Psi$ given, set $\Phi$ as a random matrix $\Phi_0$.

**Loop:** Set $k = 1$ and repeat $iter$ times.

- **Step I:** Compute the equivalent dictionary $A = \Phi_k \Psi$, normalized its columns, and compute its Gram matrix $G^k = A^T A$. 

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Step II: Apply shrinkage technique to the off-diagonal elements of $G^k$ to obtain $\hat{G}^k$
Step III: Find the best rank $m$ approximation of $\hat{G}^k$ using singular value decomposition
Step IV: Extract square root $A_k$, where $\hat{G}^k = A_k^T A_k$
Step V: Choose $\Phi_k = A_k \Psi_k^T$, i.e., minimizing $||A_k - \Phi_k \Psi_k||_F$
Step VI: End while

From Step III to Step IV, the shrinkage operation on $G^k$ could also significantly change the rank of the Gram matrix $G^k$. We could see this optimization shrinkage operation just drop the rank constraint of the Gram matrix of $G^k$. Even though by this operation the off-diagonal elements of the Gram matrix $G^k$ can be greatly lower down, we can regard it just as searching the optimal target not in a feasible signal space. Moreover, the solution $\hat{G}^k$ obtained by Step III may be not positive semidefinite, which is contradictory the fact $\hat{G}^k = A_k^T A_k$ which means $\hat{G}^k$ is positive semidefinite. What's more, after the Step IV, we can not guarantee the diagonal elements of $\hat{G}^k$ to be one. Since in the Step II our initial equivalent dictionary has been normalized, that leads the diagonal elements of our initial Gram matrix to be one, however, the optimized equivalent dictionary doesn’t have the normalized columns.

In [7] Duarte-Carvajalino and Sapiro produced a approach to learning the projection matrix for a given dictionary as

$$\min_{\Phi} ||\Pi_d^2 - \Pi_d^2 U_d^T \Phi^T U_d \Pi_d^2||_F$$

(7)

where $|| \cdot ||_F$ denotes the Frobenius norm and $\Psi = U_d \begin{bmatrix} \Pi_d & 0 \end{bmatrix} V_d^T$ is a singular value decomposition (SVD) of the dictionary $\Psi$. The numerical procedure, though not globally optimal, was reported to be faster. However, this numerical procedure used in (7) lost the original intention of making the Gram matrix as close to the identity matrix as possible due to several approximation procedures involved in. Also, in his optimization problem the rank constraint and the normalization constraint, which in nature is the normalization of the columns of the equivalent dictionary, have been dropped way. This optimization may lead the searching not within the feasible solution space. Thus the solution obtained by this algorithm may differ from the best solution in a great distance. In order to further improve the performance of CS systems, we can change the optimization target in the above optimization problem as a more "good" target in the mutual coherence measure, which we will give details in the next section.

B. Problem formation and the proposed method

In this section proposed optimization problem is formulated, which is divided into two parts. The first part is related with our optimization target and in the second part after given the optimization target we will formulate the joint rank and normalization constrained optimization problem for optimizing the projection matrix.

1) Optimization target: an Equiangular Tight Frame (ETF) of size $N \times M$ with $N \leq M$ is a matrix with normalized columns such that its Gram matrix $G = A^T A$ satisfies

$$\forall k \neq j, ||G_{ij}|| = \sqrt{\frac{L - M}{M(L - 1)}}$$

(8)

It can be shown [9] that for a matrix $A \in \mathbb{R}^{M \times N}$, $\mu(A)$ is bounded with

$$\mu \leq \mu(A) \leq 1$$

(9)

with the low bound given by

$$\mu = \sqrt{\frac{L - M}{M(L - 1)}}$$

(10)

This means the ETF can obtain the least mutual coherence. Thus ETF has a very nice mutual coherence behavior. However, it is difficult to make the equivalent dictionary $A = \Phi \Psi$ an ETF with $\Phi$ only as the degrees of freedom, compared with those in a totally free $A$, are much reduced. Therefore, we extend the searching space to a more convex set $\Lambda_e$

$$\Lambda_e \triangleq \{ G_{ea} \in \mathbb{R}^{L \times \ell} : G_{ea} = G_{ea}', G_{ea}(k,k) = 1, \forall k \}$$

(11)

in which $\ell > 0$ is a constant to control the searching space. When $\epsilon = \mu$, the ideal ETF Grams of dimension $L$ are confined in $\Lambda_e$.

The optimization target $G_{ea} \in \Lambda_e$ here is dependent on the given Gram matrix $G$. In fact, $G_{ea}$ is the projection of $G$ on space $\Lambda_e$, i.e.,

$$G_{ea} = \arg \min_{G_{ea} \in \Lambda_e} ||\hat{G} - G||_F$$

(12)

The solution of this problem is given in [11] as follows:

$$G_{ea}(i,j) = \begin{cases} \frac{G(i,j)}{|G(i,j)|} \leq \mu, & \text{if } i = j \\ 1, \text{otherwise} \end{cases}$$

(13)

which means by using the above shrinkage method we can effectively and efficiently find the optimization target $G_{ea}$ corresponding to the Gram matrix $G$.

2) Optimization problem formulation: with the obtained the optimization target $G_{ea}$, one may ask how can we effectively optimize the distance between $G$ and $G_{ea}$ in a more reasonable way?

As is discussed above, a more reasonable formulation of the optimization problem should consider both the rank constraint, normalization constraint and the positive semidefinite constraint simultaneously. Thus we can formulate the optimal
sensing matrix design problems as below (with the obtained the optimization target $G_{ea}$):

$$\min \quad ||G - G_{ea}||_F$$

$$s.t. \quad G(i,i) = 1, \quad i = 1, \ldots, L,$$

$$G \in S^L_+,$$

$$\text{rank}(G) \leq M.$$  \hspace{1cm} (14)

where $G = \Psi^T \Phi^T \Phi \Psi$, $\Psi$ is given, $M$ is the number of the rows of projection matrix $\Phi$, $L$ is the number of the columns of dictionary $\Psi$, and $S^L_+$ is the cone of positive semidefinite matrices in the space of $L \times L$ symmetric matrices $S^L$.

If the positive semidefinite constraint and the normalization constraint are dropped, one way just done by Elad [6] and Duarte [7], the above problem degrades into the follows:

$$\min \quad ||G - G_{ea}||_F$$

$$s.t. \quad \text{rank}(G) \leq M$$  \hspace{1cm} (15)

which admits an analytical solution for a given $G_{ea}$ [13]:

$$G = \sum_{i=1}^{M} \sigma_i(G_{ea}) u_i v_i^T$$

where $G_{ea}$ has the following SVD decomposition:

$$G_{ea} = U[diag(\sigma(G_{ea}))] 0|V^T$$

since $G_{ea} \in S^L_+$, hence $U = V$.

Although the degraded optimization problem has an analytical solution, as is stated previously, if we consider the property of the Gram matrix $G$, the obtained optimization solution may contradict the nature of such a Gram matrix. Thus in order to have a more reasonable solution the joint rank, normalization and positive semidefinite constraint optimization problem (14) should be attacked.

In order to solve problem (14), we first present its equivalent form:

$$\min \quad ||G - G_{ea}||_F$$

$$s.t. \quad G(i,i) = 1, \quad i = 1, \ldots, L,$$

$$G \in S^L_+,$$

$$\sigma_i(G) = 0, i = M + 1, \ldots, L.$$  \hspace{1cm} (16)

We first relax the rank constrained optimization problem into a regularized one by adding a penalized term to the objective function:

$$\min \quad \frac{1}{2} ||G - G_{ea}||_F + \sum_{i=M+1}^{L} \sigma_i(G)$$

$$s.t. \quad G(i,i) = 1, \quad i = 1, \ldots, L,$$

$$G \in S^L_+.$$  \hspace{1cm} (17)

where $c > 0$ is the regularized coefficient. Equivalently

$$\begin{align*}
\min \quad & f(G) = \frac{1}{2} ||G - G_{ea}||_F + c(I,G) - cs_M(G) \\
\text{s.t.} \quad & G(i,i) = 1, \quad i = 1, \ldots, L, \\
\quad & G \in S^L_+.
\end{align*}$$  \hspace{1cm} (18)

where $s_M(G) \triangleq \Sigma_{i=1}^{M} \sigma_i(G)$, $\langle A,B \rangle \triangleq \text{trace}(A^TB)$ and $I$ is the identity matrix of size $L \times L$. Referring to the book Convex Optimization [14] (see pp. 96), the sum of the $M$ largest singular values of a matrix, $s_M(G)$, is convex. Therefore, the above problem is non-convex optimization problem (18) as there is a concave component $-s_M(G)$ embedded in the objective function.

In the following, some techniques are employed to convert the above non-convex optimization problem (18) to a solvable convex one.

If

$$G = U[diag(\sigma(G))] 0|V^T$$

let

$$J_M(G) = \Sigma_{i=1}^{M} u_i v_i^T \in \partial(s_M(G))$$

where $\partial$ is denoted as the partial operator and $u_i$, $v_i$ is the column of $U$ and $V$, respectively. In fact

$$s_M(G) = \langle J_M(G), G \rangle$$  \hspace{1cm} (19)

As $s_M(G)$ is a convex function, for any given $G^k \in \mathbb{R}^{L \times L}$, we have

$$s_M(G) \geq s_M(G^k) = s_M(G^k) + \langle J_M(G^k), G - G^k \rangle$$  \hspace{1cm} (20)

Thus, at $G^k$, $f(G)$ is majorized by

$$f(G) \leq f^k(G) = \frac{1}{2} ||G - G_{ea}||_F + c(I,G) - cs_M(G)$$  \hspace{1cm} (21)

Relax the non-convex objection function in (18) by replacing the function $s_M(G)$ by $s_M^k(G)$:

$$\begin{align*}
G^{k+1} = \arg \min_G f^k(G) \\
\text{s.t.} \quad G(i,i) = 1, \quad i = 1, \ldots, L, \\
\quad & G \in S^L_+.
\end{align*}$$  \hspace{1cm} (22)

where $s_M^k(G) = s_M(G^k) + \langle J_M(G^k), G - G^k \rangle$. By this way, the original non-convex optimization problem (18) can be attacked by solving a serious solvable and convex regularized sub-optimization problem.

Initialize $G$ with appropriate $G^0$, then by solving problem (22) with $k = 0$, we can obtain

$$\min f^1(G) \leq \min f^0(G)$$

Then we will have the following:

$$\min f(G) \leq \ldots \leq \min f^{k+1}(G) \leq \min f^k(G) \leq \ldots \leq \min f^0(G)$$

Combine (20), (21), (22) and (III-B2), with some manipulation, the above inequality (III-B2) essentially implies the following:

$$\langle J_M(G), G \rangle \geq \ldots \geq \langle J_M(G^k), G \rangle \geq \ldots \geq \langle J_M(G^0), G \rangle$$  \hspace{1cm} (23)

When $k$ is large enough, $\langle J_M(G^k), G \rangle$ can be convergent to $\langle J_M(G), G \rangle$, which is just a generalization of traditional gradient algorithm.

In order to guarantee that the rank of the final $G^*$ is less or equal to $M$, one should increase $c$ in every iteration. That is,
to have a sequence of \( \{c_k\} \) with \( c_{k+1} \geq c_k \). There exists known algorithm named Pencorr[15] just for attacking the constrained sub-optimization (22).

We summarize our algorithm as below:

**Objective:** To optimize \( \Phi \)

**Input:** Parameters to be set:
- \( \Psi \in \mathbb{R}^{N \times L} \): the dictionary
- \( \Phi \in \mathbb{R}^{M \times N} \): the projection
- \( \text{iter} \): number of iteration

**Initialization:** With \( \Psi \) given, set \( \Phi \) as a random matrix.

**Loop:** Set \( k = 1 \) and repeat \( \text{iter} \) times.
- **Step I:** While \( 1 \leq k \leq \text{iter} \), compute \( G = (\Phi \Psi)^T (\Phi \Psi) \), then normalize it.
- **Step II:** Find the optimization target by
  \[
  G_{ca} = \prod_{\lambda_{G} > 0} G
  \]
- **Step III:** With \( G_{ca} \) obtained above, attack the constrained optimization problem stated in (14), by solving a series sub-optimization problem in (22).
- **Step IV:** After getting the positive semidefinite \( G \), we can factorizing it as \( G = A^T A \). Choose \( \Phi = A \Psi^T \), i.e. minimizing \( ||A - \Phi \Psi||_F \)
- **Step V:** If \( ||G_{ca} - \Psi^T \Phi^T \Phi \Psi||_F < ||G_{ca} - \Psi^T \Phi^T \Phi \Psi||_F \), then \( \Phi = \tilde{\Phi} \) and go to **Step I** with \( k \rightarrow k + 1 \)
- **Step VI:** End while

**IV. Computer Simulation Results**

To illustrate the behavior of the proposed algorithm and compare it with other sensing matrix designs including Gaussian matrix, Elad’s algorithm \(^1\) [6] and DCS’s algorithm [7] we present some numerical experiments to evaluate the performance of the optimized projections via the signal recovery accuracy. We choose a dictionary \( \Psi \in \mathbb{R}^{N \times L} \) and synthesize 1000 test signals \( \{y_j\}_{j=1}^{1000} \) by randomly generating \( K \)-sparse \( L \times 1 \) vectors \( \{s_j\}_{j=1}^{1000} \), and computing \( y_j = \Psi s_j \). Then we apply random sensing projection and designed projections to get measurements with \( z = \Phi y_j \). OMP method is used to recover the sparse vectors \( \hat{s}_j \) from the measurements by approximating the solution of

\[
\hat{s}_j = \arg \min ||s||_0 \ \text{ s.t. } z_j = \Phi s
\]

Then we reconstruct the signal \( y_j = \Psi \hat{s}_j \) and test the recovery error of the relevant CS system via

\[
e_r = \frac{1}{1000} \sum_{k=1}^{1000} \frac{||y_k - \hat{y}_k||_2^2}{||y_k||_2^2}, \ \hat{y}_k = \Psi \hat{s}_j
\]

In the first experiment, the size of the CS system is \( M = 25, N = 80 \) and \( L = 80 \). The sparsity \( K \) varies in the range \([1, 12]\). The results are depicted in Fig. 1. As seen, our proposed algorithm can yield a better recovery accuracy than others for all sparsity levels.

The second experiment is similar to the first one, this time fixing sparsity \( K = 4 \), we vary \( M \) from 16 to 40. The results are shown in Fig. 2. As expected, the results improves as \( M \) increases for all projections. Once again it is evident that our proposed sensing matrix outperforms the other sensing matrices.

In the third experiment, the size of the CS system is \( M = 25, N = 80, L = 80 \), with random projection and optimized projections. We should point out that the settings of the three experiments are the same as [6].

\[\text{Fig. 1. Reconstruction error } e_r \text{ as a function of the signal sparsity } K \text{ for } M = 25, N = 80, L = 80, \text{ with random projection and optimized projections. Note: a vanishing graph implies a zero error rate.}\]

\[\text{Fig. 2. Reconstruction error } e_r \text{ as a function of the number of measurement } M, N = 80, L = 80, K = 4, \text{ with random projection and optimized projections. Note: a vanishing graph implies a zero error rate.}\]

\(^1\)This algorithm has two parameters \( \gamma \) and \( t \). In our simulations, we set \( t = 20\%; \gamma = 0.95 \), unless there is additional instruction.
V. CONCLUSIONS

This paper considers the problem of designing the projection matrix $\Phi$ for a compressive sensing system in which the dictionary $\Psi$ is assumed to be given. We first provide some related work on the sensing matrix optimization problem and analyze some drawbacks in their algorithms. These works don’t consider both the rank constraint, normalization constraint and the positive semidefinite constraint simultaneously, which lead to a not very reasonable formulation of the original sensing matrix optimization problem. To address this shortcoming in their optimization problem, a novel algorithm based on joint rank and positive semidefinite constrained optimization for optimal projection matrix searching is proposed. Simulation results reveal that the signal recovery performance of sensing matrix obtained by proposed algorithm surpasses that of other standard sensing matrix designs.

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