1. **Adrien-Marie Legendre (18 September 1752 - 10 January 1833): A French Mathematician**

(1) \[(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0, \quad \nu \in \mathbb{C}.\]

This equation, (1), is called Legendre’s equation in commonly encountered in the study of potential fields in spherical coordinates. The solution to (1) can be found by power series techniques and leads to the orthogonal Legendre Polynomials.\(^1\)

1.1. **Recurrence Relations.** Assume that \(y(x)\) has a power-series representation about \(x=0\) and solve (1).

To do this assume that,

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n, \]

and derive formula for \(y', y''\). After this substitute the series representations into (1) to find,

\[ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \nu(\nu+1)a_n x^n = 0, \]

which yields the relations,

\[ x^0 : \quad 2 \cdot 1a_2 + \nu(\nu+1)a_0 = 0 \]
\[ x^1 : \quad 3 \cdot 2a_3 + [-2 + \nu(\nu+1)]a_1 = 0 \]
\[ x^k : \quad (n+2)(n+1)a_{n+2} + [-n(n-1) - 2n + \nu(\nu+1)]a_n = 0, \quad k = 2, 3, 4, 5, \ldots \]

1.2. **Legendre’s Polynomials.** Now assume that \(\nu \in \mathbb{N}^+\) and using the initial conditions \((y(0), y'(0)) = (1, 0)\) or \((y(0), y'(0)) = (0, 1)\), show that there are polynomial solutions \(P_\nu(x)\) of Legendre’s equation.\(^2\)

2. **A Modicum of Sturm-Liouville**

Jacques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882), developed the theory for real second-order linear differential operators of the form,

\[ L[y] = \frac{1}{w(x)} \left( -\frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + q(x)y \right), \]

by looking at its linear scalings \(L[y] = \lambda y\) where \(\lambda \in \mathbb{R}\).

2.1. **From Separation of Variables in Cartesian Coordinates.** Find the form of \(L[y] = \lambda y\) where \(p(x) = 1, q(x) = 0, \quad w(x) = 1.\)

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\(^1\)Adrien-Marie Legendre, 1752 - 1833, was French mathematician who is commemorated on the Eiffel Tower along with 71 other French scientists. Some career highlights include a proof that \(\pi\) is an irrational number and advancements with the elliptic integrals associated with Celestial mechanics.

\(^2\)To produce polynomial solutions if \(\nu\) is a positive integer, consider two cases:

1. If \(\nu\) is even, use the initial condition \((y(0), y'(0)) = (1, 0)\). Then \(a_1 = 0 \) (why?) and all the coefficients of odd powers are zero. Moreover, the formula that relates \(a_{n+2}\) to \(a_n\) implies that \(a_{\nu+2} = 0\). Therefore, all of the coefficients of higher even powers are also zero, and the solution is a polynomial (with no odd powers of \(x\)).

2. If \(\nu\) is odd, use the initial condition \((y(0), y'(0)) = (0, 1)\) and obtain a polynomial solution with no even powers of \(x\).
2.2. From Separation of Variables in Cylindrical Coordinates. Show that if \( p(x) = -q(x) = [-w(x)]^{-1} = x \) then \( L[y] = \lambda y \) is nothing more than Bessel’s equation of order \( \lambda \).

2.3. From Separation of Variables in Spherical Coordinates. Show that if \( p(x) = (1 - x)^2, q(x) = 0, w(x) = 1 \) then \( L[y] = \lambda y \) is nothing more than Legendre’s equation where \( \lambda = \nu(\nu + 1) \).

2.4. Solutions in Cartesian Coordinates. Find the general solution to \( L[y] = \lambda y \) in when \( p(x) = 1, q(x) = 0, w(x) = 1 \) for arbitrary \( \lambda \).

3. Our Fundamental Boundary Value Problem

Boundary value problems (BVP) typically arise within the context of PDE, which are equations modeling the evolution of a quantity in both space and time. There are important general results for BVP, which are set within the context of Sturm-Liouville problems.\(^3\) What can be efficiently done by hand tends to be limited. The problem, in Cartesian coordinates, is to find all solutions to,

\[
y'' + \lambda y = 0, ~ \lambda \in \mathbb{R}, ~ x \in (0, L),
\]

which also satisfy,

\[
\begin{align*}
\text{Case I:} & \quad l_1 y(0) + l_2 y'(0) = 0, \\
\text{Case II:} & \quad r_1 y(L) + r_2 y'(L) = 0.
\end{align*}
\]

This problem is intractable, by hand, for general values of \( l_1, l_2, r_1, r_2 \). However, the following set of values,

<table>
<thead>
<tr>
<th></th>
<th>( l_1 )</th>
<th>( l_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Case II</td>
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<td>1</td>
</tr>
<tr>
<td>Case III</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Case IV</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

lead to BVP that can be solved by hand.

3.1. BVP Hand Calculations. Find all nontrivial eigenfunctions associated with the previous BVP for each of the four cases.\(^4\)

4. An Eigenfunction Scavenger Hunt

So, the point is this:

- Solutions to Sturm-Liouville problems form a orthogonal set vectors index by the parameter \( \lambda \). Moreover, these vectors form a complete basis for the vector-space they define. That is, their vector space is also closed under the finite and infinite linear combinations of vectors from the space.\(^5\)

For bases linear independence is required but orthogonality is better since it yields tidy orthogonality statements, which make coefficient relations simple. While there is important mathematics showing why this is true, the most important thing to remember is how to make use of such relations. To aid in this future study we collect important results right now.

4.1. The List. The equations of importance are:

- The fundamental ODE for this course
- Bessel’s equation
- Legendre’s equation

\(^3\)Some of these results can be found in long homework number 7

\(^4\)Case I and Case II will be discussed in class.

\(^5\)Their vector-space, at this point, really lacks definition. This will be resolved in chapter 12.
4.2. **The Requirements.** Make a table with the following information scavenged for each of the items on the list.\(^6\)

- Mathematical form of the equation
- Forms for its fundamental solutions
- Orthogonality relations for its fundamental solutions
- Notes: Physical, Mathematical or Otherwise

5. **Back to Vector Spaces**

The following classic text has a couple of sections that are worth it for us to read: Theory of Linear Operators in a Hilbert Space, Akhiezer and Glazman, *Dover Publications*, 1963. Our library has a copy of the text. I have placed my copy outside my office for use.

5.1. **Review.** Please review the language of vector spaces found on the homework assignment ODE Part I.

5.2. **Acquire.** We are interested in sections 1.9-1.11 (pgs. 17-27) of my version of this text. Find this material and read it. To aid you I have written some notes on it. You can find these on the blog.

5.3. **Vocabulary.** Define the following words:

- Inner-Product
- Orthonormal
- Hilbert Space
- Complete
- \(L^2(a, b)\), where \(a, b \in \mathbb{R}\) such that \(a < b\)

The key logic is:

- Homogeneous linear problems define vector-spaces.
- These vector-spaces have bases.
- When bases are concerned orthonormality is best.
- Certain equations yield such bases.
- Such bases often contain infinitely-many elements.
- If this vector-space is a subspace of another such that no more orthogonal vectors can be found, then the subspace is said to be complete within the ‘larger’ space and the aforementioned bases holds in this extended space.\(^7\)
- A space is separable if it contains a countable bases.\(^8\)
- \(L^2(a, b)\) is a space of functions with finite length/energy and is the space of chapter 11 and chapter 12. This space is separable.

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\(^6\)The orthogonality relation for the fundamental ODE/BVP can be found in chapter 11 theorem 1. The rest were seen on wikipedia at some point or another.

\(^7\)Short Take: If you have a complete orthonormal set then you have exactly what you need to start writing down ‘vectors’.

\(^8\)By countable we mean any amount of vectors so long as you can enumerate them with an integer. A great example is the set of solutions to the previous BVP.