MACS-332A - Linear Algebra

Homework #7: 4.2-4.4 Due: Wednesday 3/28

True or False. You must justify your answer.

1. The column space of an \( m \times n \) matrix is in \( \mathbb{R}^m \).
   \[ \text{True. See Theorem 3.} \]

2. \( \text{Nul}A \) is the kernel of the mapping \( x \mapsto Ax \).
   \[ \text{True. See the paragraph after the definition of a linear transformation.} \]

3. A linearly independent set in a subspace \( H \) is a basis for \( H \).
   \[ \text{False. The subspace spanned by the set must also coincide with } H. \text{ See the definition of basis.} \]

4. If \( B \) is an echelon form of a matrix \( A \), then the pivot columns of \( B \) form a basis for \( \text{Col}A \).
   \[ \text{False. See the warning after Theorem 6.} \]

5. If \( \mathbb{B} \) is the standard basis for \( \mathbb{R}^n \), then the \( \mathbb{B} \)-coordinate vector of an \( x \) in \( \mathbb{R}^n \) is \( x \) itself.
   \[ \text{True. See Example 2.} \]

6. In some cases, a plane \( \mathbb{R}^3 \) can be isomorphic to \( \mathbb{R}^2 \).
   \[ \text{True. If the plane passes through the origin, as in Example 7, the plane is isomorphic to } \mathbb{R}^2. \]

1. Let \( A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \) and \( w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \). Determine if \( w \) is in \( \text{Col}A \). Is \( w \) in \( \text{Nul}A \)?

   Consider the augmented matrix, \([ A \ w]\) which is row equivalent to \( \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Since this system is consistent, \( w \) is in \( \text{Col}A \). Since \( Aw = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), \( w \) is in \( \text{Nul}A \).

2. Let \( T : V \Rightarrow W \) be a linear transformation from a vector space \( V \) into a vector space \( W \). Prove that the range of \( T \) is a subspace of \( W \). \textbf{Hint}: Typical elements of the range have the form \( T(x) \) and \( T(w) \) for some \( x, w \) in \( V \).

   Let \( R = \text{range of } T \). To show \( R \) is a subspace we need to show:
   
   (a) \( 0 \in R: T(0_V) = 0_W \Rightarrow 0 \in R \).
   
   (b) \( R \) is closed under vector addition: Let \( T(x), T(y) \) be in \( R \). By properties of a linear transformation, \( T(x) + T(y) = T(x + y) \in R \Rightarrow R \) is closed under vector addition.
   
   (c) \( R \) is closed under scalar multiplication: Let \( T(x) \) be in \( R \) and \( c \) be any scalar. Then \( cT(x) = T(cx) \Rightarrow R \) is closed under scalar multiplication.

   Thus \( R \), the range of \( T \), is a subspace of \( W \).
3. Let $v_1 = \begin{bmatrix} 7 \\ -4 \\ -9 \\ -5 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 2 \\ -7 \\ 2 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 4 \end{bmatrix}$. Note that $v_1 - 3v_2 + 5v_3 = 0$.

Find a basis for $H = \text{Span}\{v_1, v_2, v_3\}$.

Since $v_1 - 3v_2 + 5v_3 = 0$, each of the vectors are a linear combination of the others. Since none of the vectors are multiples of each other, the sets $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$ are all linearly independent and thus each forms a basis for $H$.

4. Let $p_1(t) = 1 + t$, $p_2(t) = 1 - t$, and $p_3(t) = 2$. Give a linear dependence relation among $p_1$, $p_2$ and $p_3$.

Find a basis for $\text{Span}\{p_1, p_2, p_3\}$.

$p_1 + p_2 - p_3 = 0$ or $p_3 = p_1 + p_2$. Thus, the $\text{Span}\{p_1, p_2, p_3\} = \text{Span}\{p_1, p_2\}$. Since neither are multiples of the other, the set $\{p_1, p_2\}$ is a basis for $\text{Span}\{p_1, p_2, p_3\}$

5. Find the coordinate vector $[x]_B$ of $x = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$ relative to the basis, $B = \{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}\}$.

The augmented matrix $[b_1 b_2 b_3 x]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, so $[x]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$.

6. Use coordinate vectors to test the linear independence of the set of polynomials, $\{1 - 2t^2 - 3t^3, t + t^3, 1 + 3t - 2t^2\}$.

The coordinate mapping produces the vectors $(1, 0, -2, -3)$, $(0, 1, 0, 1)$, and $(1, 3, -2, 0)$. To test for linear independence, write these vectors as the columns of a matrix and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ Since the matrix does not have a pivot in each column, its column (and therefore the corresponding polynomials) are linearly independent.