Chapter 1.7.3

SCATTERING OF SURFACE WAVES

Roel Snieder
Department of Geophysics and Center for Wave Phenomena,
Colorado School of Mines,
Golden, Colorado 80401,
U.S.A.

PART 1 SCATTERING OF WAVES BY MACROSCOPIC TARGETS

Topic 1.7 Elastodynamic Wave (Elastic) Scattering: Theory

Contents

In this chapter the theory of scattering of elastic surface waves is described. Since surface waves are guided along the surface of materials, the scattering properties of surface waves are useful for probing the heterogeneity of the material near the surface as well as perturbations of the free surface. This has applications in the detection of surface defects (Steg and Klemens, 1974) and in seismological studies for the determination of the internal structure of the Earth (Nakanishi, 1993; Snieder, 1993). Many aspects of the propagation of elastic surface waves in laterally inhomogeneous media are treated in the textbooks of Malischewsky (1987) and Keilis-Borok *et al.* (1989).

§ 1. The Surface Wave Green's Tensor in the Far Field

In scattering theory the Green's tensor for the reference model in which the scatterers are embedded plays a crucial role. For elastic waves the role of this Green's tensor is shown in §5 of Chapter 1.7.1. For surface waves the natural reference model is a model where the parameters depend on the distance to the surface only. Such a reference model is free of lateral heterogeneities, and the invariance of the medium for translations in the horizontal direction allows a solution of the elastic wave equations using either a Fourier transform (in a rectangular geometry), a Fourier–Bessel transform (in a cylindrical geometry) or an expansion in spherical harmonics (in a spherical geometry). For the Earth, the latter situation is most appropriate. However, for surface waves that do not penetrate deep into the Earth and that do not propagate to the other side of the Earth, the sphericity is not very important. For this reason the theory is developed here for a Cartesian geometry. A formulation of the surface wave Green's function in a spherical geometry is given by Takeuchi and Saito (1972) and by Dahlen and Tromp (1998). Most of the material shown in this section is derived in detail by Aki and Richards (1980).

In a layered isotropic elastic model, the wave motion separates into the SH waves and the P–SV waves (Aki and Richards, 1980). The SH waves are shear waves that are horizontally polarised in the direction perpendicular to the direction of propagation. The P–SV waves consist of both compressive and shear motion that is coupled by the vertical gradient of the material properties. The polarisation of the P–SV waves is in the vertical plane in the direction of wave propagation. The surface waves of the SH system are called *Love waves*, while the surface wave solutions of the P–SV system are called *Rayleigh waves*. The Love waves are linearly polarised in the horizontal plane perpendicular to the direction of propagation while the Rayleigh waves are elliptically polarised in the vertical plane in the direction of propagation.

In the analysis of this section a Fourier transform over the horizontal coordinates is assumed. The x axis is aligned with the direction of wave propagation. This means that a plane wave in frequency–wave-number space corresponds with a solution $\mathbf{u}(k, z, \omega)\, e^{i(kx-\omega t)}$ in the x, z, t domain. Note that this plane-wave solution does not depend on the y direction perpendicular to the path of propagation. Inserting this special solution in Eqs. (9) and (11) of Chapter 1.7.1 for the special case of an isotropic medium, one can derive expressions for the z derivatives of the stress and displacement. For the Love waves the quantities that need to be accounted for are the displacement u_y in the y direction and the

This chapter appears in Scattering and Inverse Scattering in Pure and Applied Science, Eds., Pike, R. and P. Sabatier, Academic Press, San Diego, 562-577, 2002.

stress component τ_{yz}. Following Aki and Richards (1980) the associated eigenfunctions are denoted by:

$$u_y(z) = l_1(z)\,, \qquad \tau_{yz}(z) = l_2(z)\,. \tag{1}$$

These functions satisfy the following system of differential equations

$$\frac{d}{dz}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/\mu \\ k^2\mu - \rho\omega^2 & 0 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \tag{2}$$

where ρ is the mass density and μ is the shear modulus. The functions $l_1(\underline{z})$ and $l_2(\underline{z})$ are continuous at the interface between different layers. Furthermore the traction τ_{yz} vanishes at the surface and the displacement vanishes for great depth. This implies that $l_1(z)$ and $l_2(z)$ satisfy the boundary conditions

$$l_1(z) = 0 \quad \text{as} \quad z \to \infty\,, \qquad l_2(z = 0) = 0\,. \tag{3}$$

The differential equations (2) with the boundary conditions (3) can only be satisfied for selected values of k and ω. In this chapter we will assume the frequency ω to be fixed. In that case the equations can only be satisfied for certain discrete values of the wave number k. These correspond to the surface wave modes that propagate through the system. It is possible that there are no modes; for a homogeneous half-space there are no Love wave solutions. As in §7 of Chapter 1.7.1 the modes are labelled with Greek indices. For a mode ν, the phase velocity c_ν and the group velocity U_ν are related to the wave number k_ν of that mode by

$$c_\nu = \frac{\omega}{k_\nu}, \qquad U_\nu = \frac{\partial\omega}{\partial k_\nu}\,. \tag{4}$$

For the Rayleigh waves the quantities that describe the wave propagation are the components of the displacement in the x and z directions and the stress components τ_{zx} and τ_{zz}. These quantities are described by the functions

$$u_x(z) = r_1(z)\,, \quad u_z(z) = ir_2(z)\,, \quad \tau_{zx}(z) = r_3(z)\,, \quad \tau_{zz}(z) = ir_4(z)\,. \tag{5}$$

The factors i are inserted because this leads to a real system of equations for the functions $r_1, \cdots, r_4$. Physically this factor i accounts for the elliptical polarisation of the particle motion. The functions satisfy the differential equation

$$\frac{d}{dz}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 & k & 1/\mu & 0 \\ -k\lambda(\lambda+2\mu)^{-1} & 0 & 0 & (\lambda+2\mu)^{-1} \\ k^2\zeta - \rho\omega^2 & 0 & 0 & k\lambda(\lambda+2\mu)^{-1} \\ 0 & -\rho\omega^2 & -k & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}, \tag{6}$$

where ζ is related to the Lamé parameters λ and μ by the relation $\zeta = 4\mu(\lambda+\mu)/(\lambda+2\mu)$. The boundary conditions are that the tractions vanish at the free surface and that the displacement vanishes at great depth:

$$r_1(z) = r_2(z) = 0 \quad \text{as} \quad z \to \infty\,, \qquad r_3(z = 0) = r_4(z = 0) = 0\,. \tag{7}$$

Just as with the Love waves, Eq. (6) with the boundary conditions (7) has for a fixed value of ω only solutions for certain discrete wave numbers k_ν. These correspond to the Rayleigh wave mode in the system. The phase and group velocity of these modes are given by (4).

Up to this point the modes are defined using a plane-wave dependence $\exp(ikx)$ of the wave field. However, in a cylindrical coordinate system one arrives at the same equations (2) and (6) for the Love and Rayleigh waves, respectively (Aki and Richards, 1980), using the Fourier–Bessel transform. This means that the modes that are derived here can also be used to account for the surface-wave response to a point source.

When a surface wave is excited at a point $\mathbf{r}'$ and propagates to a location $\mathbf{r}$, the wave propagates in the horizontal direction and is trapped in the vertical direction. For this reason a new variable X is used to denote the horizontal distance between these points and the azimuth of the horizontal path is denoted by φ; both quantities are defined in Fig. 1. With these coordinates and with the surface wave eigenfunctions defined in the expressions (1) and (5), the Green's tensor of the surface waves can be formulated. As shown in expression (7.145) of Aki and Richards (1980) the Green's tensor for Love waves is given in the far field ($k_\nu X \gg 1$) by

$$\mathbf{G}^{\text{Love}}(\mathbf{r},\mathbf{r}') = \sum_\nu \frac{l_1^\nu(z)l_1^\nu(z')}{8c_\nu U_\nu I_1^\nu}\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}} \times \begin{pmatrix} \sin^2\varphi & -\sin\varphi\,\cos\varphi & 0 \\ -\sin\varphi\,\cos\varphi & \cos^2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

In this expression I_1^ν is the kinetic energy of the mode

$$I_1 = \frac{1}{2}\int_0^\infty \rho l_1^2\, dz\,, \tag{9}$$

which effectively normalises the modal eigenfunctions. The summation in (8) is over all the Love modes of the system. In order to simplify the resulting expressions we use the following normalisation condition for the surface wave modes

$$8cUI_1 = 1\,. \tag{10}$$

It is shown in §7 of Chapter 1.7.1 that the Green's tensor for elastic waves can in many cases be written as a sum of dyads. The Green's tensor (8) can also be written as a sum of dyads. To see this, define the following unit vectors in the horizontal plane

$$\hat{\Delta} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \qquad \hat{\varphi} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \tag{11}$$

see Fig. 1. The vector $\hat{\Delta}$ points in the horizontal plane in the direction of wave propagation while the vector $\hat{\varphi}$ is directed in the horizontal direction perpendicular to the propagation direction. (In this chapter a caret denotes a unit vector.) The 3×3 matrix in the Green's tensor (8) is equal to the dyad $\hat{\varphi}\hat{\varphi}^{\mathrm{T}}$. The vectors $\hat{\Delta}$ and $\hat{\varphi}$ can be used to define the polarisation vectors for Love and Rayleigh waves:

$$\begin{aligned} \mathbf{p}^{\mathrm{Love}}(z,\varphi) &= l_1(z)\hat{\varphi}\,, \\ \mathbf{p}^{\mathrm{Rayleigh}}(z,\varphi) &= r_1(z)\hat{\Delta} + i r_2(z)\hat{\mathbf{z}}\,. \end{aligned} \tag{12}$$

The Green's tensor for Love waves can then be written as

$$\mathbf{G}^{\mathrm{Love}}(\mathbf{r},\mathbf{r}') = \sum_\nu \mathbf{p}^\nu(z,\varphi)\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\mathbf{p}^{\nu\dagger}(z,\varphi)\,, \tag{13}$$

where the dagger denotes the Hermitian conjugate. Since $\mathbf{p}$ is real for Love waves, this is the same as the transpose.

The Rayleigh wave Green's tensor is for the far field ($k_\nu X \gg 1$) given in expression (7.146) of Aki and Richards (1980)

$$\mathbf{G}^{\mathrm{Rayleigh}}(\mathbf{r},\mathbf{r}') = \sum_\nu \frac{1}{8c_\nu U_\nu I_1^\nu}\mathbf{M}^\nu(z,z',\varphi)\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\,, \tag{14}$$

with the 3×3 matrix $\mathbf{M}^\nu$ given by

$$\mathbf{M}^\nu = \begin{pmatrix} r_1^\nu(z)r_1^\nu(z')\cos^2\varphi & r_1^\nu(z)r_1^\nu(z')\sin\varphi\cos\varphi & -ir_1^\nu(z)r_2^\nu(z')\cos\varphi \\ r_1^\nu(z)r_1^\nu(z')\sin\varphi\cos\varphi & r_1^\nu(z)r_1^\nu(z')\sin^2\varphi & -ir_1^\nu(z)r_2^\nu(z')\sin\varphi \\ ir_2^\nu(z)r_1^\nu(z')\cos\varphi & ir_2^\nu(z)r_1^\nu(z')\sin\varphi & r_2^\nu(z)r_2^\nu(z') \end{pmatrix}. \tag{15}$$

In (14) the kinetic energy integral for Rayleigh waves is given by

$$I_1 = \frac{1}{2}\int_0^\infty \rho\left(r_1^2 + r_2^2\right)dz\,. \tag{16}$$

The matrix $\mathbf{M}$ can then be written as a dyad of the polarisation vectors of Rayleigh waves as defined in expression (12): $\mathbf{M}(z,z',\varphi) = \mathbf{p}(z,\varphi)\mathbf{p}^\dagger(z',\varphi)$. Note that the complex conjugate is crucial in this identity because the polarisation vectors for Rayleigh waves are complex. Using the condition (10) to normalise the Rayleigh wave modes, the corresponding Green's tensor can be written as

$$\mathbf{G}^{\mathrm{Rayleigh}}(\mathbf{r},\mathbf{r}') = \sum_\nu \mathbf{p}^\nu(z,\varphi)\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\mathbf{p}^{\nu\dagger}(z',\varphi)\,. \tag{17}$$

This expression has the same form as the Love wave Green's tensor of expression (13); hence the total surface wave Green's tensor follows by summing over both Love and Rayleigh waves

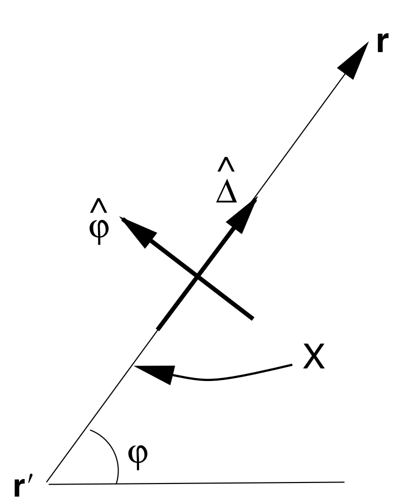
$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \sum_\nu \mathbf{p}^\nu(z,\varphi)\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\mathbf{p}^{\nu\dagger}(z',\varphi)\,. \tag{18}$$

When a force $\mathbf{f}$ at location $\mathbf{r}'$ excites the wave field, the displacement is given by

$$\mathbf{u}^{(0)}(\mathbf{r}) = \sum_\nu \mathbf{p}^\nu(z,\varphi)\frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\left\langle \mathbf{p}^\nu(z',\varphi)|\mathbf{f}(\mathbf{r}')\right\rangle, \tag{19}$$

with the inner product defined by $\langle \mathbf{u}|\mathbf{v}\rangle \equiv \sum_i u_i^* v_i$. Expression (19) has the same physical interpretation as described under Physical Interpretation of Dyadic Green's Tensor in Chapter 1.7.1; reading this expression from right to left one follows the life history of the surface wave. At the source location $\mathbf{r}'$ the surface wave mode is excited, which is described by the inner product $\left\langle \mathbf{p}^\nu|\mathbf{f}(\mathbf{r}')\right\rangle$ of the force with the polarisation vector. The surface wave mode then travels over a horizontal distance X to location $\mathbf{r}$ and experiences a phase shift and geometrical spreading as described by the terms $\exp(ik_\nu X)$ and $\sqrt{k_\nu X}$, respectively. At location $\mathbf{r}$ the particle motion is given by the polarisation vector $\mathbf{p}^\nu$. For the complete response one must

Figure 1 Definition of the geometric variables for the direct surface wave.



sum over all modes, which includes both Love and Rayleigh waves. Finally it should be remembered that the results in this section are only valid in the far field.

§ 2. The Gradient of the Surface-Wave Green's Tensor

In the scattering theory of elastic waves it is necessary to account for the gradient of the displacement field; see for example the Lippman–Schwinger equation (41) of §5 of Chapter 1.7.1. Since in the previous section only the far field surface wave Green's tensor was introduced, we consider here only the gradient of the surface wave Green's tensor in the far field. The far field is defined by the requirement that the horizontal distance is much larger than a wavelength: $k_\nu X \gg 1$.

When the point $\mathbf{r}$ in expression (18) changes, then X, z and the azimuth φ change. In the far field the dominant contribution to the gradient comes from the X derivative in the exponent in the Green's tensor and from the z derivative of the polarisation vector $\mathbf{p}^\nu(z,\varphi)$. The X derivative of the exponential term gives a contribution $ik_\nu\hat{\Delta}\,\exp\left(ik_\nu X\right)$, which means that in the far field the gradient of the Green's tensor is to leading order in $1/k_\nu X$ given by

$$\nabla\mathbf{G}(\mathbf{r},\mathbf{r}') = \sum_\nu \left(ik\hat{\Delta}\mathbf{p}^\nu(z,\varphi) + \hat{\mathbf{z}}\,\partial\mathbf{p}^\nu(z,\varphi)/\partial z\right) \times \frac{e^{i(k_\nu X+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\mathbf{p}^{\nu\dagger}(z',\varphi)\,. \tag{20}$$

The term in parentheses describes the strain associated with surface wave mode ν. It is convenient to define the strain operator for a mode ν

$$\mathbf{E}^\nu \equiv ik_\nu\hat{\Delta} + \hat{z}\frac{\partial}{\partial z}\;; \tag{21}$$

with this definition the gradient of the surface wave Green's tensor is given by

$$\partial_i G_{jk}(\mathbf{r},\mathbf{r}') = \sum_\nu \left(E_i{}^\nu p_j{}^\nu(z,\varphi)\right)\frac{e^{i(k_\nu X+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}p_k^{\nu *}(z',\varphi)\,. \tag{22}$$

It is also necessary to use the gradient ∇' of the Green's tensor with respect to the coordinate $\mathbf{r}'$. In the far field the dominant contribution comes from the X derivative of the exponential in (18) and from the z' derivative of the polarisation vector $\mathbf{p}^{\nu\dagger}(z',\varphi)$. Using the definition (21) for the strain operator at the source the gradient with respect to the source coordinate is given by

$$\partial_i' G_{jk}(\mathbf{r},\mathbf{r}') = \sum_\nu p_j{}^\nu(z,\varphi)\frac{e^{i(k_\nu X+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X}}\left(E_i^\nu p_k{}^\nu(z',\varphi)\right)^*\,. \tag{23}$$

Note the complex conjugate in the last term. This complex conjugate leads to a term $-ik_\nu\hat{\Delta}$ in the strain operator, which is due to the fact that when the source position $\mathbf{r}'$ is changed towards $\mathbf{r}$, the horizontal distance X *decreases*.

§ 3. The Surface Wave Scattering Amplitude

In this section we consider the scattering of a surface wave mode ν by a localised heterogeneity. The displacement of an incident surface mode ν is denoted by $\mathbf{u}^{(0)\nu}(\mathbf{r})$; when this wave is excited by a point force it is given by expression (19) without the summation over modes. The scattering problem is linear in the excitation; hence one can sum finally over all modes ν to obtain the response to a superposition of incident surface waves. The total displacement field for this scattering problem is denoted by $\mathbf{u}^\nu(\mathbf{r})$. It should be noted that this wave field also contains mode conversions from the incident mode ν to other surface wave modes. The total wave field satisfies the Lippman–Schwinger equation as given in expression (41) of §5 of Chapter 1.7.1:

$$u_i^\nu(\mathbf{r}) = u_i^{(0)\nu}(\mathbf{r}) + \omega^2\int G_{ij}(\mathbf{r},\mathbf{r}')\rho^{(1)}(\mathbf{r}')u_j{}^\nu(\mathbf{r}')\,dV' - \int \partial_k'\left(G_{in}(\mathbf{r},\mathbf{r}')\right)c_{nklj}^{(1)}(\mathbf{r}')\partial_l' u_j{}^\nu(\mathbf{r}')\,dV'\,. \tag{24}$$

(In this chapter the notation ∂_i stands for the partial derivative with respect to the x_i coordinate; $\partial_i f \equiv \partial f/\partial x$; and the summation convention is used where a summation over repeated indices is implied). Note that the superscript (0) in the Green's tensor is suppressed, but it is understood that we refer to the surface wave Green's tensor in the reference medium as defined in the previous two sections. In this expression the perturbation of the density and elasticity tensor is denoted by $\rho^{(1)}$ and $\mathbf{c}^{(1)}$, respectively.

In the far field, expression (18) for the Green's tensor and expression (23) for its gradient can be used. With a derivation that is analogous to the standard derivation of the scattering amplitude, one can show (Snieder, 1988) that at a distance much larger than the scatterer size the wavefield is given by

$$u_i^\nu(\mathbf{r}) = u_i^{(0)\nu}(\mathbf{r}) + \sum\nolimits_\sigma \mathbf{p}^\sigma(z,\varphi)\frac{e^{i(k_\sigma X+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\sigma X}}A^{\sigma\nu}\,, \tag{25}$$

with the scattering matrix given by

$$A^{\sigma\nu} = \omega^2\int e^{-ik_\sigma(\hat{\Delta}\cdot\mathbf{r}')}\left\langle\mathbf{p}^\sigma(z',\varphi)|\rho^{(1)}(\mathbf{r}')|\mathbf{u}^\nu(\mathbf{r}')\right\rangle dV' - \int e^{-ik_\sigma(\hat{\Delta}\cdot\mathbf{r}')}\left\langle\mathbf{E}^\sigma\mathbf{p}^\sigma|\mathbf{c}^{(1)}(\mathbf{r}')|\nabla'\mathbf{u}^\nu(\mathbf{r}')\right\rangle dV'\,. \tag{26}$$

The polarisation vector in the integral is evaluated at depth z'. Note that the full response to an incident mode ν contains a sum over outgoing modes σ. The modes $\sigma \neq \nu$ describe the mode conversions generated by the scatterer. Expression (25) therefore gives the total wave field as a superposition of the incident surface wave mode and a sum of scattered surface waves that is denoted by the summation over the mode index σ. The scattering matrix $A^{\sigma\nu}$ can be written as a volume integral over the scatterer; this integral contains the perturbation of the density $\rho^{(1)}$, the elasticity tensor $\mathbf{c}^{(1)}$ and the total wave field at the scatterer. This quantity gives the strength of the outgoing mode σ when a mode ν is incident on the scatterer.

Suppose the incident wave field is a plane-incoming surface wave mode ν that propagates in the direction given by an azimuth φ_{in}. Using the unit vector $\hat{\Delta}_{in}$ that corresponds to this azimuth through expression (11), the incident surface wave mode is given by

$$\mathbf{u}^{(0)\nu}(\mathbf{r}) = \mathbf{p}^{\nu}(z, \varphi_{in}) e^{\left(\hat{\Delta}_{in} \cdot \mathbf{r}\right)} . \tag{27}$$

In general the scattering matrix is a function of the incoming and outgoing azimuths:

$$A^{\sigma\nu} = A^{\sigma\nu}(\varphi, \varphi_{in}) . \tag{28}$$

When the scatterer displays cylindrical symmetry the scattering matrix depends only on the scattering angle $\psi = \varphi - \varphi_{in}$.

The theory developed here is completely analogous to the theory for scattering of other wave types. In fact, the optical theorem also holds for surface wave scattering. This theorem relates the forward scattering amplitude of unconverted surface waves to the total scattered power (Snieder, 1988):

$$\Im m \left(A^{\nu\nu}(\varphi_{in}, \varphi_{in})\right) = \frac{2}{\omega} P_S^{\nu} . \tag{29}$$

In this expression $P_S{}^{\nu}$ is the total power scattered through a surface that encloses the scatterer when the scatterer is illuminated with a single-incident surface wave mode ν. The reason that the imaginary component of the scattering matrix for forward scattering of *unconverted* surface waves appears on the left-hand side is related to the fact that this quantity describes decay of the amplitude of the transmitted surface wave due to the energy loss that is associated with the scattering. This energy loss is accounted for by the right-hand side of (29). The effect of scattering on the attenuation of surface waves is described by Hudson (1970) and by Brandenburg and Snieder (1989).

§ 4. The Born Approximation of Surface Waves

The scattering amplitude derived in the previous section accounts in principle for the complete scattering of surface waves. However, since the total wave field appears on the right-hand side of (26) it is very difficult to compute the scattering matrix. In many applications, scattered waves are used for imaging the heterogeneity. This imaging process is easiest when the relation between the scattered waves and the model is linear (Snieder and Trampert, 1999). For this reason the Born approximation is a valuable tool to account for the scattering of surface waves (Snieder, 1986a, b).

For the moment we ignore the perturbation of interfaces and consider volumetric perturbations of the medium only. As shown in §6 of Chapter 1.7.1, small perturbations of the interface can be included by replacing the perturbation of the interface by an equivalent volumetric perturbation. For example, for the density perturbation this amounts to making the replacement $\rho^{(1)} dV \to \rho^{(1)} dV + h \left[\rho^{(0)}\right] dS$, where h is the displacement of the interface and $\left[\rho^{(0)}\right]$ the discontinuity in the density across the interface in the reference model.

In the theory developed here we assume that the wave field is excited by a point force $\mathbf{f}$ at location $\mathbf{r}_s$. According to (19) the unperturbed wave is then given by

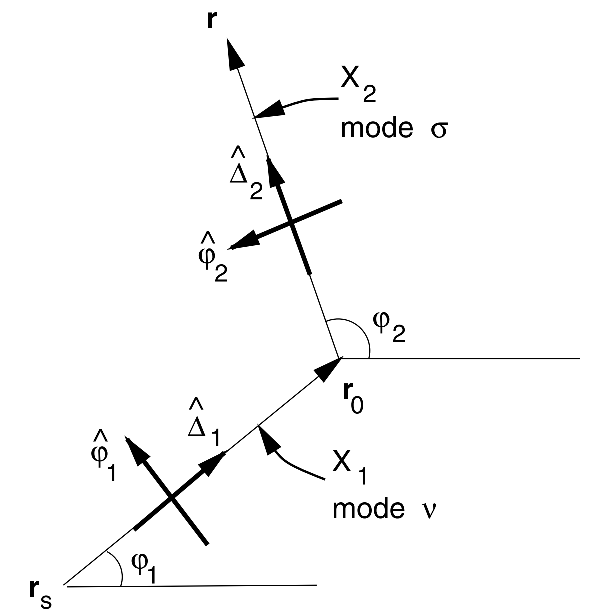
$$\mathbf{u}^{(0)}(\mathbf{r}) = \sum_{\nu} \mathbf{p}^{\nu}(z, \varphi) \frac{e^{i(k_\nu X + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\nu X}} \left\langle \mathbf{p}^{\nu}(z_s, \varphi) | \mathbf{f} \right\rangle . \tag{30}$$

The Born approximation for the scattered surface waves follows from expression (43) of §6 of Chapter 1.7.1. Inserting expression (30) and using Eqs. (18), (22) and (23) for the surface wave Green's function and its gradient, one arrives at the following expression for the single-scattered surface waves:

$$\begin{aligned} u_i^{(1)}(\mathbf{r}) = \omega^2 \sum_{\sigma\nu} \int & p_i^{\sigma}(z, \varphi_2) \frac{e^{i(k_\sigma X_2 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\sigma X_2}} p_j^{\sigma *}(z_0, \varphi_2) \\ & \times \rho^{(1)}(\mathbf{r}_0) p_j^{\nu}(z_0, \varphi_1) \frac{e^{i(k_\nu X_1 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\nu X_1}} \\ & \left\langle \mathbf{p}^{\nu}(z_s, \varphi_1) | \mathbf{f} \right\rangle dV_0 \\ - \sum_{\sigma\nu} \int & p_i^{\sigma}(z, \varphi_2) \frac{e^{i(k_\sigma X_2 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\sigma X_2}} E_k^{\sigma *} p_n^{\sigma *}(z_0, \varphi_2) \\ & \times c_{nklj}^{(1)}(\mathbf{r}_0) E_l{}^{\nu} p_j{}^{\nu}(z_0, \varphi_1) \frac{e^{i(k_\nu X_1 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\nu X_1}} \\ & \left\langle \mathbf{p}^{\nu}(z_s, \varphi_1) | \mathbf{f} \right\rangle dV_0 . \end{aligned} \tag{31}$$

The integration over the heterogeneity is carried out by the integration over $\mathbf{r}_0$. The geometric variables that appear in this expression are defined in Fig. 2. It is convenient to divide the volume integral in an integration over the horizontal surface area dS_0 and a depth integral: $dV_0 = dS_0 dz_0$. With this change, expression (31) can be written as

Figure 2 Definition of the geometric variables for the single-scattered surface waves.



$$\mathbf{u}^{(1)}(\mathbf{r}) = \sum_{\sigma\nu} \int \mathbf{p}^{\sigma}(z,\varphi_2) \frac{e^{i(k_\sigma X_2 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\sigma X_2}} \times V^{\sigma\nu}(\mathbf{r}_0) \frac{e^{i(k_\nu X_1 + \pi/4)}}{\sqrt{\frac{\pi}{2} k_\nu X_1}} \langle \mathbf{p}^{\nu}(z_s,\varphi_1)|\mathbf{f}\rangle dS_0 . \quad (32)$$

with

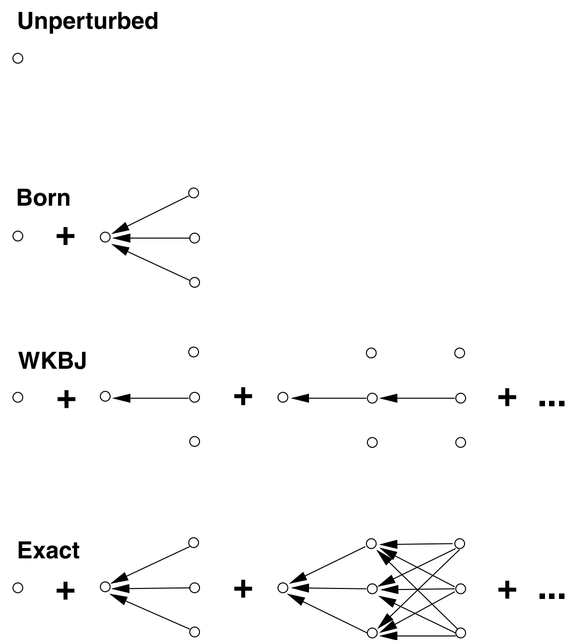
$$V^{\sigma\nu}(\mathbf{r}_0) = \omega^2 \int_0^\infty \left\langle \mathbf{p}^{\sigma}(z_0,\varphi_2)|\rho^{(1)}(\mathbf{r}_0)|\mathbf{p}^{\nu}(z_0,\varphi_1)\right\rangle dz_0 - \int_0^\infty \left\langle \mathbf{E}^{\sigma}\mathbf{p}^{\sigma}(z_0,\varphi_2)|\mathbf{c}^{(1)}(\mathbf{r}_0)|\mathbf{E}^{\nu}\mathbf{p}^{\nu}(z_0,\varphi_1)\right\rangle dz_0 . \quad (33)$$

Expression (32) can again be interpreted by reading it from right to left. At the source, mode ν is excited by the point force, which is described by the inner product $\langle \mathbf{p}^\nu|\mathbf{f}\rangle$. This mode then travels over a distance X_1 to the scattering point $\mathbf{r}_0$. During this propagation the mode experiences a phase shift $\exp(ik_\nu X_1)$ and a geometrical spreading $1/\sqrt{k_\nu X_1}$.

At the scatterer, mode ν is scattered and converted into surface wave mode σ, which is described by the scattering matrix $V^{\sigma\nu}$. The wave then travels as mode σ over distance X_2 to location $\mathbf{r}$ and obtains a phase shift $\exp(ik_\sigma X_2)$ and a geometrical spreading $1/\sqrt{k_\sigma X_2}$. The oscillation at $\mathbf{r}$ is given by the polarisation vector $\mathbf{p}^\sigma$. A summation over all pairs of incoming and outgoing modes ν and σ and an integration over the heterogeneity give the full response in the Born approximation.

It is illustrative to depict the different mode interactions as shown in Fig. 3. Each open dot denotes a mode, and each solid arrow represents an interaction

Figure 3 Diagrammatic representation of interactions between surface wave modes. The modes are indicated by open circles, the mode interactions by solid arrows.



between modes. In the figure the contribution to one particular outgoing mode is shown in different approximations. In a laterally homogeneous medium, the modes do not couple to other modes, which is shown by the top diagram in Fig. 3. In the presence of lateral variations all the modes are coupled, and mode coupling occurs to any order, which is shown in the bottom panel. In the Born approximation only the single-mode interactions are retained, which is indicated by the “Born” diagram.

§ 5. The Scattering Coefficient for an Isotropic Perturbation

The scattering matrix can be computed from (33) when the modes that are contained in the polarisation vectors and the perturbation of the density and elasticity tensor are known. This expression can be used for an arbitrary perturbation of the elasticity tensor. In this section the special case of an isotropic perturbation of the elasticity tensor is treated; for such a medium the perturbation of the elasticity tensor is related to the perturbations in the Lamé parameters through the relation

$$c^{(1)}_{ijkl} = \lambda^{(1)}\delta_{ij}\delta_{kl} + \mu^{(1)}\delta_{ik}\delta_{jl} + \mu^{(1)}\delta_{il}\delta_{jk} . \quad (34)$$

In that case the elastic properties of the medium have no intrinsic orientation. Because of rotational invariance, the scattering properties contained in the interaction matrix then depend only on the scattering angle defined as the difference in the azimuth of the incoming and outgoing surface wave,

$$\psi \equiv \varphi_2 - \varphi_1 \,; \tag{35}$$

see Fig. 2. When one inserts expression (12) for the polarisation vectors and Eq. (21) for the associated strain in Eq. (33) for the scattering matrix $V^{\sigma\nu}$, inner products of the unit vectors $\hat{\Delta}$, $\hat{\varphi}$ and $\hat{\mathbf{z}}$ for the incoming and outgoing modes appear. The inner products can be related to the scattering angle ψ by the following relations that can be deduced from expression (11) or from Fig. 2:

$$\begin{aligned}\left(\hat{\Delta}_2\cdot\hat{\Delta}_1\right) &= \left(\hat{\varphi}_2\cdot\hat{\varphi}_1\right) = \cos\psi\,,\\ \left(\hat{\Delta}_2\cdot\hat{\varphi}_1\right) &= -\left(\hat{\varphi}_2\cdot\hat{\Delta}_1\right) = \sin\psi\,,\\ \left(\hat{\mathbf{z}}\cdot\hat{\Delta}_1\right) = \left(\hat{\mathbf{z}}\cdot\hat{\Delta}_2\right) &= \left(\hat{\mathbf{z}}\cdot\hat{\varphi}_1\right) = \left(\hat{\mathbf{z}}\cdot\hat{\varphi}_2\right) = 0\,.\end{aligned} \tag{36}$$

For an isotropic perturbation of the elasticity tensor, as given by expression (34), a number of terms containing the inner products given in expression (36) appear in the interaction matrix (33). As an example, let us consider the contribution of the term $\mu^{(1)}\delta_{ik}\delta_{jl}$ in (34) to the scattering matrix. Using the summation convention this term gives the following contribution to the scattering matrix:

$$\begin{aligned}&-E_i{}^{\sigma*}p_j{}^{\sigma*}\mu^{(1)}\delta_{ik}\delta_{jl}E_k{}^{\nu}p_l{}^{\nu}\\ &= -\mu^{(1)}\left(-ik_\sigma\Delta_{2i}p_j{}^{\sigma*} + \hat{z}_i\partial p_j{}^{\sigma*}\right)\left(ik_\nu\Delta_{1i}p_j^{\nu} + \hat{z}_i\partial p_j{}^{\nu}\right)\\ &= -\mu^{(1)}\Big(k_\sigma k_\nu\left(\hat{\Delta}_2\cdot\hat{\Delta}_1\right)\langle\mathbf{p}^\sigma(\varphi_2)|\mathbf{p}^\nu(\varphi_1)\rangle\\ &\quad + \langle\partial\mathbf{p}^\sigma(\varphi_2)|\partial\mathbf{p}^\nu(\varphi_1)\rangle\Big)\,.\end{aligned} \tag{37}$$

In the last identity we used that $\hat{\mathbf{z}}$ is orthogonal to the $\hat{\Delta}$ and $\hat{\varphi}$ vectors. The result can be written as

$$-\mu^{(1)}\left(k_\sigma k_\nu\langle\mathbf{p}^\sigma(\varphi_2)|\mathbf{p}^\nu(\varphi_1)\rangle\cos\psi + \langle\partial\mathbf{p}^\sigma(\varphi_2)|\partial\mathbf{p}^\nu(\varphi_1)\rangle\right)\,. \tag{38}$$

(Note that the inner product implies taking the complex conjugate of the left vector.) At this point the specific form of the polarisation vectors as given in (12) must be used. The polarisation vectors are different for Love and Rayleigh waves. Therefore we need to distinguish four different situations: outgoing Love mode–incoming Love mode, outgoing Love mode–incoming Rayleigh mode, outgoing Rayleigh mode–incoming Love mode and outgoing Rayleigh mode–incoming Rayleigh mode. These situations will be abbreviated with the notation LL, LR, RL and RR respectively.

Consider the special case that ν is a Love wave and σ a Rayleigh wave. Then $\langle\mathbf{p}^\sigma(\varphi_2)|\mathbf{p}^\nu(\varphi_1)\rangle = r_1^\sigma\hat{\Delta}_2\cdot l_1^\nu\hat{\varphi}_1 - ir_2^\sigma\hat{\mathbf{z}}\cdot l_1^\nu\hat{\varphi}_1 = r_1^\sigma l_1^\nu\sin\psi$. The term $\langle\partial\mathbf{p}^\sigma(\varphi_2)|\partial\mathbf{p}^\nu(\varphi_1)\rangle$ can be handled in the same way. The total contribution of the term $\mu^{(1)}\delta_{ik}\delta_{jl}$ to $V_{RL}^{\sigma\nu}$ is for this special mode-pair thus given by

$$\begin{aligned}&-\mu^{(1)}\left(k_\sigma k_\nu r_1^\sigma l_1^\nu\sin\psi\cos\psi + \partial r_1^\sigma\partial l_1^\nu\sin\psi\right)\\ &= -\mu^{(1)}\left(1/2\,k_\sigma k_\nu r_1^\sigma l_1^\nu\sin 2\psi + \partial r_1^\sigma\partial l_1^\nu\sin\psi\right)\,.\end{aligned} \tag{39}$$

Treating all terms in the perturbation of the elasticity tensor in this way and including the contribution to the Born approximation (43) due to the perturbation of interfaces gives the complete scattering matrix for the four classes of incoming and outgoing surface wave modes (Snieder, 1986a):

$$\begin{aligned}V_{\mathrm{LL}}^{\sigma\nu} &= \int_0^\infty\left\{l_1^\sigma l_1^\nu\rho^{(1)}\omega^2 - (\partial l_1^\sigma)(\partial l_1^\nu)\mu^{(1)}\right\}dz\,\cos\psi\\ &\quad -k_\sigma k_\nu\int_0^\infty l_1^\sigma l_1^\nu\mu^{(1)}dz\,\cos 2\psi\\ &\quad +\sum h\left\{l_1^\sigma l_1^\nu\left[\rho^{(0)}\right]\omega^2 - (\partial l_1^\sigma)(\partial l_1^\nu)\left[\mu^{(0)}\right]\right\}\cos\psi\\ &\quad -\sum h\,k_\sigma k_\nu l_1^\sigma l_1^\nu\left[\mu^{(0)}\right]\cos 2\psi\,,\end{aligned} \tag{40}$$

$$\begin{aligned}V_{\mathrm{RL}}^{\sigma\nu} &= \int_0^\infty\left\{r_1^\sigma l_1^\nu\rho^{(1)}\omega^2 - (\partial r_1^\sigma - k_\sigma r_2^\sigma)(\partial l_1^\nu)\mu^{(1)}\right\}dz\,\sin\psi\\ &\quad -k_\sigma k_\nu\int_0^\infty r_1^\sigma l_1^\nu\mu^{(1)}dz\,\sin 2\psi\\ &\quad +\sum h\left\{r_1^\sigma l_1^\nu\left[\rho^{(0)}\right]\omega^2 - (\partial r_1^\sigma - k_\sigma r_2^\sigma)(\partial l_1^\nu)\left[\mu^{(0)}\right]\right\}\sin\psi\\ &\quad -\sum h\,k_\sigma k_\nu r_1^\sigma l_1^\nu\mu^{(1)}\sin 2\psi\,,\end{aligned} \tag{41}$$

$$V_{\mathrm{LR}}^{\sigma\nu} = -V_{\mathrm{RL}}^{\nu\sigma}\,, \tag{42}$$

$$\begin{aligned}V_{\mathrm{RR}}^{\sigma\nu} &= \int_0^\infty\Big\{r_2^\sigma r_2^\nu\rho^{(1)}\omega^2 - \left(k_\sigma r_1^\sigma + \partial r_2^\sigma\right)\left(k_\nu r_1^\nu + \partial r_2^\nu\right)\lambda^{(1)}\\ &\qquad - \left(k_\sigma k_\nu r_1^\sigma r_1^\nu + 2(\partial r_1^\sigma)(\partial r_1^\nu)\right)\mu^{(1)}\Big\}dz\\ &\quad +\int_0^\infty\Big\{r_1^\sigma r_1^\nu\rho^{(1)}\omega^2 - \left(k_\sigma r_2^\sigma - \partial r_1^\sigma\right)\\ &\qquad \times\left(k_\nu r_2^\nu - \partial r_1^\nu\right)\mu^{(1)}\Big\}dz\,\cos\psi\\ &\quad -k_\sigma k_\nu\int_0^\infty r_1^\sigma r_1^\nu\mu^{(1)}dz\,\cos 2\psi\\ &\quad +\sum h\Big\{r_2^\sigma r_2^\nu\left[\rho^{(0)}\right]\omega^2 - \left(k_\sigma r_1^\sigma + \partial r_2^\sigma\right)\\ &\qquad \times\left(k_\nu r_1^\nu + \partial r_2^\nu\right)\left[\lambda^{(0)}\right]\\ &\qquad - \left(k_\sigma k_\nu r_1^\sigma r_1^\nu + 2(\partial r_1^\sigma)(\partial r_1^\nu)\right)\left[\mu^{(0)}\right]\Big\}\\ &\quad +\sum h\Big\{r_1^\sigma r_1^\nu\left[\rho^{(0)}\right]\omega^2 - \left(k_\sigma r_2^\sigma - \partial r_1^\sigma\right)\\ &\qquad \times\left(k_\nu r_2^\nu - \partial r_1^\nu\right)\left[\mu^{(0)}\right]\Big\}\cos\psi\\ &\quad -\sum h\,k_\sigma k_\nu r_1^\sigma r_1^\nu\left[\mu^{(0)}\right]\cos 2\psi\,.\end{aligned} \tag{43}$$

It can be seen that each of these terms consists of the first terms of a Fourier series in ψ. For conversions to the same mode type (Love–Love or Rayleigh–Rayleigh) the interaction matrix is a superposition of terms $\cos m\psi$ for $m = 0, 1, 2$. For conversions between Love and Rayleigh modes the interaction matrix consists of a superposition of terms $\sin m\psi$ for $m = 1, 2$. This implies that the conversion between Love waves and Rayleigh waves vanishes for forward scattering ($\psi = 0$) and for backscattering ($\psi = \pi$).

Figure 4 Density, P velocity and S velocity for the reference medium (solid line) and for a model of a mountain root (dashed line).

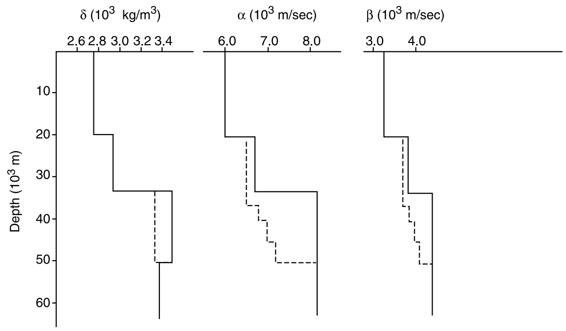
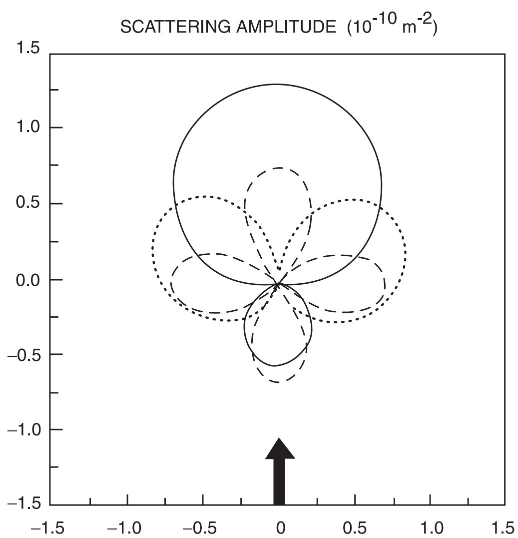


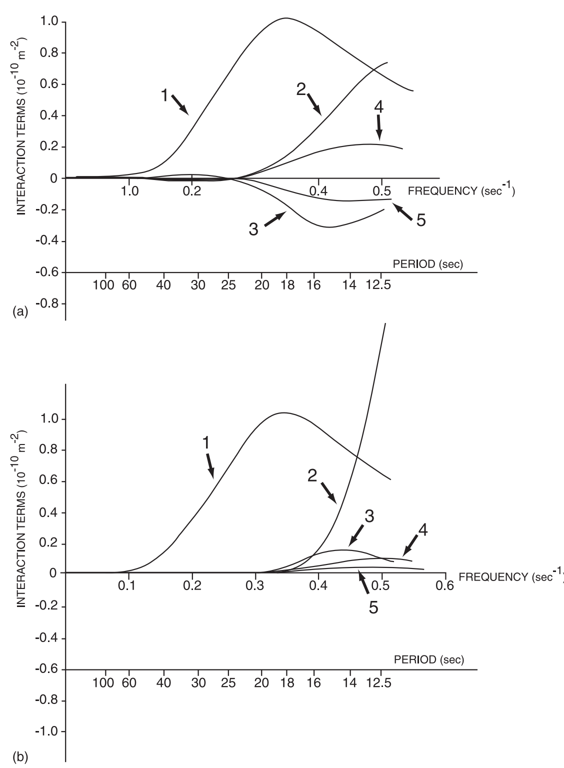
Figure 5 Radiation pattern for scattering by the mountain-root model for a period of 21 s. The direction of the incident wave is marked by the arrow. $R_1 \leftarrow R_1$ scattering is indicated by a solid line, $L_1 \leftarrow L_1$ scattering by a dashed line and $R_1 \leftarrow L_1$ conversion by a dotted line.



This is illustrated with a geophysical example of a model of a mountain root (Mueller and Talwani, 1971) shown in Fig. 4. The horizontal extent of the mountain is assumed to be $100 \times 100\,\text{km}^2$. The reference model is the M7 model as a realistic model for the Earth's mantle (Nolet, 1977). The radiation pattern due to this scatterer is shown in Fig. 5 for the scattering of the fundamental Love mode (marked L_1) and fundamental Rayleigh mode (marked R_1). In this radiation pattern the absolute value of the interaction matrix is shown as a function of the scattering angle. The direction of the incoming wave is shown with an arrow. It can be seen that the conversion between the Love and Rayleigh waves ($R_1 \leftarrow L_1$, as shown with the dotted line) vanishes in the forward and backward directions. Note the distinct difference between the $R_1 \leftarrow R_1$ radiation pattern with two lobes and the $L_1 \leftarrow L_1$ radiation pattern with a characteristic four-lobe pattern. This is related to the fact that for the interactions between Rayleigh waves the interaction matrix (43) contains an isotropic term that is independent of ψ that is not present in the interaction matrix (40)–(42) for other mode pairs.

The frequency dependence of the $\cos\psi$ terms of the scattering matrix for the mountain-root model is shown in Fig. 6. In the upper panel the interaction matrix is shown for the coupling of different Rayleigh modes with the fundamental Rayleigh mode ($R_N \leftarrow R_1$). In the lower panel the self-interaction of different Rayleigh modes ($R_N \leftarrow R_N$) is shown. Each of the terms in the scattering matrix contains two of the following terms: the frequency ω, the horizontal wave number k, and the vertical derivative of the modes. Since the vertical derivative of the modes is comparable to the horizontal wave number and since the wave number varies with frequency as ω/c, each of the terms is on the order ω^2. This ω^2 dependence leads to a contribution to the scattered power that varies with frequency as ω^4, which is characteristic of Rayleigh scattering. However, the modes also vary with frequency and this gives an additional frequency

Figure 6 The $\cos\psi$ terms of the scattering matrix for the mountain-root model as a function of frequency. The top panel depicts the interaction of the fundamental Rayleigh mode to other Rayleigh modes (indicated by the numbers), while the bottom panel depicts the self-interaction of different Rayleigh modes (indicated by the numbers).



dependence. The low-frequency behaviour of the interaction matrix is explained by this ω^2 dependence of the interaction matrix. However, this is not the whole story because the depth dependence of the modes as a function of frequency also affects the strength of the scattering. This can be seen in Fig. 6 of the interaction matrix for the scattering of the fundamental mode to itself ($R_1 \leftarrow R_1$). For frequencies higher than 0.33 Hz the $R_1 \leftarrow R_1$ scattering decreases with frequency. The physical reason for this is that for these high frequencies the penetration depth of the modes becomes so small that they do not penetrate deeply enough to sample the mountain-root heterogeneity of Fig. 4. This shows that in general the frequency dependence of surface wave scattering can be complicated.

§ 6. Scattering Coefficients for Surface Topography

Surface waves are very sensitive to perturbations of the free surface. For the detection of surface defects it is important to consider the effect of surface irregularities on surface waves (Steg and Klemens, 1974). When the perturbation of the surface is much smaller than a wavelength, expressions (40)–(43) can be used to account for the perturbation of the free surface (Snieder, 1986b). For this type of perturbation the volumetric perturbations $\rho^{(1)}$, $\lambda^{(1)}$ and $\mu^{(1)}$ vanish, and the discontinuity of the density and Lamé parameters at the surface is equal to the parameters in the surface layer; $\left[\rho^{(0)}\right] = \rho^{(0)}$ because the density and Lamé parameters vanish above the free surface.

In addition to these specific values for the perturbation of the medium, we can exploit the fact that the surface wave modes have special properties at the surface that are associated with the fact that the tractions vanish at that depth. It follows from (1) and (2) that for Love waves the traction in the y direction is given by $\tau_{yz} = l_2 = \mu\partial l_1/\partial z$, so that at the free surface

$$\frac{\partial l_1}{\partial z}(z=0) = 0\,. \tag{44}$$

For Rayleigh waves the stress components τ_{xz} and τ_{zz} vanish at the surface. It follows from (5) and (6) that $\tau_{xz} = r_3 = \mu\left(\partial r_1/\partial z - kr_2\right)$ and that $-i\tau_{zz} = r_4 = (\lambda+2\mu)\partial r_2/\partial z + k\lambda r_1$. At the free surface both τ_{xz} and τ_{zz} vanish so that

$$\begin{aligned}\frac{\partial r_1}{\partial z}(z=0) &= kr_2(z=0)\,,\\ \frac{\partial r_2}{\partial z}(z=0) &= \frac{-k\lambda}{\lambda+2\mu}r_1(z=0)\,.\end{aligned} \tag{45}$$

Using these results one obtains the following expressions for the interaction matrix for surface wave scattering by surface topography (Snieder, 1986b)

$$V_{\mathrm{LL}}^{\sigma\nu} = h\left\{l_1^\sigma l_1^\nu \rho^{(0)}\omega^2\cos\psi - k_\sigma k_\nu l_1^\sigma l_1^\nu \mu^{(0)}\cos 2\psi\right\}, \tag{46}$$

$$V_{\mathrm{RL}}^{\sigma\nu} = h\left\{r_1^\sigma l_1^\nu \rho^{(0)}\omega^2\sin\psi - k_\sigma k_\nu r_1^\sigma l_1^\nu \mu^{(0)}\sin 2\psi\right\}, \tag{47}$$

$$V_{\mathrm{LR}}^{\sigma\nu} = -V_{\mathrm{RL}}^{\nu\sigma}, \tag{48}$$

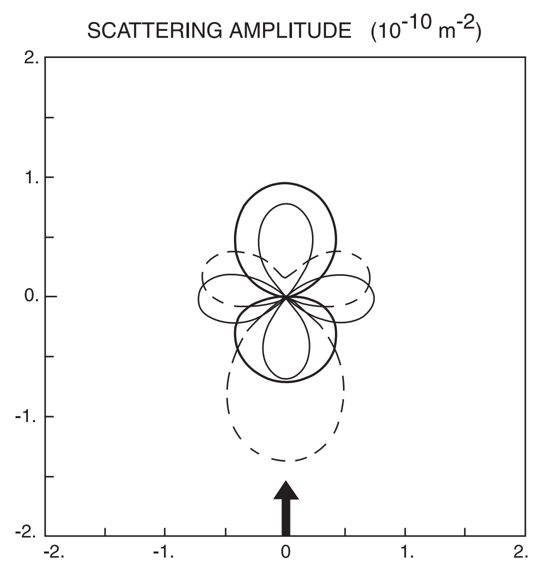
$$\begin{aligned}V_{\mathrm{RR}}^{\sigma\nu} = h\Bigg\{&r_2^\sigma r_2^\nu \rho^{(0)}\omega^2 - k_\sigma k_\nu \mu^{(0)}\frac{3\lambda^{(0)}+2\mu^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}\\ &\times r_1^\sigma r_1^\nu + 2(\partial r_1^\sigma)(\partial r_1^\nu)\left[\mu^{(0)}\right]\\ &+ r_1^\sigma r_1^\nu \rho^{(0)}\omega^2\cos\psi\\ &- k_\sigma k_\nu r_1^\sigma r_1^\nu \mu^{(0)}\cos 2\psi\Bigg\},\end{aligned} \tag{49}$$

where it is understood that all quantities are evaluated at the surface.

The radiation pattern for scattering by a mountain root and by surface topography for the fundamental Love mode to itself ($L_1 \leftarrow L_1$) is shown in Fig. 7. For surface topography (dashed line) the radiation pattern has nodes near ±120° and is very small in the forward direction. This can be understood from expression (46). For unconverted Love wave scattering one can use that $\omega = kc$ for the mode under consideration. In the second term of (46) one can use that $\mu^{(0)} = \rho^{(0)}\beta^2$, with β the shear velocity. Since the phase velocity of the fundamental mode is not too different from the shear velocity at the surface one obtains

$$\begin{aligned}V_{\mathrm{LL}}^{11} &= hl_1l_1\rho^{(0)}k^2\left\{c^2\cos\psi - \beta^2\cos 2\psi\right\}\\ &\approx hl_1l_1\rho^{(0)}\omega^2\left\{\cos\psi - \cos 2\psi\right\}.\end{aligned} \tag{50}$$

Figure 7 Radiation pattern for $L_1 \leftarrow L_1$ scattering at a period of 20 s for scattering by surface topography of 1 km height (dashed line), by the mountain-root model (thin solid line) and by the combination of both (thick solid line).



The associated radiation pattern has nodes at 0° and 120°, which is confirmed by Fig. 7. A similar analysis can be applied to the interaction between fundamental Love wave and the fundamental Rayleigh wave (Snieder, 1986b):

$$V_{\mathrm{RL}}^{11} \approx h r_1 l_1 \rho^{(0)} \omega^2 \{\sin\psi - \sin 2\psi\} . \tag{51}$$

The corresponding radiation pattern has nodes at 0°, 180°, and approximately ±60°.

§ 7. Relation between Scattering Coefficients and Phase Velocity Perturbation

In this section the relation between the interaction matrix and the phase velocity perturbation of surface waves is discussed. The linearised phase velocit perturbation of surface waves can be expressed in the perturbation of the medium using Rayleigh's principle (Snieder and Trampert, 1999). As shown in expression (7.71) of Aki and Richards (1980) the first-order perturbation of the phase velocity of surface waves is given by

$$\left(\frac{\delta c}{c}\right)^{\mathrm{L}} = \frac{\int_0^\infty \left\{k^2 l_1^2 + \left(\frac{\partial l_1}{\partial z}\right)^2\right\}\mu^{(1)} dz - \int_0^\infty \omega^2 l_1^2 \rho^{(1)}\, dz}{2k^2 \int_0^\infty \mu^{(0)} l_1^2 dz} . \tag{52}$$

Using expressions (7.66) and (7.70) of Aki and Richards (1980), the integral in the denominator can be written as $2cUI_1$, where I_1 is defined in (9) of this chapter. This means that for Love waves

$$\left(\frac{\delta c}{c}\right)^{\mathrm{L}} = \frac{1}{4k^2 cUI_1} \int_0^\infty \left\{k^2 l_1^2 + \left(\frac{\partial l_1}{\partial z}\right)^2\right\}\mu^{(1)} dz - \int_0^\infty \omega^2 l_1^2 \rho^{(1)} dz . \tag{53}$$

Using the normalisation condition (10) of the modes and expression (40) for the interaction between Love waves for the special case of unconverted waves ($\sigma = \nu$) and forward scattering ($\psi = 0$) one finds that these quantities are related by

$$\left(\frac{\delta c}{c}\right)^{\mathrm{L}} = -\frac{2}{k^2} V^{\mathrm{LL}}(\psi = 0) . \tag{54}$$

In this expression V^{LL} is the coefficient for *forward* scattering of *unconverted* modes.

For Rayleigh waves a similar result can be derived. As shown in expression (7.78) of Aki and Richards (1980) the first-order phase velocity perturbation is given by[1]

$$\left(\frac{\delta c}{c}\right)^{\mathrm{R}} = \frac{1}{4k^2 cUI_1} \left(\int_0^\infty \left\{k r_1 + \left(\frac{\partial r_2}{\partial z}\right)\right\}^2 \lambda^{(1)} dz + \int_0^\infty \left\{2k^2 r_1^2 + 2\left(\frac{\partial r_2}{\partial z}\right)^2 + \left(k r_2 - \left(\frac{\partial r_1}{\partial z}\right)\right)^2\right\}\mu^{(1)} dz - \int_0^\infty \omega^2 \left\{r_1^2 + r_2^2\right\}\rho^{(1)} dz \right) , \tag{55}$$

where the integral I_1 is defined in (16). Comparing this with the interaction coefficients for Rayleigh–Rayleigh wave scattering in expression (43) for the special case of forward scattering and unconverted waves one finds with the normalisation condition (10) that for Rayleigh waves

$$\left(\frac{\delta c}{c}\right)^{\mathrm{R}} = -\frac{2}{k^2} V^{\mathrm{RR}}(\psi = 0) , \tag{56}$$

where the interaction matrix is for unconverted mode interactions for forward scattering.

This result and (54) imply that for both Love and Rayleigh waves there is a direct relation between the phase velocity perturbation and the interaction matrix for forward scattering of unconverted modes:

$$\left(\frac{\delta c}{c}\right) = -\frac{2}{k^2} V(\text{forward, unconverted}) . \tag{57}$$

An alternative derivation of this general relation is given in Snieder (1986b). It is not a coincidence that the phase velocity perturbation and the scattering coefficients for forward scattering of unconverted waves are closely related. When a propagating wave front travels through a perturbed medium, the prime effect of the perturbation is a change in the propagation velocity of the wave front, which is by definition described as a change in the velocity. On the other hand, the interaction coefficients for forward scattering also account for the change in the transmission properties of a wave field. For this reason these quantities are closely related. This argument suggests that the relation between the velocity perturbation and the forward scattering coefficient of unconverted waves is not a peculiarity of surface waves. In fact, for elastic body waves the scattering coefficient for forward scattering of unconverted waves is determined by the velocity perturbation while the scattering coefficient for backward-scattered unconverted waves is determined by the perturbation in the impedance (Wu and Aki, 1985). This has important consequences for the analysis of seismic reflection data (Tarantola, 1986).

It follows from the expressions (40)–(43) that in the Born approximation the elements of the interaction matrix are real. With (57) this implies that the phase velocity perturbation is real as well. A real phase velocity perturbation cannot account for the attenuation of surface waves by scattering losses. This points to an important limitation of the Born approximation. In this approximation energy is not conserved; hence it should be used with great caution when it is used to account for the decay in the amplitude of a transmitted wave by scattering losses.

Again, this is not a peculiarity for surface waves, but due to the fact that the Born approximation in general is a *first-order* approximation, whereas the leading-order change in the energy is of *second order*. A proper treatment of surface wave attenuation due to scattering losses therefore needs to include multiple scattering effects. For 2D surface wave scattering this can be described by coupled-power equations (Kennett, 1990; Park and Odom, 1999).

§ 8. Ray Theory for Surface Waves

Up to this point no limitation has been imposed on the length scale of the variations of the medium. For a number of practical applications it is of interest to consider the case where the horizontal variation of the perturbation of the medium occurs on a length scale that is much larger than a wavelength. For such a slowly varying waveguide one can use the following small parameter as the basis of a perturbation expansion:

$$\varepsilon = \frac{\text{wavelength}}{\text{horizontal scale length of structural variation}} \ll 1 . \tag{58}$$

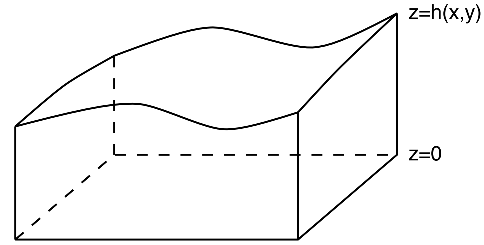
For other types of waves such as electromagnetic waves (Kline and Kay, 1965) or elastic body waves (Červený and Hron, 1980), it is known that in this approximation the wave propagation is described well by geometric-ray theory where energy travels along rays with a speed that is determined by the eikonal equation and an amplitude that is governed by the transport equation.

A similar result can be derived for elastic surface waves. However, the underlying theory is very general. Bretherton (1968) derived the propagation of waves in smoothly varying waveguides for a large class of guided waves. For elastic surface waves the theory has been formulated by Woodhouse (1974), Babich *et al.* (1976) and by Yomogida (1987). The theory outlined here is also used in ocean acoustics (Brekhovskikh and Lysanov, 1982), where it is known under the name *adiabatic mode theory*.

Because the algebraic complexity of the theory for elastic waves in slowly varying waveguides hides the physical ideas, the theory is presented here for a simple scalar analogue that contains the essential elements. We consider a layer with thickness h that extends from the bottom at $z = 0$ to an upper boundary that varies with position $z = h(x, y)$; see Fig. 8 for the geometry of this problem. We assume that within the layer scalar waves propagate that satisfy the Helmholtz equation with a constant velocity v:

$$\nabla^2 u + \frac{\omega^2}{v^2} u = 0 . \tag{59}$$

Figure 8 Geometry of the structure used in the analysis of local modes.



For simplicity it is assumed that the wave field vanishes at the top and the bottom of the layer

$$u(x, y, z = 0) = u(x, y, z = h(x, y)) = 0 . \tag{60}$$

A key concept in the theory is the use of *local modes*, which are defined at each horizontal location (x, y) as the modes that the system would have if the medium would be laterally homogeneous with the properties of the medium at that particular location (x, y). For the model used here the local modes can be found in closed form; mode number n is given by

$$u_n(x, y, z) = \sqrt{\frac{2}{h(x, y)}} \sin\left(\frac{n\pi z}{h(x, y)}\right) . \tag{61}$$

Each local mode satisfies the boundary conditions (60). Note that the local modes (61) satisfy the orthogonality relation

$$\int_0^{h(x,y)} u_n(x, y, z) u_m(x, y, z)\, dz = \delta_{nm} . \tag{62}$$

when the mode (61) is inserted in the Helmholtz equation (59) the local wave number of the mode follows: ${k_n}^2 = \omega^2/v^2 - (n\pi/h)^2$. The associated local phase velocity $c_n(x, y)$ of mode n is given by

$$c_n(x, y) = \frac{v}{\left(1 - \left(\frac{n\pi v}{\omega h(x, y)}\right)^2\right)^{1/2}} . \tag{63}$$

The concepts of local modes and the local phase velocity are crucial. In general the modes are coupled by the heterogeneity, but the crux of the theory is that for smoothly varying perturbations ($\varepsilon \ll 1$), this coupling vanishes and the modes propagate independently with the local phase velocity.

In order to verify this statement we consider the wave field as a sum of propagating modes where the modes are defined to be the local modes of the system at each location. The coefficients that multiply each mode have a phase $\varphi_n(x, y)$ and an amplitude $A_n(x, y)$ that are at this point unknown functions of the horizontal coordinates x and y:

$$u(x, y, z) = \sum_n A_n(x, y) e^{i\varphi_n(x, y)} u_n(x, y, z) . \tag{64}$$

Since each mode satisfies the boundary conditions (60), the solution (64) satisfies these boundary conditions as well. Inserting the solution (64) in the Helmholtz equation (59), using that $\partial^2 u_n/\partial z^2 = -\left(n\pi/h\right)^2 u_n = \left(k_n^2 - \omega^2/v^2\right) u_n = \left(\omega^2/c_n^2 - \omega^2/v^2\right) u_n$, gives after some rearrangement

$$\sum_n \left\{ \nabla_H^2 (A_n u_n) - A_n |\nabla_H \varphi_n|^2 u_n + \frac{\omega^2}{c_n^2} A_n u_n \right\} e^{i\varphi_n}$$
$$+ i \sum_n \left\{ 2\nabla_H \varphi_n \cdot \nabla_H (A_n u_n) + A_n \nabla_H^2 \varphi_n \, u_n \right\} e^{i\varphi_n} = 0 . \tag{65}$$

In this section ∇_H denotes the gradient with respect to the horizontal coordinates x and y.

Up to this point no approximation has been made. Now we use that the amplitude of the modes and the local modes themselves vary slowly on a scale of a wavelength, which is formally incorporated by setting

$$A_n = A_n(\varepsilon x, \varepsilon y, z), \qquad u^{(n)} = u^{(n)}(\varepsilon x, \varepsilon y, z), \tag{66}$$

with the perturbation parameter ε defined in (58). Using this in (65) gives

$$\sum_n \left\{ \varepsilon^2 \nabla_H^2 (A_n u_n) - A_n |\nabla_H \varphi_n|^2 u_n + \frac{\omega^2}{c_n^2} A_n u_n \right\} e^{i\varphi_n}$$
$$+ i \left\{ 2\varepsilon^2 \nabla_H \varphi_n \cdot \nabla_H (A_n u_n) + A_n \nabla_H^2 \varphi_n \, u_n \right\} e^{i\varphi_n} = 0 . \tag{67}$$

It will be shown later that the term $\nabla_H^2 \varphi_n$ is of order ε. Using this, the terms independent of ε are

$$\sum_n \left\{ -|\nabla_H \varphi_n|^2 + \frac{\omega^2}{{c_n}^2} \right\} A_n u_n e^{i\varphi_n} = 0 . \tag{68}$$

Equation (68) consists of a sum of contributions from different modes. The orthogonality of the modes can be used to split this expression into separate expressions for the different modes. Multiplying (68) by u_m, integrating over z and using the orthogonality relation (62) of the modes gives

$$\left|\nabla_H \varphi_m(x, y)\right|^2 = \frac{\omega^2}{c_m^2(x, y)} . \tag{69}$$

This expression does not contain a sum over modes. This means that the phase of each mode does not depend on other modes. Equation (69) is nothing but the eikonal equation of geometric ray theory (Kline and Kay, 1965; Aki and Richards, 1980; Kravtsov, 1988). The wave velocity that appears in the eikonal equation is the local phase velocity of the mode under consideration. The rays that are associated with this mode are defined as the curves that are everywhere perpendicular to the phase $\varphi_m(x, y)$. Using a standard derivation (Aki and Richards, 1980) it follows that these rays satisfy the equation of kinematic ray tracing:

$$\frac{d}{ds}\left(\frac{1}{c_n(x, y)} \frac{d\mathbf{r}}{ds} \right) = \nabla \left(\frac{1}{c_n(x, y)} \right) . \tag{70}$$

Since the phases of different modes are decoupled, every mode travels along its own ray. Using $k_n = \omega/c_n$ the phase of mode n given by

$$\varphi_n = \int_{ray\, n} k_n ds , \tag{71}$$

where the integral is over the ray for that mode.

It follows from the eikonal equation (69) that $\nabla_H \varphi_n$ depends weakly on x and y: $\nabla_H \varphi_n = \nabla_H \varphi_n (\varepsilon x, \varepsilon y)$, which means that $\nabla_H^2 \varphi_n$ is indeed of order ε. Note that when the variation in the properties of the medium or in the amplitude is not smooth on a scale of a wavelength, mode coupling occurs because in that case ε is not a small parameter. In that case the term $\varepsilon^2 \nabla_H^2 (A_n u_n)$ in (67) gives rise to mode coupling because u_n and the horizontal gradient $\nabla_H(u_m)$ of another mode are *not* orthogonal.

The analysis given here can be extended to more complex system of modes (Bretherton, 1968; Woodhouse, 1974; Babich *et al.*, 1976; Yomogida, 1987). A more detailed analysis reveals not only that the phase is described by the eikonal equation (69) but that the amplitude satisfies the transport equation that is characteristic for geometric-ray theory. The crucial elements in the analysis are that (i) there are local modes with an appropriate orthogonality relation, (ii) the coordinate system can be divided into coordinates that carry the guided waves and transverse coordinates in which the medium varies slowly and (iii) the perturbation procedure outlined here can be applied. The end result is that for media that vary smoothly in the horizontal direction:

- The modes decouple, during the propagation each mode is given by the local mode, which is defined as the mode in a system that is translationally invariant and which has the properties of the model at that location.
- Each mode propagates with a local phase velocity that is the phase velocity of the local mode. The phase of the propagating mode is accounted for by the eikonal equation (69).
- The energy associated with each mode travels along rays defined by the local phase velocity. The amplitude of each mode contribution is governed by the transport equation of geometric-ray theory.

It is of interest to consider a medium that consists of a reference medium with wave number $k_n^{(0)}$ for mode n that is weakly perturbed. The wave number of mode n is then perturbed with a local change $\delta k_n(x, y)$. Using Fermat's principle (Aldridge, 1994) and ignoring a perturbation of the amplitude, the wave field is given by

$$u = \sum_n A_n^{(0)} \exp\left(i \int \left(k_n^{(0)} + \delta k_n \right) ds \right). \tag{72}$$

This is called in surface wave propagation the WKBJ approximation (seismologists like to add the name of Jeffreys (1924) to the names of Wentzel, Krames and Brillouin). Note that this approximation is not quite the same as the WKB approximation (Bender and Orszag, 1978) because the perturbation of the amplitude is not accounted for and because only the first-order change of the wave number is taken into account. The WKBJ approximation (72) forms the basis of the partitioned waveform inversion of Nolet (1990), which is an important technique in the large-scale inversion of surface wave data in seismology.

In the WKBJ approximation (72) the modes do not interact with other modes. However, the wave field depends nonlinearly on the perturbation of the medium because this perturbation enters the term δk_n in the exponent. This means that the WKBJ approximation accounts for self-interactions of the mode under consideration to *any* order. This is shown diagrammatically in Fig. 3.

§ 9. Role of Mode Coupling

The theory of scattering of surface waves does not only play a role in the analysis of surface wave data; if the set of surface wave modes is complete, the complete response of the system can be given as a sum over surface wave modes. For a finite body, such as the Earth, one can describe the full response exactly as a sum over the normal modes of the Earth (Dahlen and Tromp, 1998). For an infinite body such as a layered half-space, the surface wave modes unfortunately do *not* form a complete set. As an example consider a homogeneous half-space, this system carries no Love modes and only one Rayleigh mode (Aki and Richards, 1980). This single Rayleigh mode cannot account for the full response of the half-space, which is due to the fact that in an infinite system, waves can radiate to infinity, whereas the surface wave modes are trapped near the surface. This radiation is accounted for by propagating body waves that are associated with the continuous part of the spectrum. There is in fact a choice whether one accounts for these wave phenomena by a summation of modes or by a integration over rays (Tain-Fu and Er-Chang, 1982; Felsen, 1984; Haddon, 1986).

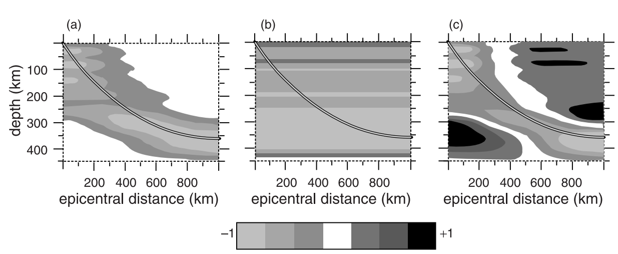
However, not all body wave phases radiate energy to infinity. In seismological applications body waves are refracted towards the Earth's surface because of the strong increase of the velocity with depth or because they are reflected or refracted upwards by discontinuities. These body waves can be described well by a superposition of surface wave modes (Nolet *et al.*, 1989; Marquering and Snieder, 1995). As an alternative, one can approximate an infinite system by a finite system by placing a reflector at a depth that is so great that the reflected waves arrive too late to be of importance; this approximation is called the *locked-mode* approximation (Harvey, 1981). The key point is that when a certain wave can be accounted for by a superposition of surface wave modes, then it is possible to use the theory of this chapter to account for the perturbation of the wave field by the perturbation of the medium.

The theory of the previous section seems to imply that mode coupling does not play an important role. However, mode coupling is crucial to account for some physical phenomena. This is illustrated with an example shown in Fig. 9. In a realistic Earth model a wave field is excited in the upper left-hand corner. A ray (indicated with the curved line) dives into the Earth and has a turning point near the lower-right corner. Finally the ray propagates to the surface, but since this part of the ray is identical to the down-going part it is not shown. In many applications it is important to know the sensitivity of the wave field to perturbations of the medium. In a linear approximation this is described by a sensitivity function

$$\delta u = \int\int K(x,z)\,\delta\beta(x,z)\,dx\,dz, \tag{73}$$

where for simplicity only the shear velocity β is perturbed. When the response of the system can be written by a superposition of surface wave modes, then the Born approximation (32) can be used to compute the sensitivity kernel $K(x,z)$. This kernel, based on the Born approximation of surface waves in two dimensions (Marquering and Snieder, 1995), is shown in the left panel of Fig. 9. In the sum over modes only the modes that have a phase slowness (defined as the reciprocal of the phase velocity) that is close to the slowness of the body wave that propagates along the

Figure 9 Spatial distribution of the sensitivity kernel $K(x,z)$ to the shear velocity perturbation computed by summation of surface wave modes when (a) all the modes are coupled (all σ,ν), (b) coupling to other modes is ignored ($\sigma=\nu$ only) and (c) only coupling to other modes is taken into account ($\sigma \neq \nu$). The source is in the upper left corner. An *S* wave propagates along the path indicated by the curved line. Only the modes with a phase velocity close to the horizontal velocity of the *S* wave are used in the computation.



ray were taken into account. The resulting sensitivity function is centred around the ray. The finite width of the sensitivity function is associated with the finite frequency of the body wave, which results in a Fresnel zone with a finite width (Kravtsov, 1998).

In the Born approximation the perturbation of the wave field consists of a double sum over modes. Suppose the mode conversion between surface wave modes is switched off; that is, the contributions $\sigma \neq \nu$ are removed from the sum (32). In that case the sensitivity function is given by the middle panel of Fig. 9. The sensitivity does not depend on the horizontal coordinates, and the resulting sensitivity function has the unphysical property that it is not centred on the geometric ray. The mathematical reason for this artefact is not difficult to find. Let us assume that the wave propagates over a horizontal distance L and that we consider the contribution of a heterogeneity at an intermediate location x. In the Born approximation (32) the incoming mode σ gives a contribution $\exp ik_\sigma x$ and the outgoing mode ν gives a contribution $\exp ik_\nu(L-x)$. This means that as far as the dependence on the horizontal coordinates is concerned, the contribution of the mode pair (σ, ν) to the sensitivity function is given by

$$K(x,z) = \sum_{\sigma,\nu} (\cdots)\, e^{ik_\nu L}\, e^{i(k_\sigma - k_\nu)x} . \tag{74}$$

When only the modes $\sigma = \nu$ are taken into account in this sum, the sensitivity kernel does not depend on the horizontal coordinate x. It is the interference term $\exp\left(i\left(k_\sigma - k_\nu\right)x\right)$ that gives the horizontal dependence of the sensitivity function that is needed to localise the sensitivity function near the ray. The coupling between different modes is crucial for achieving this. In fact, the right panel of Fig. 9 shows the contribution to the sensitivity function by the mode pairs that are different ($\sigma \neq \nu$). The sum of the right and middle panels gives the left panel. It can be seen that the sum of the intramode coupling ($\sigma = \nu$) and the intermode coupling ($\sigma \neq \nu$) leads to a cancellation of the sensitivity function far from the geometric ray. The description of the perturbation of body waves by a sum over coupled modes has found important applications in seismology (Li and Romanowicz, 1995; Marquering *et al.*, 1996, 1999; Zhao and Jordan, 1998).

§ 10. Recent Developments

Although much progress has been made in the development of surface wave scattering theory, two main problems can be defined. The first is the coupling of surface waves to body waves; the second is the multiple scattering of surface waves. As argued in the previous section, surface waves can be coupled to body waves that radiate energy to infinity (or transport energy from infinity). Expression (43) of §6 of Chapter 1.7.1 gives the general Born approximation for elastic waves. The main problem for incorporating these body waves in the theory is not the coupling of the body waves and the surface waves, which is described in the terms proportional to $\rho^{(1)}$ and $c^{(1)}_{nklj}$ of Eq. (43) of Chapter 1.7.1. The problem lies in finding an adequate description of the Green's tensors $\mathbf{G}$ in that expression. For surface waves, the Green's tensor (18) can be used. For an arbitrary model the modes can be computed numerically and the surface wave Green's tensor is then known. For body waves, there is no expression for the Green's tensor in closed form for an arbitrary layered medium.

One approach that has been taken is to assume that the body waves propagate through a homogeneous half-space. In that case the body wave Green's tensor is known in closed form. This was used by Hudson (1967) in his description of the coupling of body waves and surface waves by the topography on a homogeneous half-space. Similarly, Odom (1986) used a layer with variable thickness over a homogeneous half-space to account for the interaction between surface waves and body waves by the topography of internal interfaces. A different approach was taken by Maupin (1996), who introduced the concept of *radiation modes* in an inhomogeneous half-space. The interaction of these radiation modes with the surface wave modes accounts for the interaction between body waves and surface waves.

In general, multiple scattering problems are difficult to solve and closed form solutions are known only for idealised situations. However, a number of techniques have been formulated that allow for a relatively efficient calculation of multiple-scattered surface waves. One approach consists in applying *invariant embedding* to surface wave scattering. Suppose one has a system where the scatterers are located in a finite region of space. The idea of invariant embedding is that one does not account for the scattering properties of the whole medium. Instead one describes how the scattering properties change when the scattering region is enlarged. This technique can account in a very efficient way for a variety of scattering and diffusion problems (Bellmann and Kalaba, 1960) as well as for the propagation of elastic body waves through an inhomogeneous stack of layers (Tromp and Snieder, 1989). Invariant embedding is well suited to account for the multiple scattering of surface waves by horizontal variations in the properties of the medium. This theory has been formulated using a basis of fixed surface wave modes (Kennett, 1984) or a basis of local surface wave modes that depend only on the local structure of the medium (Odom, 1986; Maupin, 1988).

Another technique that has been used to account for the multiple scattering of surface waves is the multiple-scattering theory of Waterman (1968), where the full scattering properties of an isolated scatterer are computed. This approach has been applied to scattering of surface waves by an isolated scatterer by Bostock (1991). The technique of Waterman has been extended by Bostock (1992) to a system of discrete scatterers that scatter surface waves in a theory that accounts for multiple scattering of surface waves by a conglomerate of discrete scatterers. Using appropriate modal expansions in a Cartesian geometry (Kennett, 1998) or on a sphere (Friederich, 1999), coupled equations for the modal coefficients can be derived to account for multiple scattering of surface waves in 3D. Multiple forward scattering can be handled efficiently by integrating these coupled equations recursively in the direction of the propagating wave front (Friederich *et al*., 1993; Friederich, 1999).

Acknowledgements

I thank Valérie Maupin, Wolgang Friederich, Bob Odom and Bob Nowack for their critical and constructive comments on this manuscript.

Note

1. Expression (7.78) of Aki and Richards (1980) should contain a term k^2 in the denominator.

References

Aki, K. and Richards, P. G. 1980, *Quantitative Seismology*. Freeman, San Francisco.

Aldridge, D. F. 1994, Linearization of the eikonal equation. *Geophysics* **59**, 1631–1632.

Babich, V. M., Chikhachev, B. A. and Yanovskaya, T. B. 1976, Surface waves in a vertically inhomogeneous elastic half-space with a weak horizontal inhomogeneity. *Izv. Earth Phys.* **12**, 242–245.

Bellmann, R. and Kalaba, R. 1960, Invariant embedding and mathematical physics I. Particles processes. *J. Math. Phys. 1*, 280–308.

Bender, C. M. and Orszag, S. A. 1978, *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill, New York.

Bostock, M. G. 1991, Surface wave scattering from 3-D obstacles. *Geophys. J. Internat.* **104**, 351–370.

Bostock, M. 1992, Multiple scattering of surface waves, *Geophys. J. Internat.* **108**, 52–70.

Brandenburg, A. and Snieder, R. 1989, The attenuation of surface waves due to scattering. *Geophys. J.* **98**, 183–194.

Brekhovskikh, L. M. and Lysanov, Yu. 1982, *Fundamental of Ocean Acoustics*. Springer-Verlag, Berlin.

Bretherton, F. P. 1968, Propagation in slowly varying waveguides. *Proc. R. Soc. A* **302**, 555–576.

Červený, V. and Hron, F. 1980, The ray series method and dynamical ray tracing system for three-dimensional inhomogeneous media. *Bull. Seismol. Soc. Am.* **70**, 47–77.

Dahlen, F. A. and Tromp, J. 1998, *Theoretical Global Seismology*. Princeton Univ. Press, Princeton.

Felsen, L. B. 1984, Progressing and oscillatory waves for hybrid synthesis of source-excited propagation in layered media. *Geophys. J. R. Astron. Soc.* **79**, 11–33.

Friederich, W. 1999, Propagation of seismic shear and surface waves in a laterally heterogeneous mantle by multiple forward scattering. *Geophys. J. Internat.* **136**, 180–204.

Friederich, W., Wielandt, E. and Stange, S. 1993, Multiple forward scattering of surface waves: Comparison with an exact solution and Born single-scattering methods. *Geophys. J. Internat.* **112**, 264–275.

Haddon, R. A. W. 1986, Exact evaluation of the response of a layered elastic medium to an explosive point source using leaking modes. *Bull. Seismol. Soc. Am.* **76**, 1755–1775.

Harvey, D. J. 1981, Seismogram synthesis using normal mode superposition: The locked mode approximation. *Geophys. J. R. Astron. Soc.* **66**, 37–70.

Hudson, J. A. 1967, Scattered surface waves by a surface obstacle. *Geophys. J. R. Astron. Soc.* **13**, 441–458.

Hudson, J. A. 1970, The attenuation of surface waves by scattering. *Proc. Camb. Phil. Soc.* **67**, 215–223.

Jeffreys, H. 1924, On certain approximate solutions of linear differential equations of second order. *Phil. London Math. Soc.* **23**, 428–436.

Keilis-Borok, V.I., Levshin, A. L., Yanovskaya, T. B., Lander, A. V., Bukchin, B. G., Barmin, M. P., Ratnikova, L. I. and Its, E. 1989, *Seismic Surface Waves in a Laterally Inhomogeneous Earth*. Kluwer, Dordrecht.

Kennett, B. L. N. 1984, Guided wave propagation in laterally varying media – I. Theoretical development. *Geophys. J. R. Astron. Soc.* **79**, 235–255.

Kennett, B. L. N. 1990, Guided wave attenuation in laterally varying media. *Geophys. J. Internat.* **100**, 415–422.

Kennett, B. L. N. 1998, Guided waves in three-dimensional structures. *Geophys. J. Internat.* **133**, 159–174.

Kline, M. and Kay, I. W. 1965, *Electromagnetic Theory and Geometrical Optics*. Interscience, New York.

Kravtsov, Ya. A. 1988, Rays and caustics as physical objects. In *Prog. in Optics, XXVI* (E. Wolf, Ed.), pp. 227–348. Elsevier, Amsterdam.

Li, X. D. and Romanowicz, B. 1995, Comparison of global waveform inversions with and without considering cross-branch modal coupling. *Geophys. J. Internat.* **121**, 695–709.

Malischewsky, P. 1987, *Surface Waves and Discontinuities*. Akademie Verlag, Berlin.

Marquering, H. and Snieder, R. 1995, Surface-wave mode coupling for efficient forward modelling and inversion of body-wave phases. *Geophys. J. Internat.* **120**, 186–208.

Marquering, H., Snieder, R. and Nolet, G. 1996, Waveform inversions and the significance of surface wave mode coupling. *Geophys. J. Internat.* **124**, 258–278.

Marquering, H., Dahlen, F. A. and Nolet, G. 1999, Three-dimensional sensitivity kernels for finite-frequency traveltimes: The banana-doughnut paradox. *Geophys. J. Internat.* **137**, 805–815.

Maupin, V. 1988, Surface waves across 2-D structures: A method based on coupled local modes. *Geophys. J.* **93**, 173–185.

Maupin, V. 1996, The radiation modes of a vertically varying halfspace: A new representation of the complete Green's function in terms of modes. *Geophys. J. Internat.* **126**, 762–780.

Mueller, S. and Talwani, M. 1971, A crustal section of the eastern Alps based on gravity and seismic refraction data. *Pure Appl. Geophys.* **85**, 226.

Nakanishi, I. 1993, Surface wave tomography: Velocity and Q. In *Seismic Tomography*, (H. M. Iyer and K. Hirahara, Eds.), pp. 92–132. Prentice-Hall, London.

Nolet, G. 1977, The upper mantle under Western Europe inferred from the dispersion of Rayleigh modes. *J. Geophys.* **43**, 265–286.

Nolet, G. 1990, Partitioned waveform inversion and two-dimensional structure under the network of autonomously recording seismographs. *J. Geophys. Res.* **95**, 8499–8512.

Nolet, G., Sleeman, R., Nijhof, V. and Kennett, B. L. N. 1989, Synthetic reflection seismograms in three dimensions by a locked-mode approximation. *Geophysics* **54**, 350–358.

Odom, R. I. 1986, A coupled mode examination of irregular waveguides including the continuous spectrum. *Geophys. J. R. Astron. Soc.* **86**, 425–453.

Park, M. and Odom, R. I. 1999, The effect of stochastic rough interfaces on coupled-mode elastic waves. *Geophys. J. Internat.* **136**, 123–143.

Snieder, R. 1986a, 3D linearized scattering of surface waves and a formalism for surface wave holography. *Geophys. J. R. Astron. Soc.* **84**, 581–605.

Snieder, R. 1986b, The influence of topography on the propagation and scattering of surface waves. *Phys. Earth Planet. Inter.* **44**, 226–241.

Snieder, R. 1988, The optical theorem for surface waves and the relation with surface wave attenuation. *Geophys. J.* **95**, 293–302.

Snieder, R. 1993, Global inversions using normal modes and long-period surface waves. In *Seismic tomography* (H. M. Iyer and K. Hirahara, Eds.), pp. 23–63. Prentice-Hall, London.

Snieder, R. and Trampert, J. 1999, Inverse problems in geophysics. In *Wavefield Inversion* (A. Wirgin, Ed.), pp. 119–190. Springer-Verlag, New York.

Steg, R. G. and Klemens, P. G. 1974, Scattering of Rayleigh waves by surface defects. *J. Appl. Phys.* **45**, 23–29.

Tain-Fu, G. and Er-Chang, S. 1982, The transformation between the mode representation and the generalized ray representation of a sound field. *J. Sound and Vibration* **80**, 105–115.

Takeuchi, H. and Saito, M. 1972, Seismic surface waves. In *Seismology: Surface Waves and Earth Oscillations, (Methods in Computational Physics, Vol. 11)*, (B. A. Bolt, Ed.). Academic Press, New York.

Tarantola, A. 1986, A stategy for nonlinear inversion of seismic reflection data. *Geophysics* **51**, 1893–1903.

Tromp, J. and Snieder, R. 1989, The reflection and transmission of plane P- and S-waves by a continuously stratified band: A new approach using invariant embedding. *Geophys. J.* **96**, 447–456.

Waterman, P. C. 1968, New formulation of acoustic wave scattering. *J. Acoust. Soc. Am.* **45**, 1417–1429.

Woodhouse, J. H. 1974, Surface waves in a laterally varying layered structure. *Geophys. J. R. Astron. Soc.* **37**, 461–490.

Wu, R. S. and Aki, K. 1985, Scattering characteristics of elastic waves by an elastic heterogeneity. *Geophysics* **50**, 582–595.

Yomogida, K. 1987, Gaussian beams for surface waves in transversely isotropic media. *Geophys. J. R. Astron. Soc.* **88**, 297–304.

Zhao, L. and Jordan, T. H. 1998, Sensitivity of frequency-dependent traveltimes to laterally heterogeneous, anisotropic Earth structure. *Geophys. J. Internat.* **133**, 683–704.