MATHEMATICS REVIEW SHEET

SCOTT A. STRONG

Abstract. In 2013, after a near decade hiatus from teaching calculus to first-year students, I returned from an instructional schedule that was almost completely consumed by teaching a growing student body topics from Fourier analysis, differential equations and linear algebra to one that emphasized multi-variate calculus. This document contains a list of topics with problems that I would expect someone who has studied single–variable calculus with series and sequence to have either seen or be able to digest. The narrative around the associated problems, whose answers are nearly always given, is meant to give some derivations that might be missing in ones recall and quickly summarize my own thoughts on the topics.

Please report any errors to sstrong@mines.edu

1. Algebra

1.1. Quadratic forms of a single variable. The study of quadratic form is a deep mathematical topic and speaks to the fact that nonlinearities significantly impede mathematical analysis. With that in mind we review properties of the simplest of all quadratic forms.

1. Prove that the equation \( f(x) = ax^2 + bx + c \) has roots given by \( x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

2. Plot \( f(x) = ax^2 + bx + c \) assuming that \( a, b, c \in \mathbb{R} \).

1.2. Partial fraction decomposition. Polynomials functions are not closed under the operation of division. While differential calculus applied to such rational functions is straightforward, integral calculus does not feel the same. This is because rational functions must be reduced to other algebraically equivalent forms so that the results of integral calculus interface nicely. The fundamental theorem of algebra says that for a reciprocated polynomial of degree \( N \) we have

\[
(1) \quad f(x) = \frac{1}{\sum_{n=0}^{N} a_n x^n} = \prod_{n=1}^{N} \frac{a_n^{-1}}{x - x_n}
\]

where \( x_n \) is the \( n^{th} \) root of the polynomial \( [f(x)]^{-1} \). Partial fractions tells us that there exist \( \alpha_n \) such that

\[
(2) \quad \prod_{n=1}^{N} \frac{a_n^{-1}}{x - x_n} = \sum_{n=1}^{N} \frac{\alpha_n}{x - x_n}
\]

which is useful for the purposes of integration since each term in this sum is integrable through the use of logarithms. To find the coefficients \( \alpha_n \) we must find a common denominator to relate both sides of the previous equality. This is the procedural essence of constructing partial fraction decompositions. It is important to note here that some of the roots \( x_n \) may be complex numbers, which means that the coefficients \( \alpha_n \) may be complex. In this case one may decompose the polynomial up to linear terms multiplying quadratic polynomials. While this makes the integration more difficult, it is often desirable when one wants to work in only the real number system. The following problems provide some practice on the procedure of partial fractions.

1. Using partial fractions, show that

\[
\frac{1}{x^2 + 2x - 3} = \frac{1}{4} \left( \frac{1}{x - 1} - \frac{1}{x + 3} \right).
\]

2. Using partial fractions and synthetic division, show that

\[
\frac{x^3 + 16}{x^3 - 4x^2 + 8x} = 1 + 2 \left( \frac{1}{x} + \frac{x}{x^2 - 4x + 8} \right).
\]
3. Using partial fractions and synthetic division, show that
\[
\frac{1}{(x-1)^3} + \frac{x+1}{x^2+1} + \frac{1}{(x^2+1)^2}.
\]

1.3. **Exponential and trigonometric functions.** If look ahead to section 5.2 of this document then you will find that the name of this subsection should really just be exponential functions. You will also revisit the following questions which are easily answered using the concept of power–series representations.

1. Graph \(e^x, \sin(x), \cos(x)\). Make sure to include both \(x\) and \(y\) intercepts.
2. Explain why \(e^{a+b} = e^a e^b\).
3. Prove that \(\sin^2(x) + \cos^2(x) = 1\) and using this show that \(\sec^2(x) - \tan^2(x) = 1\).
4. Show that \(\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta\).
5. Show that \(\sin(2x) = 2\sin(x) \cos(x)\).
6. Show that \(\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1\).

2. LIMITS

While both differential and integral calculus are useful/usable to/in applied science and engineering, they exist as a consequence of the mathematical notion of a limit. For those that study mathematics the exact language of a limit is discussed. Through this language we justify the existence of a limit. For most this is a topic best left alone in favor of developing trust in the concept of continuity. Before we mention this property we note the following properties of limits,

(4) \(\lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)\),
(5) \(\lim_{x \to p} (f(x) - g(x)) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)\),
(6) \(\lim_{x \to p} (f(x) \cdot g(x)) = \lim_{x \to p} f(x) \cdot \lim_{x \to p} g(x)\),
(7) \(\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} f(x) / \lim_{x \to p} g(x)\), provided that \(\lim_{x \to p} g(x) \neq 0\),

where \(f, g\) are functions such that all stated limits exist.

2.1. **Continuity.** For a function, \(f\), to be continuous at a point, \(x = x_0 \in \mathbb{R}\), we must have

(8) \(\lim_{x \to x_0} f(x) \) exists,
(9) \(\lim_{x \to x_0} f(x) = f(x_0)\).

Roughly speaking, a continuous function takes in “small” changes and produces “small” changes. Otherwise, a function is said to be a discontinuous function. When one says that a function is continuous, it is implied at all points in its domain. It is not horrible to show that the power–function, \(f(x) = x^n\), for \(n \in \mathbb{R}\) is a continuous function. With this in hand one should show that:

1. \(f_n(x) = \sum_{n=0}^{N} a_n x^n\) where \(a_n \in \mathbb{R}\) for \(n = 0, 1, 2, \ldots\) is a continuous function.
2. \(R(x) = f_n(x) / f_k(x)\) is a continuous function.

A similar logic cannot be applied to the exponential and trigonometric functions but rest assured, they are continuous functions.

2.2. **Squeeze theorem.** Two policemen are escorting a drunk prisoner between them, and both officers go to a cell, then (regardless of the path taken, and the fact that the prisoner may be wobbling about between the policemen) the prisoner must also end up in the cell. Mathematically, for \(f, g, h\) defined on \(I \subset \mathbb{R}\), except possibly at \(x_0 \in I\), we suppose that for every \(x \in I, x \neq x_0\) we have

(10) \(g(x) \leq f(x) \leq h(x)\)

such that

(11) \(\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L \in \mathbb{R}\)

implies that \(\lim_{x \to x_0} f(x) = L\). Using this theorem, show that:

1. \(\lim_{x \to 0} \frac{\sin x}{x} = 1\).
2. \(\lim_{x \to 0} \frac{1 - \cos x}{x} = 0\).
3. Differential Calculus

With the notion of limit, it is possible to consider limits of difference quotients which are used to define the slope between two points. The result is an operation that will take in a list of points and output the slope between points infinitesimally close to one another,

\[
\frac{d}{dx}[f] = \frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

where, if you like, \( h = \Delta x \). If the derivative of a function exists at a point then we say that the function is differentiable at this point. Calling a function differentiable without referencing to a particular point is meant to imply differentiability at all points in the functions domain. If a function is differentiable at a point then it is continuous at that point. In the event that you can take \( n \)-many derivatives, without arriving at a function that has discontinuities, we say that the function is \( n \)-smooth. If this can be done infinitely—many times then we just say the function is smooth. We will not consider simple derivatives through limit definition here, instead we remind the reader of some properties of the derivative.

3.1. The product rule. The derivative is an operator which takes in a function and returns a function. Its operation is represented symbolically by \( \frac{d}{dx}[f] = f'(x) \) which makes us think that the operator, \( \frac{d}{dx} \) takes in the data \( f \) and returns its instantaneous rate of change structure \( f' \). Upon defining a new mathematical operation, it is common to ask how it affects pre–existing mathematical rules/structures. For example, the operation of differentiation maps sums to sums in the sense of

\[
\frac{d}{dx}[f + g] = \frac{df}{dx} + \frac{dg}{dx}
\]

However, the same statement seems faulted for products, which we can see to be the case symbolically through the limit definition of the derivative where \( h = fg \).

\[
\begin{align*}
\lim_{a \to 0} (f(a) + g(a)) - (f(a)g(a)) &= \lim_{a \to 0} f(a)g(a + a) - f(a)g(a) \\
&= \lim_{a \to 0} f(a)g(a) + f(a)g(a + a) + f(a)g(a + a) - f(a)g(a) \\
&= \lim_{a \to 0} [f(a)g(a + a) - f(a)g(a)] \cdot g(a + a) + f(a) \cdot [g(a + a) - g(a)] \\
&= \lim_{a \to 0} f(a)g(a) + f(a) \cdot g(a + a) - f(a)g(a) \\
&= f'(a)g(a) + f(a)g'(a).
\end{align*}
\]

The following problems make use of this rule of differentiation.

1. Do the problems listed here [here](https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/productruledirectory/ProductRule.html)

3.2. The chain rule. Function composition is a simple way to create new functions from pre-existing ones and is denoted as \( (f \circ g)(x) = f(g(x)) \) and we say that \( f \) is composed with \( g \). We might like to know, generally, how we can express the derivative of such of a function by knowing the derivatives of the functions in composition. The proof has subtleties which I will not go into here. Instead,

\[
|x| = \begin{cases} 
-x, & x < 0 \\
x, & x \geq 0
\end{cases}
\]

is the prototypical counterexample.

Although, the following is a nice use of our previous problems,

\[
\begin{align*}
\frac{\sin(x + h) - \sin(x)}{h} &= \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\
&= \frac{\cos(x) \sin(h)}{h} - \sin(x) \frac{1 - \cos(h)}{h} - \cos(x), & \text{as } h \to 0,
\end{align*}
\]

which shows that the derivative of sine is cosine.

Another example of an operator on functions is matrix multiplication, which takes in and returns a vector function. Generalizing, it is possible to take in two functions and return a single function. Examples of such a binary operator are scalar addition and multiplication.
the following gives you a symbolic idea of why a rule like the chain rule might exist.

\[
\frac{d}{dx}(f \circ g)(x) \bigg|_{x=a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \\
= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
= f'(g(a))g'(a)
\]

What is not addressed in these formal manipulations is the existence and value of the above limits. Skipping past this hurdle we note the following equivalent notations for the chain rule,

\[ (f \circ g)'(x) = f'(g(x))g'(x) \]

\[ \left[f'(g(x))\right]' = f'(g(x))g'(x) \]

\[ \frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx} \]

the last of which is probably the most useful since it shows that one can think of \( g \) as an intermediate variable. This allows one to extend to the case of multiple compositions quite nicely,

\[ \frac{d}{dx} \left[f(g_1(g_2(g_3(x))))\right] = \frac{df}{dg_1} \frac{dg_1}{dg_2} \frac{dg_2}{dg_3} \frac{dg_3}{dx} \\
= f'(g_1(g_2(g_3(x))))g'_1(g_2(g_3(x)))g'_2(g_3(x))g'_3(x)
\]

To practice the chain rule, given \( f(x) = e^{\sin(x^2)} \):

1. Show that \( f'(x) = 2e^{\sin(x^2)} \cos(x^2)\).
2. How many functions are in composition? How would your write this out using Leibniz notation of derivative?

3.3. L'Hôpital's rule. In the context of limits, an indeterminate form is an expression that cannot be resolved after substitution of limits into sub-expressions. For instance, if \( f(x) = x^2 \) and \( g(x) = x \) then both \( f(x)/f(x) \) and \( f(x)/g(x) \) tend to 0/0 as \( x \to 0 \) but upon algebraic simplifications, \( f(x)/f(x) = 1 \) while \( f(x)/g(x) = x \) clearly have different limiting values. L'Hôpital's rule treats these problems for differentiable functions by considering instead their rates of change. Simplifying a bit, we have that if \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \) or \( \pm \infty \) and \( \lim_{x \to c} \frac{f'(x)}{g'(x)} \) exists then

\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \]

Using this theorem, show that:

1. \( \lim_{x \to 0} \frac{\sin x}{x} = 1. \)
2. \( \lim_{x \to 0} \frac{1 - \cos x}{x} = 0. \)

4. INTEGRAL CALCULUS

The notion of a limit also makes possible extending finite arithmetic to the infinite. While infinite products are important, the most natural place to start is with summation,

\[ \lim_{N \to \infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n. \]

The relevant mathematical question to answer is whether the limit of partial sums converges to a finite value. For this to occur it is necessary for \( a_n \to 0 \) as \( n \to \infty \) but those who have studied series knows that this is not sufficient for convergence.

What we are interested in here is a specific sort of sum known as the Reimann sum,

\[ \sum_{n=1}^{N} f(x_n^*) \Delta x \]

where \( x_n^* \) is a test-value in the \( n^{th} \) partition of the domain \([a,b] = D \subset \mathbb{R}\). There are many suitable partitions of the domain but the most natural is a sub–division into \( N \)–many parts of width \( \Delta x = (b-a)/N \). Geometrically, this summation computes the area of \( N \)–many rectangles of height \( f(x_n^*) \) and width \( \Delta x \). Taking the limit of the series as \( N \to \infty \) computes an area associated with infinitely–many rectangles of infinitesimal width. The second point is the most important and this is that the domain has been

---

8 We used that the limit of a product is the product of limits, provided that the limits exist, which was never proven.

9 Consider a sum whose summands are elements of the sequence \( \{1,2^{-1},3^{-1},\ldots\} \), which is known as the harmonic series, \( \sum_{n=1}^{\infty} \frac{1}{n} \). We have that the summand tends to zero as we move further down the list but the series itself is known to diverge.

10 Hence, if we believe this area to be finite then we can feel comfortable with the convergence of the infinite series. This is, however, not a proof.
decomposed into a continuum of points instead of $N$—many partitions. It is this continuum that leads to our definition of integral,

\begin{equation}
\lim_{N \to \infty} \sum_{n=1}^{N} f(x_n^*) \Delta x = \int_{a}^{b} f(x) \, dx.
\end{equation}

That is, the limit of the Reimann sum is what we call a definite integral and answers questions about areas bounded by curves that cannot be studied through the geometry of the Greeks. While computing the value of a Reimann sum is difficult, the fundamental theorem of calculus,

\begin{equation}
F(x) = \int_{a}^{x} f(t) \, dt \implies \frac{dF}{dx} = f(x)
\end{equation}

for continuous $F$ has a nice corollary

\begin{equation}
\int_{a}^{b} f(x) \, dx = F(b) - F(a),
\end{equation}

which lets us compute the value of the Reimann sum through the use of anti–derivatives.\footnote{An important point that comes from this is that derivatives and integrals are, in some sense, inverse operations. Continued study of mathematics questions the exact sense in which one can consider the operations to be inversely related. In these studies one finds out that there exist functions which are integrable but nowhere differentiable. Such investigations are closely related to the analysis of partial differential equations.} While this is useful and important we note that while it is trivial to take derivatives of nearly any reasonable function we might think to write down, the same is not true of anti–differentiation.\footnote{For instance, if $f(x) = e^{-x}$ and $g(x) = x^2$ then one can easily compute both the derivatives and anti-derivatives of $f$ and $g$. Also, through the chain–rule it is possible to compute the derivative of the composition, $f(g(x))$ but it is not known what the anti–derivative of this composition is.}

The following sub–sections outlines some problems from integral calculus a student of applied science and engineering should be familiar with.

### 4.1. Fun integrations/anti–derivatives

The following is a collection of integration problems that do not fit into a standard group that I have found useful over the years.

1. \[
\int \cos^2(5x)\,dx
\]

2. \[
\int_{1}^{9} \frac{1}{t\sqrt{3t}}\,dt
\]

3. \[
\int \frac{2x+4}{x^3 - 2x^2}\,dx
\]

4. \[
\int \frac{x + 1}{x - 1}\,dx
\]

5. Find the anti–derivatives for all of the functions from section\footnote{An important point that comes from this is that derivatives and integrals are, in some sense, inverse operations. Continued study of mathematics questions the exact sense in which one can consider the operations to be inversely related. In these studies one finds out that there exist functions which are integrable but nowhere differentiable. Such investigations are closely related to the analysis of partial differential equations.}\footnote{For instance, if $f(x) = e^{-x}$ and $g(x) = x^2$ then one can easily compute both the derivatives and anti-derivatives of $f$ and $g$. Also, through the chain–rule it is possible to compute the derivative of the composition, $f(g(x))$ but it is not known what the anti–derivative of this is composition is.}

6. Using implicit differentiation show that \[
\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)
\]

7. Using implicit differentiation show that \[
\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x)
\]

8. Using implicit differentiation show that \[
\int \frac{1}{1+x^2}\,dx = \arctan(x)
\]

### 4.2. Integration by parts

If the product rule tells us how to differentiate products then integration by parts is its undoing. For differentiable functions $f, g$ we have that $[fg]' = f'g + fg'$. Integrating both sides with respect to $x$ and remembering that the definition of differentials gives us $g'(x)\,dx = \frac{dg}{dx}\,dx = dg$

\begin{equation}
\int f(x)g'(x)\,dx = \int f(x)\,dg = f(x)g(x) - \int g(x)f'(x)\,dx.
\end{equation}

While this is symbolically nice, what we need is a way to compute with it. So, for an example let’s take

\begin{equation}
\int x^2 e^{2x}\,dx
\end{equation}

We let $f(x) = x^2$ and $g'(x) = e^{2x}$ to get that

\begin{equation}
\int x^2 e^{2x}\,dx = \frac{x^2 e^{2x}}{2} - \int 2xe^{2x}\,dx.
\end{equation}

One step of integration by parts has not found the general anti–derivative but has instead generated part of it and a second simpler integral that will require another by–parts step to do. You should notice that you will again choose the power–function for $f(x)$ and the exponential for $g'(x)$. That is, the second step is not procedurally different than the first, which is often the case. For problems multi–step integration by parts problems, it makes sense to encode the procedure with a table.
An integration by parts table has two columns: the left column starts with your choice for \( f(x) \) and the right starts with your choice for \( g'(x) \). The left column is differentiated and the right is anti–differentiated. If the function in the left column vanishes after enough differentiation then right column is filled correspondingly. For our problem, this looks like

<table>
<thead>
<tr>
<th>D</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( + e^{2x} )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( - \frac{1}{2} e^{2x} )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( + \frac{1}{4} e^{2x} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( - \frac{1}{8} e^{2x} )</td>
</tr>
</tbody>
</table>

Every time you multiply entries going from upper–left to lower–right you are forming \( f(x)g(x) \) terms, i.e. you are doing integrals.

All that has to be remembered is that the negative sign in Eq. (38) results as a sign alternation as you do the multi–step integration. So for our problem, we have

\[
\int x^2 e^{2x} \, dx = \frac{x^2 e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C. \tag{41}
\]

Any time you want to stop the multi–step integration you can form a term across columns, which yields an integral that waits to be computed. For example,

\[
\int x^2 e^{2x} \, dx = \frac{x^2 e^{2x}}{2} - \int \frac{2xe^{2x}}{2} \, dx. \tag{42}
\]

You should practice these methods on the following problems:

1. \( \int \ln(2x) \, dx \)
2. \( \int_0^1 t^2 e^{2t} \, dt \)
3. \( \int e^{\alpha x} \cos(\beta x) \, dx \) and \( \int e^{\alpha x} \sin(\beta x) \, dx \)

4.3. Integration by substitution. If integration by parts is the inversion of the product rule for differentiation then the chain rule must have a similar statement. Starting from the fundamental theorem of calculus with the assumption that \( f(x) \) is continuous on the interval \( I = (a, b) \) and \( g'(x) \) on \( I^* = [a, b] \) we have that

\[
\int_{g(a)}^{g(b)} f(x) \, dx = F(g(b)) - F(g(a)) = (F \circ g)(t) \bigg|_a^b = \int_a^b \frac{d}{dt} [F \circ g](t) \, dt = \int_a^b f(g(t))g'(t) \, dt, \tag{43}
\]

where \( F \) is the anti–derivative of \( f \). Again, this is a generic symbolic representation of the idea that if you find composite function whose argument also appears in the integrand as a derivative, then the integral rests on the anti–derivative of \( f \). Some examples:

1. \( \int_0^1 t^2 \sin(2t^3) \, dt \)
2. \( \int t \sqrt{t^2 - 9} \, dt \)

4.4. The fundamental theorem of calculus with chain–rule. Sometimes it is the case that a bound of a definite integral is a function of the independent variable. Taking a derivative in this setting requires both the fundamental theorem of calculus and the chain rule.

\[
\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = \frac{d}{dx} \left[ F(g(x)) - F(g(a)) \right] = \frac{d}{dx} \left[ F(g(x)) \right] = f(g(x))g'(x) \quad \tag{44}
\]

Using this

1. Show that \( \frac{d}{dt} \int_{x-ct}^{x+ct} v(u) \, du = c(v(x+ct) - v(x-ct)) \)
5. Power-Series

A common theme of mathematics is re-representation of data where the end user trades a complicated mathematical object for simpler ones at some cost. For instance one could ask, given a function \( f \), do there exist constants, \( a_n \) such that

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

If \( f \) is a polynomial function then the problem is simple. What if \( f \) is not a polynomial function? Well, the most the series on the right-hand side can define is a polynomial function but what if \( N \to \infty \)? Then the first thing we should ask is whether the infinite-series converges. The next thing that we ask is if it converge, does it converge to \( f \)? We do not recall the many tests of series convergence here. Instead we remind the reader that if \( f \) is smooth at \( x = a \) then we have the power-series representation,

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n
\]

known as a Taylor series. What we see here is that we have traded the complicated function \( f \) for the sum, albeit infinite, of simpler power functions. This is a powerful concept and tool that is ubiquitous in mathematics. The following subsections contain useful results for students in applied science and engineering.

5.1. Taylor/Maclaurin series for common functions. There are some Taylor series which are good to know off the top of your head. In the following contains six standard relationships, four if you count exponential and trigonometric functions as related. To cement these forms, show the following relationships:

1. \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \) for all \( x \)
2. \( \log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \) for \( |x| < 1 \)
3. \( \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \) for \( |x| < 1 \)
4. \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) for \( |x| < 1 \)
5. \( \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \) for \( |x| < 1 \)
6. \( \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n \) for \( |x| < 1 \)
7. \( \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n-1)nx^{n-2} \) for \( |x| < 1 \)
8. \( \frac{2x^2}{(1-x)^3} = \sum_{n=0}^{\infty} (n-1)nx^n \) for \( |x| < 1 \)
9. \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \) for all \( x \)
10. \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \) for all \( x \)
11. \( \sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n)!^2(4^n)} x^n = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \cdots \) for \( |x| \leq 1 \)

5.2. Complex exponential functions. The imaginary number is notated \( i = \sqrt{-1} \) and lets us define the field of complex numbers \( z \in \mathbb{C} \) such that \( z = a + bi \) where \( a, b \in \mathbb{R} \). While off-putting to some, this number system is immensely useful and allows us to connect several important objects heavily studied in differential and integral calculus.

1. Show that \( i^2 = -1 \)

2. Show, formally\(^{13}\) that

\[
\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}
\]

3. Show that \( e^{ix} = \cos x + i \sin x \)
4. Show that \( 2 \cos(x) = e^{ix} + e^{-ix} \) and \( 2i \sin(x) = e^{ix} - e^{-ix} \).

\(^{13}\)In math when someone speaks of a formal series, they mean to say that mathematical operations will be conducted without regard to convergence properties. In this problem, that the series can be split is a consequence of its absolute convergence.
What we see here is two-fold. First, the exponential function is related to the trigonometric functions through the use of imaginary numbers. Second, the imaginary number is not something that should be considered dubious. In this context it is seen as a placeholder for a sign alternation pattern. That is, the imaginary number allows every other even/odd number in the summand sequence to change sign. Nothing imaginary about that!

Using these relationships, and assuming that standard exponential rules hold true for complex argument, one can make some quick work of trigonometric identities. First we define the following notations for $z \in \mathbb{C}$

\begin{align}
\text{Re} \{ z \} &= \text{Re} \{ a + bi \} = a, \\
\text{Im} \{ z \} &= \text{Im} \{ a + bi \} = b.
\end{align}

Now it is simple to show that

\begin{equation}
\cos(x + y) = \text{Re} \left[ e^{i(x+y)} \right] = \text{Re} \left[ e^{ix} e^{iy} \right] = \cos(x) \cos(y) - \sin(x) \sin(y).
\end{equation}

With these tricks in mind:

1. Prove that $\sin^2(x) + \cos^2(x) = 1$ and using this show that $\sec^2(x) - \tan^2(x) = 1$.
2. Prove that $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$.
3. Prove that $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$.
4. Show that $\sin(2x) = 2 \sin(x) \cos(x)$.
5. Show that $\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1$.
6. Using the notations of real and imaginary compute the final integrals in section 4.2 without using integration by parts. [14]

5.3. Hyperbolic trigonometric functions. To show Euler’s formula, $e^{ix} = \cos(x) + i \sin(x)$, we appealed to the Maclaurin series representation of the exponential function, powers of the imaginary numbers and re-ordering the summed sequence of an infinite series. If we omit the imaginary argument we get,

\begin{equation}
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{equation}

These series are similar to the cosine/sine series but lack a sign alternation in the summand. The following problems illustrate some of their important properties.

1. Show that $\sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} = \frac{e^x + e^{-x}}{2}$ and $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x - e^{-x}}{2}$.

2. Show that $\frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} = \frac{e^x - e^{-x}}{2}$ and that $\frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x + e^{-x}}{2}$.

3. Let $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Show that $\cosh(-x) = \cosh(x)$ and $\cosh(i x) = \cosh(x)$.

4. Let $\sinh(x) = \frac{e^x - e^{-x}}{2}$. Show that $\sinh(-x) = -\sinh(x)$ and $\sinh(i x) = i \sin(x)$.

5. Show that $\cosh^2(t) - \sinh^2(t) = 1$.

6. If $x(t) = \cos(t)$ and $y(t) = \sin(t)$ then what shape does the 2D parametric plot of $(x(t), y(t))$ produce? What if these circular trigonometric function were replaced by their hyperbolic variants?

7. First show that $\cosh^2(x) - \sinh^2(x) = 1$.

8. Using this relation and implicit differentiation show that $\int \frac{1}{\sqrt{1 + x^2}} \, dx = \arcsinh(x)$.

9. Using implicit differentiation show that $\int \frac{1}{\sqrt{x^2 - 1}} \, dx = \text{arccosh}(x)$.

10. If we define $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ and $\text{sech}(x) = 1/\cosh(x)$, then using implicit differentiation show that $\int \frac{1}{1 - x^2} \, dx = \text{arctanh}(x)$.

[14] An alternate way of avoiding integration by parts is to replace the circular trigonometric functions with their exponential forms.