Linear Algebra. Pre-final practice

Major Topics
Review:

- *Matrices*: how to calculate sum and products.
- *Special matrices*: transpose, inverse, symmetric, skew-symmetric, diagonal, orthogonal.
- *Determinants*: calculate (2x2). Simple properties [det (*AB*)= det (*A*) det (*B*)]
- *Eigenvalues*: how to find them
- *Eigenvectors*: definition, how to find them, properties
- *Diagonalization*: how to do it
- *Gaussian elimination*: how to do it, rank

1. Two matrices are given:

$$A = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Find A^2

$$A^2 = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

Find *AB*.

$$AB = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & 8 \end{pmatrix}$$

Verify that $A(A+B) = A^2 + AB$.

$$A + B = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix},$$

$$A(A+B) = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 17 \end{pmatrix};$$

$$A^2 + AB = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 17 \end{pmatrix}$$

Find A^{-1}: [$A^{-1} = \frac{1}{\det(A)}(A_{jk})^T$, using cofactors; for 2x2]:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$\det(A) = -9;$

$$A^{-1} = \frac{1}{-9}\begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}$$

Find $\det(A^{-1}) = \dfrac{1}{\det A} = -1/9;$

Find the eigenvalues of A:

$$\det(A-\lambda I)=\begin{vmatrix}-1-\lambda & 2\\ 4 & 1-\lambda\end{vmatrix}=-(1-\lambda^2)-8=\lambda^2-9;$$

$\lambda_1=3;\ \lambda_2=-3;$

Find the trace: $trace(A)=\lambda_1+\lambda_2=a_{11}+a_{22}=3+(-3)=-1+1=0;$

Note: $\det A=\lambda_1\lambda_2$

Find the linearly independent eigenvectors of A. Show that you can choose them to be:

$\mathbf{x}_1{}^T=(1\quad 2);\ \mathbf{x}_2{}^T=(1\quad -1).$

Using the eigenvectors, let P be a 2x2 matrix.

$$\lambda_1=3\Rightarrow\begin{pmatrix}-1-3 & 2\\ 4 & 1-3\end{pmatrix}=\begin{pmatrix}-4 & 2\\ 4 & -2\end{pmatrix}\Rightarrow\begin{pmatrix}-4 & 2\\ 4 & -2\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow -4x_{11}+2x_{12}=0$$

$x_{11}=1\Rightarrow x_{12}=2$

$$\lambda_2=-3\Rightarrow\begin{pmatrix}-1+3 & 2\\ 4 & 1+3\end{pmatrix}=\begin{pmatrix}2 & 2\\ 4 & 4\end{pmatrix}\Rightarrow\begin{pmatrix}2 & 2\\ 4 & 4\end{pmatrix}\begin{pmatrix}x_{21}\\ x_{22}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow 2x_{21}+2x_{22}=0$$

$x_{11}=1\Rightarrow x_{12}=-1$

Verify that P can be used to diagonalize A.

$$P=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$$

$$P^{-1}AP=D;\quad \det(P)=-1-2=-3;\quad P^{-1}=\frac{1}{-3}\begin{pmatrix}-1 & -2\\ -2 & 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$$

$$P^{-1}AP=\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 2\\ 4 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1+4 & -1-2\\ 4+2 & 4-1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}3 & -3\\ 6 & 3\end{pmatrix}=$$

$$\frac{1}{3}\begin{pmatrix}3+6 & 0\\ 6-6 & -6-3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}9 & 0\\ 0 & -9\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix};$$

Solutions of selected problems of Homework from Chapter 6

6.2. 3
For

$$C = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

find C^2, C^TC, CC^T.

The matrix C is symmetric, therefore
$C^2 = C^TC = CC^T$. Moreover, C^2 is symmetric because C is symmetric.
Proof.
$C^2 = CC^T$, thus $(C^2)^T = (CC^T)^T = (C^T)^TC^T = CC^T = C^2$,
using the property $(AB)^T = B^TA^T$

6.2. 5
For

$$a = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad d = \begin{pmatrix} 4 & 3 & 0 \end{pmatrix}$$

find a^Td, a^Td^T , da, ad.

a^Td is not defined.
$a^Td^T = (da)^T = 1*4+4*3+3*0=16$.

$$ad = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1*4 & 1*3 & 1*0 \\ 4*4 & 4*3 & 4*0 \\ 3*4 & 3*3 & 3*0 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{pmatrix}$$

6.2.10(a).
Show that for any A the matrix $B=AA^T$ is square and symmetric.
Show that AB with symmetric A and B is symmetric if and only if A and B commute.
Find all real square matrices that are n=both symmetric and skew-symmetric.

Let A be mxn. Then A^T is nxm. Thus $B=AA^T$ is defined and it is a square matrix of size mxm.
To show that is symmetric:
$B^T=(AA^T)^T=AA^T=B$ – symmetric.
Next, we are given that $AB=A^TB^T$ as A and B are symmetric.
$(AB)^T=(A^TB^T)^T=BA = AB$ (as A and B commute) – AB is symmetric.
Also, in the opposite direction, if AB is symmetric, A and B are commute.

6.7.10. Show that $(A^2)^{-1}=(A^{-1})^2$

Consider $A^2 (A^2)^{-1}= A^2 (A^{-1})^2=AA A^{-1} A^{-1}=I$.

Also, $(A^2)^{-1} A^2 = (A^{-1})^2 A^2 = A^{-1} A^{-1}AA =I$.

6.7.12. Show that $(A^T)^{-1}=(A^{-1})^T$

Consider $A^T (A^T)^{-1}= A^T (A^{-1})^T = (A^{-1}A)^T= I^T = I$, and

$(A^T)^{-1} A^T = (A^{-1})^T A^T = (AA^{-1})^T=I$.

6.7.13. Show that A^{-1} is symmetric is A is symmetric

Consider $(A^{-1})^T= (A^T)^{-1}$ (see prev. problem)

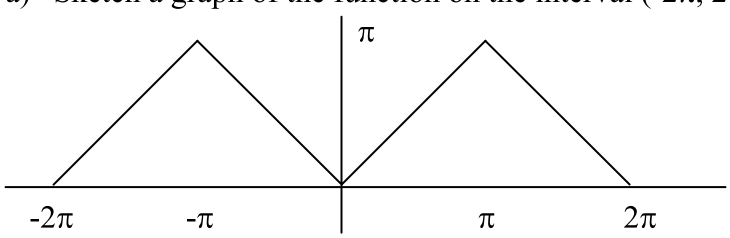
Since A is symmetric, $(A^{-1})^T= (A^T)^{-1} = A^{-1}$.

Fourier Analysis. Pre-final practice

Given is the 2π periodic function :

$f(x) = |x| \quad -\pi < x < \pi$

a) Sketch a graph of the function on the interval (-2π, 2π).



b) Characterize the function: odd, or even.

Even

c) Write the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

d) Compute the general formulae for the coefficients:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \left.\frac{2x^2}{\pi 2}\right|_0^{\pi} = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{\pi}\left[\frac{x \sin nx}{n} - \frac{1}{n}\int_0^{\pi} \sin nx dx\right] = \frac{2}{\pi}\left.\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]\right|_0^{\pi} =$$

$$\left.\frac{2}{\pi}\frac{\cos nx}{n^2}\right|_0^{\pi} = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2} = \begin{cases} \dfrac{-4}{n^2\pi}, & \text{if odd} \\ 0, & \text{if even} \end{cases}$$

e) Write the Fourier series in the general form

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2}.$$

f) Write the first three members of the Fourier series:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right), \quad |x| \le \pi.$$

Partial Differential Equations. Pre-final practice

Solve the following problem for the wave equation as an infinite sum of terms called *modes* (each mode oscillates infinitely in time *t* and dies out -- damping effect):

$$\frac{\partial^2 u}{\partial x^2} - 2r\frac{\partial u}{\partial t} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2};$$

$L=\pi$ and $c=1$.

Assume that the damping constant r satisfies the condition:

$0 < r < 1$.

Write a solution for homogeneous boundary conditions, a given initial displacement $f(x)$ and initial velocity = 0.

Solution.

Step 1

Separate variables:

$$u(x,t) = X(x)T(t);$$

$$X''T - 2rXT' = XT''.$$

Step 2

Separate PDE:

$$\frac{X''}{X} = \frac{T''}{T} + 2r\frac{T'}{T} = k.$$

Step 3

Find eigenvalues and solution of ODE in x:

$$k = -n^2 \Rightarrow X(x) = E\sin nx,$$

$$n = 1,2,\ldots,$$

Step 4

Write ODE in t:

$$T'' + 2rT' + n^2T = 0.$$

Step 5

Write characteristic equation:

$$m^2 + 2rm + n^2 = 0 \Rightarrow$$

Step 6

Find its solution

$$m = -r \pm i\sqrt{n^2 - r^2},$$

$$0 < r < 1 <= n.$$

Write general solution for ODE in t:

Step 7

$T(t) =$

$e^{-rt}\left\{C_1 \cos\sqrt{n^2 - r^2}\,t + D_1 \sin\sqrt{n^2 - r^2}\,t\right\}.$

Write general solution for PDE:

$u(x,t) =$

Step 8

$\sum_{n=1}^{\infty} e^{-rt}\left\{A_n \cos\sqrt{n^2 - r^2}\,t + B_n \sin\sqrt{n^2 - r^2}\,t\right\}\sin nx.$

Analyze initial conditions:

Step 9

$u(x,0) = f(x), and$

$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L.$

Write general solution of PDE:

Step 10

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-rt} \cos\sqrt{n^2 - r^2}\,t \sin nx.$

Problems from previous tests

1. Given a wave equation, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}$, $0 < x < \pi$, $t > 0$, with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial conditions $u(x,0) = f(x)$ and $u'(x,0) = 0$ perform the following:

(i) Using the method of separation of variables show that the general solution for a vibrating string has the form:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-t/2}\left(A_n \cos\lambda_n t + B_n \sin\lambda_n t\right)\sin nx,$$

A_n, B_n $(n = 1,2,\ldots)$ are arbitrary constants; and

$$\lambda_n = \sqrt{n^2 - \frac{1}{4}}.$$

Solution:

$u(x,t) = X(x)T(t);$

$\frac{X''}{X} = \frac{T''}{T} + \frac{T'}{T} = k.$

$k = -n^2 \Rightarrow X(x) = E \sin nx,$

$n = 1,2,\ldots,$

$T'' + T' + n^2 T = 0.$

$T(t) = e^{mt} \Rightarrow m^2 + m + n^2 = 0 \Rightarrow$

$m = -(1/2) \pm i\sqrt{n^2 - (1/2)^2} = -1/2 \pm i\lambda_n .$

$\left(\lambda_n = \sqrt{n^2 - (1/2)^2}\right)$

$T_n(t) = A_n e^{-t/2} \cos \lambda_n t + B_n e^{-t/2} \sin \lambda_n t$;

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} e^{-t/2}\left(A_n \cos\lambda_n t + B_n \sin\lambda_n t\right) \sin nx,\ n = 1,2,\ldots$$

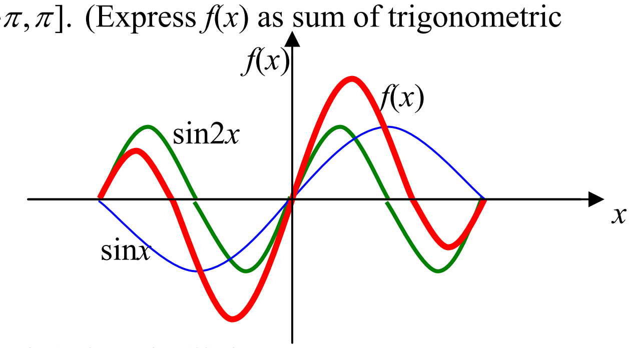
(ii) Assuming that the initial deflection is

$$f(x) = 2\sin\frac{3}{2}x\cos\frac{1}{2}x,$$

which extension of the half-range Fourier series obtained in (i) should be used?
Answer. Odd extension.

(iii) Sketch the graph of $f(x)$ for $x \in [-\pi, \pi]$. (Express $f(x)$ as sum of trigonometric functions.)

$$f(x) = \sin x + \sin 2x$$



(iv) Show that the solution of PDE with *f(x)* given in (ii) is

$$u(x,t) = e^{-t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)\sin x + \cos\left(\frac{\sqrt{15}}{2}t\right)\sin 2x\right].$$

Solution.

In the formula for general solution, the term $B_n \sin \lambda_n t$ must be 0 since the initial velocity is equal 0.
It becomes

$$u(x,t) = \sum_{n=1}^{\infty} e^{-t/2}\left(A_n \cos\lambda_n t\right)\sin nx .$$

For $t = 0$,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx .$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin x + \sin 2x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx \sin x dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx \sin 2x dx$$

$A_1 = A_2 = 1,\ A_n = 0,\ n = 3,4,\ldots$

as follows from orthogonality of the trigonometric system on the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}.$$

Also,

$$\lambda_1 = \sqrt{1^2 - (1/2)^2} = \frac{\sqrt{3}}{2},$$

$$\lambda_2 = \sqrt{2^2 - (1/2)^2} = \frac{\sqrt{15}}{2}.$$

2. **Half-range Fourier series for solving PDE**

For a given heat-transfer equation on the interval $0 < x < \pi$ with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial condition

$u(x,0) = f(x) = 1,$

the general solution has the following form:

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \sin nx\right) e^{-\lambda_n^2 t},$$

compute the coefficients A_n, B_n in the half-range Fourier series (odd extension).

Solution.

Expand the initial function

$f(x) = 1,$

into the Fourier series:

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{2}{\pi}\left(\frac{-(\cos n\pi - 1)}{n}\right) = \frac{2}{\pi}\left(\frac{-((-1)^n - 1)}{n}\right),,$$

For odd indexes:

$$B_1 = \frac{4}{\pi}, \quad B_3 = \frac{4}{3\pi}, \quad B_5 = \frac{4}{5\pi}, \ldots,$$

For even indexes $m = 2n$:

$$B_m = 0.$$

3. **Solution of PDE by Laplace transform**

Solve equation:

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x; \quad u(x,0) = 1,\ x \ge 0; \quad u(0,t) = 0,\ t \ge 0$$

by Laplace transform.

Recall Laplace transforms:

$f(t)$	Laplace transform of $f(t)$
1	$\frac{1}{s}$

t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$

Solution.

Apply LT in *t*. Obtain:

$$x\frac{\partial U}{\partial x} + sU - 1 = \frac{x}{s};$$

$$U(0,s) = 0,$$

or, in standard form,

$$\frac{\partial U}{\partial x} + \frac{s}{x}U = \frac{1}{s} + \frac{1}{x}.$$

Integrating factor:

$$F = e^{\int (s/x)dx} = e^{s\ln x} = x^s.$$

$$\frac{\partial (x^s U)}{\partial x} = \frac{1}{s}x^s + x^{s-1} \Rightarrow$$

$$x^s U = \frac{1}{s(s+1)}x^{s+1} + \frac{x^s}{s} + A(s).$$

$$U(0,s) = 0 \Rightarrow A(s) = 0 \Rightarrow$$

$$U(x,s) = \frac{x}{s(s+1)} + \frac{1}{s} =$$

$$= x\left\{\frac{a}{s} + \frac{b}{s+1}\right\} + \frac{1}{s}$$

$$= x\left\{\frac{as + a + bs}{s(s+1)}\right\} + \frac{1}{s} \Rightarrow$$

$$a = 1,\ b = -a = -1.$$

$$U(x,s) = x\left\{\frac{1}{s} - \frac{1}{s+1}\right\} + \frac{1}{s} \Rightarrow$$

$$u(x,t) = x(1 - e^{-t}) + 1.$$