Fractals, Dynamical Systems and Chaos

MACS325 - Spring 2007
Outline

- Introduction
- Fractals
- Dynamical Systems and Chaos
- Conclusions
Introduction

- When beauty is abstracted then ugliness is implied. When good is abstracted then evil has been implied. - Lao Tzu

- A mathematical model is an abstraction of a natural/physical system which uses a formal symbolic language to derive knowledge pertaining to the system.

- In abstraction much is lost. However, this does not imply that the knowledge gained from these models is simple.
Geometric Complexity

- What is geometrically straightforward?
  - Euclidian Geometry
  - Non-Euclidian Geometry

- What is not geometrically straightforward?
  - Fractal Geometry
Fractals

A fractal is a set with non-integral Hausdorff Dimension greater than its Topological Dimension.

The topological dimension of the real line is one. The plane is has dimension two.

Fractals, once thought to be pathological creations, have connections to the chaos found in dynamical systems.
History of Fractal Geometry

- Canto Set
- Peano Curves
- Koch Curve
- Gaston Julia
- Benoit Mandelbrot
Cantor Set

Take the interval [0,1] and remove sub-intervals ad infinitum.

Introduced in 1883 by Georg Cantor.

The set of points which is left behind has a topological dimension of 0 and a Hausdorff dimension of approximately 0.6309.
Peano Curve

A curve is the geometric object described by a continuous mapping whose domain is $[0,1]$. Often the range of such a function is, at most, the entire real line.

In 1890 Giuseppe Peano constructed a continuous mapping from $[0,1]$ to the unit square.
Koch Curve

Helge von Koch in 1904 constructed a curve that is continuous but differentiable nowhere!

The Koch curve has topological dimension 1 and Hausdorff dimension of approximately 1.26.
Gaston Julia’s Sets

Gaston Julia (1893–1978) studies the complex function of complex parameter \( c \), \( f_c(z) = z^2 + c \).

For different values of \( c \) the set which is created from the sequence
\( \{z, f_c(z), f_c(f_c(z)), f_c(f_c(f_c(z))), \ldots \} \) is

- Disconnected
- Simply Connected
- Connected
Mandelbrot

From 1975-1982 Mandelbrot coins the term ‘fractal’ and publishes *The Fractal Geometry of Nature*.

Coastline Paradox - Measure the coastline of a land-mass with a ruler of length $L$. Compare this to the measurement given by a ruler of length $l$, $l<L$. - L. F. Richardson (1881-1953)

Plotting the length of the ruler against the measurement on a log-log scale gives a straight line. The slope of this line is the fractal dimension of the coastline.
Mandelbrot's Set

Mandelbrot set derives from the complex function of complex parameter:

\[ f_c(z) = z^2 + c \]

The behavior of the sequence \((0, f_c(0), f_c(f_c(0)), f_c(f_c(f_c(0))), \ldots )\) defines the Mandelbrot Set.
Fractal Conclusions

Fractals are complicated geometric sets which can be characterized as having:

- A Hausdorff Dimension greater than its topological dimension.
- A self-similar structure.
Dynamical Systems

A dynamical system is the mathematical formalism/rule which describes the deterministic evolution of a point in phase-space.

The long term behavior of solutions to linear dynamical systems is completely understood in terms of eigenvalue/eigenvector decomposition.

In general the effects of dissipation causes trajectories to tend to typical behavior. The geometry of phase space corresponding to this typical behavior is called an attractor.
Van der Pol Oscillator

\[ y'' - u(1-y^2)y' + y = 0 \]

- Electrical circuits using vacuum tubes
- Action potentials for neurons
- Seismological modeling of two plates in a geologic fault
Discrete Logistics Equation

\[ x_{n+1} = r x_n (1 - x_n) \]

Models the discrete time evolution of a population subject to a carrying capacity.
Bifurcation and Period Doubling

For certain values of the growth rate ‘r’ ‘strange’ things happen to the population evolution.

Period doubling is a bifurcation in which the system switches to a new behavior with twice the period of the original system.
Period doubling is a typical characteristic of a system entering into chaos.

For $r$-values between 3.5 and 4 the logistics map has sensitive dependence on initial conditions, which is another characteristic of chaos.
Lorenz Attractor

\[
x' = a(y-x) \\
y' = bx - y - xz \\
z' = -cz + xy
\]

For \(a=10\), \(b=28\), \(c=8/3\), the Lorenz equations have exponentially divergent trajectories and thus are sensitive to initial conditions.
Chaos II

The trajectories for the discrete logistics equation and Lorenz equations become so complicated/chaotic because the geometry of the attracting set has been topologically mixed/folded. This generally causes the appearance of fractal geometries.
Chaos III

When dynamical systems are characterized as having either,

- a sensitive dependance on initial conditions (chaos),

or

- an attracting set with non-integer Hausdorff dimension (fractal),

then the dynamical systems attracting set is called strange.
Conclusions

- Though mathematical models are approximations of physical/natural systems, math's simplest nonlinear systems give rise to sophisticated and complex behavior,

- Understanding these models and their interpretations requires some of the more advanced geometric, algebraic and numerical techniques currently being studied.

- He accepts the ebb and flow of thing. Nurtures them, but does not own them and lives but does not dwell. - Lao Tzu