Fractal Antenna Engineering: The Theory and Design of Fractal Antenna Arrays

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1. Abstract

A fractal is a recursively generated object having a fractional dimension. Many objects, including antennas, can be designed using the recursive nature of a fractal. In this article, we provide a comprehensive overview of recent developments in the field of fractal antenna engineering, with particular emphasis placed on the theory and design of fractal arrays. We introduce some important properties of fractal arrays, including the frequency-independent multi-band characteristics, schemes for realizing low-sidelobe designs, systematic approaches to thinning, and the ability to develop rapid beam-forming algorithms by exploiting the recursive nature of fractals. These arrays have fractional dimensions that are found from the generating subarray used to recursively create the fractal array. Our research is in its infancy, but the results so far are intriguing, and may have future practical applications.

2. Introduction

The term fractal, which means broken or irregular fragments, was originally coined by Mandelbrot [1] to describe a family of complex shapes that possess an inherent self-similarity in their geometrical structure. Since the pioneering work of Mandelbrot and others, a wide variety of applications for fractals has been found in many branches of science and engineering. One such area is fractal electrodynamics [2-6], in which fractal geometry is combined with electromagnetic theory for the purpose of investigating a new class of radiation, propagation, and scattering problems. One of the most promising areas of fractal electrodynamics research is in its application to antenna theory and design.

Traditional approaches to the analysis and design of antenna systems have their foundation in Euclidean geometry. There has been a considerable amount of recent interest, however, in the possibility of developing new types of antennas that employ fractal rather than Euclidean geometric concepts in their design. We refer to this new and rapidly growing field of research as fractal antenna engineering. There are primarily two active areas of research in fractal antenna engineering, which include the study of fractal-shaped antenna elements, as well as the use of fractals in antenna arrays. The purpose of this article is to provide an overview of recent developments in the theory and design of fractal antenna arrays.

The first application of fractals to the field of antenna theory was reported by Kim and Jaggard [7]. They introduced a methodology for designing low-sidelobe arrays that is based on the theory of random fractals. The subject of time-harmonic and time-dependent radiation by bifractal dipole arrays was addressed in [8]. It was shown that, whereas the time-harmonic far-field response of a bifractal array of Hertzian dipoles is also a bifractal, its time-dependent far-field response is a unifractal. Lakhakia et al. [9] demonstrated that the diffracted field of a self-similar fractal screen also exhibits self-similarity. This finding was based on results obtained using a particular example of a fractal screen, constructed from a Sierpinski carpet. Diffraction from Sierpinski-carpet apertures has also been considered in [6], [10], and [11]. The related problems of diffraction by fractally serrated apertures and Cantor targets have been investigated in [12-17].
The fact that self-scaling arrays can produce fractal radiation patterns was first established in [18]. This was accomplished by studying the properties of a special type of nonuniform linear array, called a Weierstrass array, which has self-scaling element spacings and current distributions. It was later shown in [19] how a synthesis technique could be developed for Weierstrass arrays that would yield radiation patterns having a certain desired fractal dimension. This work was later extended to the case of concentric-ring arrays by Liang et al. [20]. Applications of fractal concepts to the design of multi-band Koch arrays, as well as to low-sidelobe Cantor arrays, are discussed in [21]. A more general fractal geometric interpretation of classical frequency-independent antenna theory has been offered in [22]. Also introduced in [22] is a design methodology for multi-band Weierstrass fractal arrays. Other types of fractal array configurations that have been considered include planar Sierpinski carpets [23-25] and concentric-ring Cantor arrays [26].

The theoretical foundation for the study of deterministic fractal arrays is developed in Section 3 of this article. In particular, a specialized pattern-multiplication theorem for fractal arrays is introduced. Various types of fractal array configuration are also considered in Section 3, including Cantor linear arrays and Sierpinski carpet planar arrays. Finally, a more general and systematic approach to the design of deterministic fractal arrays is outlined in Section 4. This generalized approach is then used to show that a wide variety of practical array designs may be recursively constructed using a concentric-ring circular subarray generator.

3. Deterministic fractal arrays

A rich class of fractal arrays exists that can be formed recursively through the repetitive application of a generating subarray. A generating subarray is a small array at scale one ($P = 1$) used to construct larger arrays at higher scales (i.e., $P > 1$). In many cases, the generating subarray has elements that are turned on and off in a certain pattern. A set formula for copying, scaling, and translation of the generating subarray is then followed in order to produce the fractal array. Hence, fractal arrays that are created in this manner will be composed of a sequence of self-similar subarrays. In other words, they may be conveniently thought of as arrays of arrays [6].

The array factor for a fractal array of this type may be expressed in the general form [23-25]

$$AF_P(\psi) = \prod_{p=1}^{P} GA(\delta^{P-1}\psi),$$

(1)

where $GA(\psi)$ represents the array factor associated with the generating subarray. The parameter $\delta$ is a scale or expansion factor that governs how large the array grows with each recursive application of the generating subarray. The expression for the fractal array factor given in Equation (1) is simply the product of scaled versions of a generating subarray factor. Therefore, we may regard Equation (1) as representing a formal statement of the pattern-multiplication theorem for fractal arrays. Applications of this specialized pattern-multiplication theorem to the analysis and design of linear as well as planar fractal arrays will be considered in the following sections.

3.1 Cantor linear arrays

A linear array of isotropic elements, uniformly spaced a distance $d$ apart along the $z$ axis, is shown in Figure 1. The array factor corresponding to this linear array may be expressed in the form [27, 28]

$$AF(\psi) = \begin{cases} I_0 + 2\sum_{n=1}^{N} I_n \cos[m\psi], & \text{for } 2N+1 \ \text{elements} \\ 2\sum_{n=1}^{N} I_n \cos[(n-1/2)\psi], & \text{for } 2N \ \text{elements} \end{cases}$$

(2)

where

$$\psi = kd[\cos\theta - \cos\theta_0]$$

(3)

and

$$k = 2\pi / \lambda.$$  

(4)

These arrays become fractal-like when appropriate elements are turned off or removed, such that

$$I_n = \begin{cases} 1, & \text{if element } n \text{ is turned on} \\ 0, & \text{if element } n \text{ is turned off}. \end{cases}$$

(5)

Hence, fractal arrays produced by following this procedure belong to a special category of thinned arrays.

One of the simplest schemes for constructing a fractal linear array follows the recipe for the Cantor set [29]. Cantor linear arrays were first proposed and studied in [21] for their potential use in the design of low-sidelobe arrays. Some other aspects of Cantor arrays have been investigated more recently in [23-25].

The basic triadic Cantor array may be created by starting with a three-element generating subarray, and then applying it repeatedly over $P$ scales of growth. The generating subarray in this case has three uniformly spaced elements, with the center element turned off or removed, i.e., 101. The triadic Cantor array is generated recursively by replacing 1 by 101 and 0 by 000 at each stage of the construction. For example, at the second stage of construction ($P = 2$), the array pattern would look like...
and at the third stage \((P = 3)\), we would have
\[101000101,\]

The array factor of the three-element generating subarray with the representation 101 is
\[G_A(\psi) = 2\cos(\psi),\]  
which may be derived from Equation (2) by setting \(N = 1, I_0 = 0,\) and \(I_1 = 1.\) Substituting Equation (6) into Equation (1) and choosing an expansion factor of three \((i.e., \delta = 3)\) results in an expression for the Cantor array factor given by
\[\hat{A}_P^P(\psi) = \prod_{p=1}^{P \hat{A}}(3^{P-1}\psi) = \prod_{p=1}^{P} \cos(3^{p-1}\psi),\]  

where the hat notation indicates that the quantities have been normalized. Figure 2 contains plots of Equation (7) for the first four stages in the growth of a Cantor array.

Suppose that the spacing between array elements is a quarter-wavelength \((i.e., d = \lambda/4)\), and that \(\theta_0 = 90^\circ\). Then, an expression for the directivity of a Cantor array of isotropic point sources may be derived from
\[D_P(u) = \frac{\hat{A}_P^P(\frac{\pi u}{2})}{\frac{1}{2} \int_{-1}^{1} \hat{A}_P^P(\frac{\pi u}{2}) du},\]  

where \(\psi = \frac{\pi}{2} u,\) with \(u = \cos \theta,\) and
Substituting Equation (9) into Equation (8) and using the fact that
\[ \frac{1}{2} \int_{-\pi}^{\pi} \left[ 1 + \cos \left( \frac{3^{p-1} \pi u}{2} \right) \right] du = 1 \] (10)
leads to the following convenient representation for the directivity:
\[ D_p(u) = \prod_{\rho=1}^{p} \left[ 1 + \cos \left( \frac{3^{\rho-1} \pi u}{2} \right) \right] = 2^p \prod_{\rho=1}^{p} \cos^2 \left( \frac{3^{\rho-1} \pi u}{2} \right) \] (11)
Finally, it is easily demonstrated from Equation (11) that the maximum value of directivity for the Cantor array is
\[ D_p = D_p(0) = 2^p, \quad P = 1, 2, \ldots \] (12)
or
\[ D_p(dB) = 3.01 P, \quad P = 1, 2, \ldots \] (13)
Locations of nulls in the radiation pattern are easy to compute from the product form of the array factor, Equation (1). For instance, at a given scale \( P \), the nulls in the radiation pattern of Equation (9) occur when
\[ \cos \left( \frac{3^{p-1} \pi u}{2} \right) = 0. \] (14)
Solving Equation (14) for \( u \) yields
\[ u^*_{p} = \pm (2k-1)(1/3)^{p-1}, \quad k = 1, 2, \ldots, \left(3^{p-1} + 1\right)/2. \] (15)
Hence, from this we may easily conclude that the radiation patterns produced by triadic Cantor arrays will have a total of \( 3^{p-1} + 1 \) nulls.

The generating subarray for the triadic Cantor array discussed above is actually a special case of a more general family of uniform Cantor arrays. The generating subarray factor for this general class of uniform Cantor arrays may be expressed in the form
\[ \hat{G}(\psi) = \frac{2}{\delta + 1} \frac{\sin \left( \frac{\delta + 1}{2} \psi \right)}{\sin [\psi]}, \] (16)
where
\[ \delta = 2n + 1 \quad \text{and} \quad n = 1, 2, \ldots \] (17)
Hence, by substituting Equation (16) into Equation (1), it follows that these uniform Cantor arrays have fractal array factor representations given by [21,23-25]
\[ \hat{A}(\psi) = \frac{2}{\delta + 1} \prod_{\rho=1}^{p} \frac{\sin \left( \frac{\delta + 1}{2} \psi \right)}{\sin \left[ \frac{\delta^\rho - 1}{2} \psi \right]}, \] (18)
For \( n = 1 \) (\( \beta = 3 \)), the generating subarray has a pattern 101 such that Equation (18) reduces to the result for the standard triadic Cantor array found in Equation (7). The generating subarray pattern for the next case, in which \( n = 2 \) (\( \beta = 5 \)), is 10101 and, likewise, when \( n = 3 \) (\( \beta = 7 \)), the array pattern is 1010101. The fractal dimension \( D \) of these uniform Cantor arrays can be calculated as [21]
\[ D = \frac{\log \left( \frac{\delta + 1}{2} \right)}{\log (\delta)}. \] (19)
This suggests that \( D = 0.6309 \) for \( n = 1 \), \( D = 0.6826 \) for \( n = 2 \), and \( D = 0.7124 \) for \( n = 3 \).

As before, if it is assumed that \( d = \lambda/4 \) and \( \theta_0 = 90^\circ \), then the directivity for these uniform Cantor arrays may be expressed as [23-25]
\[ D_p(u) = \left( \frac{2}{\delta + 1} \right)^p \prod_{\rho=1}^{p} \frac{\sin^2 \left( \frac{\pi (\delta + 1) \delta^\rho - 1}{2} u \right)}{\sin^2 \left( \frac{\pi \delta^\rho - 1}{2} u \right)}, \] (20)
where use has been made of the fact that
\[ \frac{1}{2} \int_{-\pi}^{\pi} \left[ \frac{\pi (\delta + 1) \delta^\rho - 1}{4} \right] \frac{\sin^2 \left( \frac{\pi \delta^\rho - 1}{2} u \right)}{\sin^2 \left( \frac{\pi \delta^\rho - 1}{2} u \right)} du = \left( \frac{\delta + 1}{2} \right)^p. \] (21)
The corresponding expression for maximum directivity is
\[ D_p = D_p(0) = \left( \frac{\delta + 1}{2} \right)^p, \quad P = 1, 2, \ldots \] (22)

Figure 3. A directivity plot for a uniform Cantor fractal array with \( P = 1 \) and \( \beta = 7 \). The spacing between elements of the array is \( d = \lambda/4 \). The maximum directivity for stage 1 is \( D_1 = 6.02 \text{ dB} \).
Figure 4. A directivity plot for a uniform Cantor fractal array with $P=2$ and $\delta=7$. The spacing between elements of the array is $d=\lambda/4$. The maximum directivity for stage 2 is $D_2=12.04$ dB.

Figure 5. A directivity plot for a uniform Cantor fractal array with $P=3$ and $\delta=7$. The spacing between elements of the array is $d=\lambda/4$. The maximum directivity for stage 3 is $D_3=18.06$ dB.

$AF(\psi) = \prod_{p=0}^{\infty} GA(\delta^{p+1}\psi)$.  

(24)

Next, by making use of Equation (24), we find that

$AF(\delta^{q}\psi) = \prod_{p=0}^{\infty} GA(\delta^{p+q+1}\psi) = \prod_{n=0}^{\infty} GA(\delta^{n+1}\psi) = AF(\psi)$,  

(25)

which is valid provided $q$ is a positive integer (i.e., $q=0,1,2,...$). The parameter $\delta^{q}$, introduced in Equation (25), may be interpreted as a frequency shift that obeys the relation

$f_{aq} = \delta^{q} f_0$, for $q=0,1,...$,  

(26)

where $f_0$ is the original design frequency. Finally, for a finite size array, Equation (1) may be used in order to show that

$AF_p(\delta^{q}\psi) = \left[ \prod_{n=0}^{P-1} GA(\delta^{n+1}\psi) \right] \frac{\prod_{n=0}^{P-1} GA(\delta^{n+1}\psi)}{\prod_{n=0}^{P-1} GA(\delta^{n+1}\psi)}$ for $q=1,2,...$,  

(27)

where the bracketed term in Equation (27) clearly represents the “end-effects” introduced by truncation.

One of the more intriguing attributes of fractal arrays is the possibility for developing algorithms, based on the compact product representation of Equation (1), which are capable of performing extremely rapid radiation pattern computations [23]. For example, if Equation (2) is used to calculate the array factor for an odd number of elements, then $N$ cosine functions must be evaluated, and $N$ additions performed, for each angle. On the other hand, however, using Equation (7) only requires $P$ cosine-function evaluations and $P-1$ multiplications. In the case of an 81 element triadic Cantor array, the fractal array factor is at least $N/P=40/4=10$ times faster to calculate than the conventional discrete Fourier transform.

$D_P(\psi) = 10P\log\left(\frac{\delta+1}{2}\right)$, where $P=1,2,...$.  

(23)

A plot of the directivity for a uniform Cantor array, with $d=\lambda/4$, $\theta_0=90^\circ$, $\delta=7$, and $P=1$, is shown in Figure 3. Figures 4 and 5 show the directivity plots that correspond to stages of growth for this array of $P=2$ and $P=3$, respectively.

The multi-band characteristics of an infinite fractal array may be demonstrated by following a procedure similar to that outlined in [21] and [22]. In other words, suppose that we consider the array factor for a doubly infinite linear fractal (uniform Cantor) array, which may be defined as

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Figure 6. The geometry for a symmetric planar array of isotropic sources with elements uniformly spaced a distance $d_x$ and $d_y$ apart in the $x$ and $y$ directions, respectively.
3.2 Sierpinski carpet arrays

The previous section presented an application of fractal geometric concepts to the analysis and design of thinned linear arrays. In this section, these techniques are extended to include the more general case of fractal planar arrays. A symmetric planar array of isotropic sources, with elements uniformly spaced a distance \( d_x \) and \( d_y \) apart in the \( x \) and \( y \) directions, respectively, is shown in Figure 6. It is well known that the array factor for this type of planar array configuration may be expressed in the following way [28]:

\[
AF'(\psi_x,\psi_y) = I_{11} + 2 \sum_{m=2}^{M} \left[ I_{1m} \cos\left( m \psi_x \right) + I_{m1} \cos\left( m \psi_y \right) \right] + 4 \sum_{n=2}^{M} \sum_{m=2}^{M} I_{mn} \cos\left( m \psi_x \right) \cos\left( n \psi_y \right),
\]

(28)

for \((2M-1)^2\) elements

\[
AF'(\psi_x,\psi_y) = 4 \sum_{n=1}^{M} \sum_{m=1}^{M} I_{mn} \cos\left[ (m-1/2) \psi_x \right] \cos\left[ (n-1/2) \psi_y \right],
\]

(29)

for \((2M)^2\) elements

where

\[
\psi_x = kd_x \left[ \sin \theta \cos \phi - \sin \theta_0 \cos \phi_0 \right],
\]

(30)

\[
\psi_y = kd_y \left[ \sin \theta \sin \phi - \sin \theta_0 \sin \phi_0 \right].
\]

(31)

As before, these arrays can be made fractal-like by following a systematic thinning procedure, where

\[
I_{mn} = \begin{cases} 
1, & \text{if element } (m,n) \text{ is turned on} \\
0, & \text{if element } (m,n) \text{ is turned off}
\end{cases}
\]

(32)

A Sierpinski carpet is a two-dimensional version of the Cantor set [29], and can similarly be applied to thinning planar arrays. Consider, for example, the simple generating subarray

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

The normalized array factor associated with this generating subarray, for \( d_x = d_y = \lambda/2 \), is given by

\[
\hat{A}(u_x,u_y) = \frac{1}{4} \cos(\pi u_x) \cos(\pi u_y) + 2 \cos(\pi u_x) \cos(\pi u_y),
\]

(33)

where

\[
u_x = \sin \theta \cos \phi - \sin \theta_0 \cos \phi_0,
\]

(34)

\[
u_y = \sin \theta \sin \phi - \sin \theta_0 \sin \phi_0.
\]

Substituting Equation (32) into Equation (1) with an expansion factor of \( \delta = 3 \) yields the following expression for the fractal array factor at stage \( P \):

\[
\hat{A}_F(u_x,u_y) = \frac{1}{4^P} \prod_{p=1}^{P} \left[ \cos(3^{p-1} \pi u_x) + \cos(3^{p-1} \pi u_y) \right. + 2 \cos(3^{p-1} \pi u_x) \cos(3^{p-1} \pi u_y). \]

(35)

The geometry for this Sierpinski carpet fractal array at various stages of growth is illustrated in Figure 7, along with a plot of the corresponding array factor. A comparison of the array factors for the first four stages of construction shown in Figure 7 reveals the self-similar nature of the radiation patterns.

![Figure 7. The geometry for the Sierpinski carpet fractal array at various stages of growth. Scale 1 is the generator subarray. Column 2 is the geometrical configuration of the Sierpinski carpet array: white blocks represent elements that are turned on, and black blocks represent elements that are turned off. Column 3 is the corresponding array factor, where the angle phi is measured around the circumference of the plot, and the angle theta is measured radially from the origin at the lower left.](image-url)
An expression for the directivity of the Sierpinski carpet array, for the case in which \( \theta_0 = 0^\circ \), may be obtained from

\[
D_P(\theta, \phi) = \frac{1}{\frac{1}{4\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |AF_P^2(\theta, \phi)|^2 \sin \theta \, d\theta \, d\phi}
\]

which follows directly from Equation (35). The double integral that appears in the denominator of Equation (36) does not have a closed-form solution in this case, and therefore must be evaluated numerically. However, this technique for evaluating the directivity is much more computationally efficient than the alternative approach, which involves making use of the Fourier-series representation for the Sierpinski carpet array factor given by Equations (28) and (31), with

\[
\psi_x = \pi \sin \theta \cos \phi,
\]

\[
\psi_y = \pi \sin \theta \sin \phi,
\]

\[AF_P^2(\theta, \phi) = \frac{1}{16 \pi^2} \prod_{p=1}^{P} \left[ \cos \left( 3^{p-1} \pi \sin \theta \cos \phi \right) + \cos \left( 3^{p-1} \pi \sin \theta \sin \phi \right) + 2 \cos \left( 3^{p-1} \pi \sin \theta \cos \phi \right) \cos \left( 3^{p-1} \pi \sin \theta \sin \phi \right) \right]^2,
\]

which is the Sierpinski carpet array factor, is the array factor associated with its complement, and \( AF \) denotes the array factor of the full planar array. It can be shown that the array factor for a uniformly excited square planar array (i.e., \( I_m = 1 \) for all values of \( m \) and \( n \)) may be expressed in the form

\[
AF_P(u_x, u_y) = \overline{AF_P}(u_x, u_y) = AF(u_x, u_y),
\]

where

\[
AF_P(u_x, u_y) = 2 \prod_{p=1}^{P} \left[ \cos \left( 3^{p-1} \pi u_x \right) + \cos \left( 3^{p-1} \pi u_y \right) 
+ 2 \cos \left( 3^{p-1} \pi u_x \right) \cos \left( 3^{p-1} \pi u_y \right) \right]
\]

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\[
AF(u_x, u_y) = \left[ 1 - 2 f_P(u_x) \right] \left[ 1 - 2 f_P(u_y) \right],
\]

where

\[
f_P(x) = \frac{\cos \left( \frac{\pi}{4} \left( 3^P - 1 \right) x \right) \sin \left( \frac{\pi}{4} \left( 3^P + 1 \right) x \right)}{\sin \left( \frac{\pi}{2} x \right)}.
\]
Finally, by using Equation (43) together with Equation (42), an expression for the complementary array factor \( \bar{AF}_P \) may be obtained directly from Equation (41). Plots of the directivity for a \( P = 4 \) Sierpinski carpet array and its complement are shown in Figures 10 and 11, respectively.

The multi-band nature of the planar Sierpinski carpet arrays may be easily demonstrated by generalizing the argument presented in the previous section for linear Cantor arrays. Hence, for doubly infinite carpets, we have

\[
AF(\psi_x, \psi_y) = \prod_{\mu=-\infty}^{\infty} GA(\delta^{\mu} \psi_x, \delta^{\mu} \psi_y),
\]

from which we conclude that

\[
AF(\delta^q \psi_x, \delta^q \psi_y) = AF(\psi_x, \psi_y) \quad \text{for } q = 0, 1, \ldots.
\]

4. The concentric circular ring subarray generator

4.1 Theory

An alternative design methodology for the mathematical construction of fractal arrays will be introduced in this section. The technique is very general, and consequently provides much more flexibility in the design of fractal arrays when compared to other approaches previously considered in the literature [21, 23-26]. This is primarily due to the fact that the generator, in this case, is based on a concentric circular ring array.

The generating array factor for the concentric circular ring array may be expressed in the form [30]

\[
GA(\theta, \phi) = \sum_{m=1}^{M} \sum_{n=1}^{N_m} I_{mn} e^{j\phi_{mn}}(\theta, \phi),
\]

where

\[
\psi_{mn}(\theta, \phi) = kr_m \sin \theta \cos(\phi - \phi_{mn}) + \alpha_{mn}
\]

\[
k = 2\pi/\lambda
\]

\[M = \text{Total number of concentric rings}\]

\[N_m = \text{Total number of elements on the } m\text{th ring}\]

\[r_m = \text{Radius of the } m\text{th ring}\]

\[I_{mn} = \text{Excitation current amplitude of the } n\text{th element on the } m\text{th ring located at } \phi = \phi_{mn}\]

\[\alpha_{mn} = \text{Excitation current phase of the } n\text{th element on the } m\text{th ring located at } \phi = \phi_{mn}\]

A wide variety of interesting as well as practical fractal array designs may be constructed using a generating subarray of the form given in Equation (47). The fractal array factor for a particular stage of growth \( P \) may be derived directly from Equation (47) by following a procedure similar to that outlined in the previous section. The resulting expression for the array factor was found to be

\[
AF_P(\theta, \phi) = \prod_{\mu=1}^{P} \left[ \sum_{m=1}^{M} \sum_{n=1}^{N_m} I_{mn} e^{j\phi_{mn}(\theta, \phi)} \right],
\]

where \( \delta \) represents the scaling or expansion factor associated with the fractal array.

A graphical procedure, which is embodied in Equation (55), can be used to conveniently illustrate the construction process for fractal arrays. For example, suppose we consider the simple four-element circular array of radius \( r \), shown in Figure 12a. If we regard this as the generator (stage 1) for a fractal array, then the
The next stage of growth (stage 2) for the array would have a geometrical configuration of the form shown in Figure 12b. Hence the first step in the construction process, as depicted in Figure 12, is to expand the four-element generator array by a factor of \( \delta \). This is followed by replacing each of the elements of the expanded array by an exact copy of the original unscaled four-element circular subarray generator. The entire process is then repeated in a recursive fashion, until the desired stage of growth for the fractal array is reached.

It is convenient for analysis purposes to express the fractal array factor, Equation (55), in the following normalized form:

\[
\hat{A}_F(\theta, \phi) = \prod_{p=1}^P \left[ \frac{\sum_{m=1}^{M} N_m e^{j2\pi \delta^p \theta_m(\theta, \phi)}}{\sum_{n=1}^{M} \sum_{m=1}^{N_m} I_{mn}} \right].
\]  

(56)

Taking the magnitude of both sides of Equation (56) leads to

\[
|\hat{A}_F(\theta, \phi)| = \prod_{p=1}^P \left| \frac{\sum_{m=1}^{M} N_m e^{j2\pi \delta^p \theta_m(\theta, \phi)}}{\sum_{n=1}^{M} \sum_{m=1}^{N_m} I_{mn}} \right|
\]

which has a corresponding representation in terms of decibels given by

\[
|\hat{A}_F(\theta, \phi)|_{dB} = 20 \prod_{p=1}^P \log \left( \frac{\sum_{m=1}^{M} N_m e^{j2\pi \delta^p \theta_m(\theta, \phi)}}{\sum_{n=1}^{M} \sum_{m=1}^{N_m} I_{mn}} \right)
\]

(58)

For the special case when \( \delta = 1 \), we note that Equations (57) and (58) reduce to

\[
\hat{A}_F(\theta, \phi) = \prod_{p=1}^P \left[ \frac{\sum_{m=1}^{M} N_m e^{j2\pi \theta_m(\theta, \phi)}}{\sum_{n=1}^{M} \sum_{m=1}^{N_m} I_{mn}} \right]
\]

(59)

\[
|\hat{A}_F(\theta, \phi)|_{dB} = 20 \prod_{p=1}^P \log \left( \frac{\sum_{m=1}^{M} N_m e^{j2\pi \theta_m(\theta, \phi)}}{\sum_{n=1}^{M} \sum_{m=1}^{N_m} I_{mn}} \right)
\]

(60)

Another unique property of Equation (56) is the fact that the conventional co-phasal excitation [30],

\[
\alpha_{mn} = -kr_m \sin \theta_0 \cos (\phi_0 - \phi_m),
\]

where \( \theta_0 \) and \( \phi_0 \) are the desired main-beam steering angles, can be applied at the generating subarray level. To see this, we recog-
Figure 14. The first four stages in the construction process of a uniformly spaced binomial array.

\[
AF_p(\varphi) = \frac{1}{2^P} \prod_{p=1}^{P} \sum_{n=1}^{2} e^{j\delta p \pi \left[ \frac{\pi}{2} \cos(\varphi - \phi_n) + \alpha_n \right]},
\]

(62)

where, without loss of generality, we have set \( \theta = 90^\circ \) and

\[
\phi_n = (n-1)\pi,
\]

(63)

\[
\alpha_n = -\frac{\pi}{2} \cos(\phi_0 - \phi_n).
\]

(64)

Substituting Equations (63) and (64) into Equation (62) results in a simplified expression for the fractal array factor, given by

\[
AF_p(\varphi) = \prod_{p=1}^{P} \cos\left[ \delta^{p-1} \frac{\pi}{2} (\cos \varphi - \cos \phi_0) \right].
\]

(65)

If we choose an expansion factor equal to one (i.e., \( \delta = 1 \)), then Equation (65) may be written as

\[
AF_p(\varphi) = \cos\left[ \frac{\pi}{2} (\cos \varphi - \cos \phi_0) \right].
\]

(66)

This represents the array factor for a uniformly spaced \( d = \lambda/2 \) linear array with a binomial current distribution, where the total number of elements, \( N_p \), for a given stage of growth, \( P \), is \( N_p = P + 1 \). The first four stages in the construction process of these binomial arrays are illustrated in Figure 14. The general rule in the case of overlapping array elements is to replace each of those elements by a single element that has a total excitation-current amplitude equal to the sum of all the individual excitation-current amplitudes. For example, in going from stage 1 to stage 2 in the binomial-array construction process illustrated in Figure 14, we find that two elements will share a common location at the center of the resulting three-element array. Since each of these two array elements has one unit of current, they may be replaced by a single equivalent element that is excited by two units of current.

Next, we will consider the family of arrays that are generated when \( \delta = 2 \). The expression for the array factor in this case is

\[
AF_p(\varphi) = \cos\left[ \frac{\pi}{2} (\cos \varphi - \cos \phi_0) \right].
\]

(65)

4.2 Examples

Several different examples of recursively generated arrays will be presented and discussed in this section. These arrays have in common the fact that they may be constructed via a concentric circular ring subarray generator of the type considered in the previous section. Hence, the mathematical expressions that describe the radiation patterns of these arrays are all special cases of Equation (56).

4.2.1 Linear arrays

Various configurations of linear arrays may be constructed using a degenerate form of the concentric circular ring subarray generator introduced in Section 4.1. For instance, suppose we consider the two-element circular subarray generator with a radius of \( r = \lambda/4 \), shown in Figure 13. If the excitation-current amplitudes for this two-element generating subarray are assumed to be unity, then the general fractal array factor expression given in Equation (56) will reduce to the form
which results from a sequence of uniformly excited, equally spaced \((d = \lambda/2)\) arrays. Hence, for a given stage of growth \(P\), these arrays will contain a total of \(N_P = 2^P\) elements, spaced a half-wavelength apart, with uniform current excitations. Finally, the last case that will be considered in this section corresponds to a choice of \(\delta = 3\). This particular choice for the expansion factor gives rise to the family of triadic Cantor arrays, which have already been discussed in Section 3.1. These arrays contain a total of \(N_P = 2^P\) elements, and have current excitations which follow a uniform distribution. However, the resulting arrays in this case are non-uniformly spaced. This can be interpreted as being the result of a thinning process, in which certain elements have been systematically removed from a uniformly spaced array in accordance with the standard Cantor construction procedure. The Cantor array factor may be expressed in the form

\[
\hat{A}F_P(\phi) = \prod_{p=1}^{P} \cos \left[ 2^{p-1} \frac{\pi}{2} (\cos \phi - \cos \phi_0) \right], \tag{67}
\]

which follows directly from Equation (65) when \(\delta = 3\).

### 4.2.2 Planar square arrays

In this section, we will consider three examples of planar square arrays that can be constructed using the uniformly excited four-element circular subarray generator shown in Figure 15. This subarray generator can also be viewed as a four-element square array. The radius of the circular array was chosen to be \(r = \lambda/(2\sqrt{2})\), in order to insure that the spacing between the elements of the circumscribed square array would be a half-wavelength (i.e., \(d = \lambda/2\)). In this case, it can be shown that the general expression for the fractal array factor given in Equation (56) reduces to

\[
\hat{A}F_P(\theta, \phi) = \frac{1}{4^P} \prod_{p=1}^{P} e^{j \beta p \pi \left[ \frac{\pi}{\sqrt{2}} \sin \theta \cos (\phi - \phi_0) + \alpha_n \right]}, \tag{69}
\]

where

\[
\phi_n = (n-1) \frac{\pi}{2}, \tag{70}
\]

\[
\alpha_n = -\frac{\pi}{\sqrt{2}} \sin \theta_0 \cos (\phi_0 - \phi_0). \tag{71}
\]

If we define

\[
\psi_n(\theta, \phi) = \frac{\pi}{\sqrt{2}} \left[ \sin \theta \cos (\phi - \phi_0) - \sin \theta_0 \cos (\phi_0 - \phi_0) \right], \tag{72}
\]

then Equation (69) may be written in the convenient form

\[
\hat{A}F_P(\theta, \phi) = \frac{1}{4^P} \prod_{p=1}^{P} e^{j \beta p \pi \psi_n(\theta, \phi)}, \tag{73}
\]
where
\[ \psi_n(\theta_0, \phi_0) = 0. \]

Now suppose we consider the case where the expansion factor \( \delta = 1 \). Substituting this value of \( \delta \) into Equation (73) leads to

\[ \hat{A}_P(\theta, \phi) = \left[ \frac{1}{4} \sum_{n=1}^{4} e^{i \frac{1}{2} \pi n^2 \phi_n(\theta, \phi)} \right]^P. \]  (74)

The first four stages of growth for this array are illustrated in Figure 16. The pattern that emerges clearly shows that this construction process yields a family of square arrays, with uniformly spaced elements \( (d = \lambda/2) \) and binomially distributed currents. For a given stage of growth \( P \), the corresponding array will have a total of \( N_p = (P+1)^2 \) elements. Figure 17 contains plots of the far-field radiation patterns that are produced by the four arrays shown in Figure 16. These plots were generated using Equation (74) for values of \( P = 1, 2, 3, \) and 4. It is evident from Figure 17 that the radiation patterns for these arrays have no sidelobes, which is a feature characteristic of binomial arrays [27].

The next case that will be considered is when the expansion factor \( \delta = 2 \). This particular choice of \( \delta \) results in a family of uniformly excited and equally spaced \( (d = \lambda/2) \) planar square arrays, which increase in size according to \( N_p = 2^{2P} \). The recursive array factor representation in this case is given by

\[ \hat{A}_P(\theta, \phi) = \frac{1}{4^P} \prod_{p=1}^{P} \sum_{n=1}^{4} e^{i 2^{p-1} \pi n \phi_n(\theta, \phi)}. \]  (75)

For our final example of this section, we return to the square Sierpinski carpet array, previously discussed in Section 3.2. The generating subarray for this Sierpinski carpet consisted of a uniformly excited and equally spaced \( (d = \lambda/2) \) planar array with the center element removed. However, we note here that this generating subarray may also be represented by two concentric four-element circular arrays. By adopting this interpretation, it is easily shown that the Sierpinski carpet array factor may be expressed in the form

\[ \hat{A}_P(\theta, \phi) = \frac{1}{8^P} \prod_{p=1}^{P} \sum_{a=1}^{2} \sum_{\ell=1}^{4} e^{i 3^{p-1} \pi \phi_{\ell a}(\theta, \phi)}, \]  (76)

where

\[ \psi_{mn}(\theta, \phi) = \sqrt{m \pi} \left[ \sin \theta \cos(\phi - \phi_{mn}) - \sin \theta_0 \cos(\phi_0 - \phi_{mn}) \right] \]  (77)

\[ \phi_{mn} = \left( \frac{mn - 1}{m} \right) \frac{\pi}{2}, \]  (78)

\[ N_p = 2^{3P}. \]  (79)

4.2.3 Planar triangular arrays

A class of planar arrays will be introduced in this section that have the property that their elements are arranged in some type of...
The geometry of a three-element circular subarray generator with a radius of \( r = \frac{\lambda}{2\sqrt{3}} \).

Recursively generated triangular lattice. The first category of triangular arrays that will be studied consists of those arrays that can be constructed from the uniformly excited three-element circular subarray generator shown in Figure 18. This three-element circular array of radius \( r = \frac{\lambda}{2\sqrt{3}} \) can also be interpreted as an equilateral triangular array, with half-wavelength spacing on a side (i.e., \( d = \lambda/2 \)). The fractal array factor associated with this triangular generating subarray is

\[
\hat{A}_P(\theta, \phi) = \frac{1}{3} \sum_{p=1}^{P} \sum_{n=1}^{3} e^{j\beta r^{-1} \left[ \frac{\pi}{\sqrt{3}} \sin \theta \cos(\phi - \phi_n) + \alpha_n \right]},
\]

where \( \phi_n = (n-1)\frac{2\pi}{3} \), \( \alpha_n = -\frac{\pi}{\sqrt{3}} \sin \theta_0 \cos(\phi_0 - \phi_n) \).

The compact form of Equation (80) is then

\[
\hat{A}_P(\theta, \phi) = \frac{1}{3} \prod_{p=1}^{P} \sum_{n=1}^{3} e^{j\alpha_p} \psi_n(\theta, \phi),
\]

where

\[
\psi_n(\theta, \phi) = \frac{\pi}{\sqrt{3}} \left[ \sin \theta \cos(\phi - \phi_n) - \sin \theta_0 \cos(\phi_0 - \phi_n) \right].
\]

If it is assumed that \( \delta = 1 \), then we find that Equation (83) can be expressed in the simplified form

\[
\hat{A}_P(\theta, \phi) = \left[ \frac{1}{3} \sum_{n=1}^{3} e^{j\psi_n(\theta, \phi)} \right]^P.
\]

This represents the array factor for a stage-\( P \) binomial triangular array.
array. The total number of elements contained in this array may be determined from the following formula:

\[
N_p = \sum_{p=1}^{P+1} p = \frac{(P+1)(P+2)}{2},
\]

(86)

which has been derived by counting overlapping elements only once. On the other hand, if we choose \( \delta = 2 \), then Equation (83) becomes

\[
\hat{A}F_p(\theta, \phi) = \frac{1}{3^P} \prod_{p=1}^{P} \sum_{n=1}^{3} e^{i2\pi n \delta_p} \psi_n(\theta, \phi).
\]

(87)

This array factor corresponds to uniformly excited Sierpinski gasket arrays, of the type shown in Figure 19. The construction process illustrated in Figure 19 assumes that the array elements are located at the center of the shaded triangles. Hence, these arrays have a growth rate that is characterized by \( N_p = 3^P \).

The second category of triangular arrays that will be explored in this section is produced by the six-element generating subarray shown in Figure 20. This generating subarray consists of two three-element concentric circular arrays, with radii \( r_1 = \frac{d}{2} \) and \( r_2 = \frac{d}{\sqrt{3}} \). The excitation current amplitudes on the inner three-element array are twice as large as those on the outer three-element array. The dimensions of this generating subarray were chosen in such a way that it forms a non-uniformly excited six-element triangular array, with half-wavelength spacing between its elements (i.e., \( d = \lambda/2 \)). If we treat the generating subarray as a pair of three-element concentric circular ring arrays, then it follows from...
Equation (56) that the fractal array factor in this case may be expressed as

$$\hat{A}\mathcal{F}_p(\theta, \phi) = \frac{1}{g^p} \prod_{p=1}^{P} \sum_{m=1}^{3} \sum_{n=1}^{3} I_{mn} e^{j\psi_{mn}^{(p)}(\theta, \phi)}.$$  

We will next consider two special cases of Equation (94), namely when $\delta = 1$ and $\delta = 2$. In the first case, Equation (94) reduces to

$$\hat{A}\mathcal{F}_p(\theta, \phi) = \frac{1}{g^p} \prod_{p=1}^{P} \sum_{m=1}^{3} \sum_{n=1}^{3} I_{mn} e^{j\phi_{mn} \cos(\theta - \phi_{mn})},$$  

where

$$I_{mn} = \frac{2}{m^2},$$ \hspace{1cm} (89)

$$k_r = \frac{m\pi}{\sqrt{3}},$$ \hspace{1cm} (90)

$$\phi_{mn} = \frac{2m + m - 3}{3},$$ \hspace{1cm} (91)

$$\alpha_{mn} = -k_r \sin \theta \cos (\phi_0 - \phi_{mn}).$$ \hspace{1cm} (92)

If we define

$$\psi_{mn}^{(p)}(\theta, \phi) = \frac{m\pi}{\sqrt{3}} \left[ \sin \theta \cos (\phi - \phi_{mn}) - \sin \theta \cos (\phi_0 - \phi_{mn}) \right],$$ \hspace{1cm} (93)

such that $\psi_{mn}^{(p)}(\theta_0, \phi_0) = 0$, then Equation (88) may be written in the form

$$\hat{A}\mathcal{F}_p(\theta, \phi) = \left[ \frac{1}{g^p} \sum_{m=1}^{3} \sum_{n=1}^{3} I_{mn} e^{j\psi_{mn}^{(p)}(\theta, \phi)} \right]^p.$$ \hspace{1cm} (95)

In the second case, Equation (94) reduces to

$$\hat{A}\mathcal{F}_p(\theta, \phi) = \frac{1}{g^p} \prod_{p=1}^{P} \sum_{m=1}^{3} \sum_{n=1}^{3} I_{mn} e^{j\phi_{mn} \cos(\theta - \phi_{mn})},$$ \hspace{1cm} (97)

where

$$N_p = \sum_{p=1}^{2^{p+1}-1} p = (P+1)(2P+1).$$ \hspace{1cm} (96)

$$N_p = \sum_{p=1}^{2^{p+1}-1} p = 2^P(P+1).$$ \hspace{1cm} (98)

Figure 22. The current distributions for the first three triangular arrays shown in Figure 21.
Figure 25. The geometry for a uniformly excited six-element circular subarray generator of radius $r = \lambda/2$.

These are both examples of fully-populated uniformly spaced ($d = \lambda/2$) and non-uniformly excited triangular arrays. Hence, this type of construction scheme can be exploited in order to realize low-sidelobe-array designs. To further exemplify this important property, we will focus here on the special case where $\delta = 2$. The first four stages in the construction process of this triangular array are illustrated in Figure 21. The triangular arrays shown in Figure 21 are made up of many small triangular subarrays that have elements located at each of their vertices. Some of these vertices overlap and, consequently, produce a nonuniform current distribution across the array, as shown in Figure 22. Figure 23 contains a sequence of far-field radiation-pattern plots, calculated using Equation (97) with $\phi = 0^\circ$, which correspond to the four triangular arrays depicted in Figure 21. It is evident from the plots shown in Figure 23 that a sidelobe level of at least $-20$ dB can be achieved by higher-order versions of these arrays. Color contour plots of these radiation patterns are also shown in Figure 24. This sequence of contour plots clearly reveals the underlying self-scalability, which is manifested in the radiation patterns of these triangular arrays.

4.2.4 Hexagonal arrays

Another type of planar array configuration in common use is the hexagonal array. These arrays are becoming increasingly popular, especially for their applications in the area of wireless communications. The standard hexagonal arrays are formed by placing elements in an equilateral triangular grid with spacings $d$ (see, for example, Figure 2.8 of [31]). These arrays can also be viewed as consisting of a single element located at the center, surrounded by several concentric six-element circular arrays of different radii. This property has been used to derive an expression for the hexagonal array factor [31]. The resulting expression, in normalized form, is given by
Figure 24. Color contour plots of the far-field radiation patterns produced by the four triangular arrays shown in Figure 21. The angle phi varies azimuthally from 0° to 360°, and the angle theta varies radially from 0° to 90°.

Figure 29. Color contour plots of the far-field radiation patterns produced by the four hexagonal arrays shown in Figures 26 and 27. The angle phi varies azimuthally from 0° to 360°, and the angle theta varies radially from 0° to 90°.

Figure 32. Color contour plots of the far-field radiation patterns produced by a series of four \((P = 1, 2, 3, 4)\) fully-populated hexagonal arrays generated with an expansion factor of \(\delta = 2\). The angle phi varies azimuthally from 0° to 360°, and the angle theta varies radially from 0° to 90°.

Figure 33. Color contour plots of the far-field radiation patterns produced by a series of four \((P = 1, 2, 3, 4)\) fully-populated hexagonal arrays generated with an expansion factor of \(\delta = 2\). In this case, the phasing of the generating subarray was chosen such that the maximum radiation intensity would occur at \(\theta_0 = 45°\) and \(\phi_0 = 90°\). The angle phi varies azimuthally from 0° to 360°, and the angle theta varies radially from 0° to 90°.
Figure 26. A schematic representation of the first four stages in the construction of a hexagonal array. The element locations correspond to the vertices of the hexagons.

Figure 27. The element locations and associated current distributions for each of the four hexagonal arrays shown in Figure 26.
where

$$r_{pm} = d \sqrt{p^2 + (m-1)^2 - p(m-1)},$$

(100)

$$\phi_{pm} = \cos^{-1} \left[ \frac{r_{pm}^2 + d^2 - d^2 (m-1)^2 - p(m-1)}{2r_{pm}d} \right] + \frac{n\pi}{3},$$

(101)

$$\alpha_{pm} = -k r_{pm} \sin \theta_0 \cos(\phi_0 - \phi_{pm}),$$

(102)

and $P$ is the number of concentric hexagons in the array. Hence, the total number of elements contained in an array with $P$ hexagons is

$$N_p = 3P(P+1) + 1.$$ (103)

At this point, we investigate the possibility that useful designs for hexagonal arrays may be realized via a construction process based on the recursive application of a generating subarray. To demonstrate this, suppose we consider the uniformly excited six-element circular generating subarray of radius $r_0 = \lambda/2$, shown in Figure 25. This particular value of radius was chosen so that the six elements in the array correspond to the vertices of a hexagon with half-wavelength sides (i.e., $d = \lambda/2$). Consequently, the array factor associated with this six-element generating subarray may be shown to have the following representation:

$$\hat{A}_F(\theta, \phi) = \frac{1}{6^P} \prod_{p=1}^{P} \sum_{m=1}^{6} e^{i \phi_{pm} \cos(\theta - \phi_0)} \alpha_m,$$ (104)

where

$$\phi_n = (n-1)\frac{\pi}{3},$$

(105)

$$\alpha_n = -\pi \sin \theta_0 \cos(\phi_0 - \phi_n).$$

(106)

The array factor expression given in Equation (104) may also be written in the form

$$\hat{A}_F(\theta, \phi) = \frac{1}{6^P} \prod_{p=1}^{P} \sum_{m=1}^{6} e^{i \phi_{pm} \cos(\theta - \phi_0)} \psi_n(\theta, \phi),$$

(107)

where

$$\psi_n(\theta, \phi) = \pi \left[ \sin \theta \cos(\phi - \phi_n) - \sin \theta_0 \cos(\phi_0 - \phi_n) \right].$$

(108)

We will first examine the special case where the expansion factor of the recursive hexagonal array is assumed to be unity (i.e., $\delta = 1$). Under these circumstances, Equation (107) reduces to

$$\hat{A}_F(\theta, \phi) = \left[ \sum_{n=1}^{6} e^{i \psi_n(\theta, \phi)} \right]^P.$$ (109)

Figure 28. Plots of the far-field radiation patterns produced by the four hexagonal arrays shown in Figures 26 and 27.
Figure 30. Plots of the far-field radiation patterns produced by a series of four \((P=1,2,3,4)\) fully-populated hexagonal arrays generated with an expansion factor of \(\delta=1\).

Figure 31. Plots of the far-field radiation patterns produced by a series of four \((P=1,2,3,4)\) fully-populated hexagonal arrays generated with an expansion factor of \(\delta=2\).
These arrays increase in size at a rate that obeys the relationship

$$N_p = 3p(P+1) + (1 - \delta_p), \quad (110)$$

where $\delta_p$ represents the Kronecker delta function, defined by

$$\delta_p = \begin{cases} 1, & P = 1 \\ 0, & P \neq 1 \end{cases} \quad (111)$$

In other words, every time this fractal array evolves from one stage to the next, the number of concentric hexagonal subarrays contained in it increases by one.

The second special case of interest to be considered in this section results when a choice of $\delta = 2$ is made. Substituting this value of $\delta$ into Equation (107) yields an expression for the recursive hexagonal array factor given by

$$\hat{A}_{hp}(\theta, \phi) = \frac{1}{6} \prod_{p=1}^P \sum_{n=1}^6 e^{2\pi i \nu_n(\theta, \phi)}, \quad (112)$$

where

$$N_p = 3\left(2^p - 1\right) - 2^{p-1}\left(2^{p-1} - 1\right) \quad (113)$$

Clearly, by comparing Equation (113) with Equation (110), we conclude that these recursive arrays will grow at a much faster rate than those generated by a choice of $\delta = 1$. Schematic representations of the first four stages in the construction process of these arrays are illustrated in Figure 26, where the element locations correspond to the vertices of the hexagons. Figure 27 shows the element locations and associated current distributions for each of the four hexagonal arrays depicted in Figure 26. Figures 26 and 27 indicate that the hexagonal arrays that result from the recursive construction process with $\delta = 2$ have some elements missing, i.e., they are thinned. This is a potential advantage of these arrays from the design point of view, since they may be realized with fewer elements. Another advantage of these arrays is that they possess low sidelobe levels, as indicated by the set of radiation pattern slices for $\phi = 90^\circ$, shown in Figure 28. Full color contour plots of these radiation patterns have also been included in Figure 29. Finally, we note that the compact product form of the array factor given in Equation (112) offers a significant advantage in terms of computational efficiency, when compared to the conventional hexagonal array factor representation of Equation (99), especially for large arrays. This is a direct consequence of the recursive nature of these arrays, and may be exploited to develop rapid beam-forming algorithms.

It is interesting to look at what happens to these arrays when an element with two units of current is added to the center of the hexagonal generating subarray shown in Figure 25. Under these circumstances, the expression for the array factor given in Equation (107) must be modified in the following way:

$$\hat{A}_{hp}(\theta, \phi) = \frac{1}{8} \prod_{p=1}^P \left\{ 2 + \sum_{n=1}^5 e^{i2\pi n\nu_n(\theta, \phi)} \right\}. \quad (114)$$

Plots of several radiation patterns calculated from Equation (114) with $\delta = 1$ and $\delta = 2$ are shown in Figures 30 and 31, respectively. These plots indicate that a further reduction in sidelobe levels may be achieved by including a central element in the generating subarray of Figure 25. Figure 32 contains a series of color contour plots that show how the radiation pattern intensity evolves for a choice of $\delta = 2$. Finally, a series of color contour plots of the radiation intensity for this array are shown in Figure 33, where the phasing of the generating subarray has been chosen so as to produce a main-beam maximum at $\theta_0 = 45^\circ$ and $\phi_0 = 90^\circ$.

5. Conclusions

Fractal antenna engineering represents a relatively new field of research that combines attributes of fractal geometry with antenna theory. Research in this area has recently yielded a rich class of new designs for antenna elements as well as arrays. The overall objective of this article has been to develop the theoretical foundation required for the analysis and design of fractal arrays. It has been demonstrated here that there are several desirable properties of fractal arrays, including frequency-independent multi-band behavior, schemes for realizing low-sidelobe designs, systematic approaches to thinning, and the ability to develop rapid beam-forming algorithms by exploiting the recursive nature of fractals.

6. Acknowledgments

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7. References


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**Introducing Feature Article Authors**

Douglas H. Werner is an Associate Professor in the Pennsylvania State University Department of Electrical Engineering. He is a member of the Communications and Space Sciences Lab (CSSL) and is affiliated with the Electromagnetic Communications Lab. He is also a Senior Research Associate in the Intelligence and Information Operations Department of the Applied Research Laboratory at Penn State. Dr. Werner received the BS, MS, and PhD degrees in Electrical Engineering from the Pennsylvania State University in 1983, 1985, and 1989, respectively. He also received the MA degree in Mathematics there in 1986. Dr. Werner was presented with the 1993 Applied Computational Electromagnetics Society (ACES) Best Paper Award, and was also the recipient of a 1993 International Union of Radio Science (URSI) Young Scientist Award. In 1994, Dr. Werner received the Pennsylvania State...
University Applied Research Laboratory Outstanding Publication Award. He has also received several Letters of Commendation from the Pennsylvania State University Department of Electrical Engineering for outstanding teaching and research. Dr. Werner is an Associate Editor of Radio Science, a Senior Member of the IEEE, a member of the American Geophysical Union, USNC/URSI Commissions B and G, the Applied Computational Electromagnetics Society (ACES), Eta Kappa Nu, Tau Beta Pi, and Sigma Xi. He has published numerous technical papers and proceedings articles, and is the author of six book chapters. His research interests include theoretical and computational electromagnetics with applications to antenna theory and design, micro-waves, wireless and personal communication systems, electromagnetic wave interactions with complex materials, fractal and knot electrodynamics, and genetic algorithms.

Randy Haupt is Professor and Department Head in the Electrical and Computer Engineering Department at Utah State University, Logan, Utah. He has a PhD in Electrical Engineering from the University of Michigan, an MS in Electrical Engineering from Northeastern University, an MS in Engineering Management from Western New England College, and a BS in Electrical Engineering from the USAF Academy. Dr. Haupt was a project engineer for the OTH-B radar and a research antenna engineer for Rome Air Development Center. Prior to coming to Utah State, he was Chair and a Professor in the Electrical Engineering Department of the University of Nevada, Reno, and a Professor of Electrical Engineering at the USAF Academy. His research interests include genetic algorithms, antennas, radar, numerical methods, signal processing, fractals, and chaos. He was the Federal Engineer of the Year in 1993, and is a member of Tau Beta Pi, Eta Kappa Nu, USNC/URSI Commission B, and the Electromagnetics Academy. He has eight patents, and is co-author of the book Practical Genetic Algorithms (John Wiley & Sons, January 1998).

Pingjuan L. Werner received her PhD degree from Penn State University in 1991. She is currently an Associate Professor with the College of Engineering, Penn State. Her research interests include antennas, wave propagation, genetic-algorithm applications in electromagnetics, and fractal electrodynamics. She is a member of the IEEE, Eta Kappa Nu, Tau Beta Pi, and Sigma Xi.

Juan Mosig Named EPFL "Extraordinary Professor"

The Council of the Federal Institutes of Technology, Switzerland, have appointed Juan R. Mosig as Extraordinary Professor in electromagnetism in the Department of Electrical Engineering of the Lausanne Federal Institute of Technology (EPFL). His appointment will become effective January 1, 2000.

Born in Cadix, Spain, Juan Mosig obtained his Diploma of Telecommunications Engineer in 1973 at the Polytechnic University of Madrid. In 1975, he received a Fellowship from the Swiss Confederation to carry out advanced studies at the then Electromagnetism and Microwaves Chair of EPFL. He worked on the design and analysis of microwave printed structures and, supported by the Hasler Foundation and the Swiss National Fund for Scientific Research, originated a new direction for research that is still being strongly pursued.

Under the direction of Prof. Fred Gardiol, Mr. Mosig completed his doctoral thesis at the Laboratory of Electromagnetism and Acoustics (LEMA) of EPFL in 1983, on the topic "Microstrip Structures: Analysis by Means of Integral Equations." In 1985, he also received the Doctor in Engineering from the Polytechnic University of Madrid. That same year, Dr. Mosig became director of a European Space Agency project to develop an optimization process for the computation of planar antenna arrays. This project was the start of long and fruitful cooperation between LEMA and the European Space Agency (ESA), leading to nine doctoral theses and a dozen research projects in collaboration with aerospace companies and other Swiss and European Universities.

Between 1984 and 1991, Dr. Mosig was an invited researcher at the Rochester Institute of Technology and at the University of Syracuse, New York, where he worked on spurious electromagnetic radiation from high-speed computer circuits. He was then an Invited Professor at the Universities of Rennes, France (1985); Nice, France (1986); Boulder, Colorado, USA (1987); and at the Technical University of Denmark in Lyngby (1990). He developed techniques to study electromagnetic fields radiated by printed circuits and antennas, some of which led to commercial software. Since 1978, Dr. Mosig has taught electromagnetism and antenna theory at EPFL. He became a Professor in 1991. He is the author of numerous monographs and publications on planar antennas, and twice received the Best Paper Award at the JINA conferences on antennas (1988 and 1998). He has been the Swiss delegate for the European COST-Telecommunications projects since 1986. He is also a member of the Political Technology Committee of the Swiss Federal Commission on Space Matters. He became a Fellow of the IEEE in 1999. At EPFL, Prof. Mosig will pursue his teaching activities in the Electrical Engineering and Communications Systems sections. His research activities will be in the fields of propagation and electromagnetic radiation from very-high-frequency antennas and circuits.

[The above item was taken from a press release by EPFL.]