1 Introduction

Some examples which illustrate how difference equations arise and the diversity of the areas in which difference equations apply.

1.1 Plane Division

Consider a plane that has lying in it $k$ nonparallel lines. Into how many separate regions will the plane be divided if not more than 2 lines intersect in the same point?

Let $N_k$ be the number of regions. Then the $(k + 1)^{th}$ line will be cut by the $k$ previous lines in $k$ points and, consequently, divides each of the $k + 1$ prior existing regions into 2. We then have,

$$N_{k+1} = N_k + (k + 1)$$

Then we see for $k = 0$, $N_0 = 1$ (since the plane is undivided. Similarly, for $k = 1$, $N_1 = 2$, since a single line divides the plane into 2 regions.

1.2 Savings Certificate

The value of a savings certificate initially worth $1000 accumulates interest paid each month at 1% per month. Then the following sequence represents the value of the certificate month by month.

$$A = (1000, 1010, 1020.10, 1030.30, \ldots)$$

For a sequence such as this, we define the $n$th first difference as

$$\Delta a_n = a_{n+1} - a_n$$

Specifically for our savings certificate we have

$$\Delta a_0 = a_1 - a_0 = 10$$  
$$\Delta a_1 = a_2 - a_1 = 10.10$$  
$$\Delta a_2 = a_3 - a_2 = 10.20$$  
$$\vdots$$

We see that the first difference is simply the interest paid during that month. Thus,

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n \Rightarrow a_{n+1} = a_n + 0.01a_n = 1.01a_n$$

Our dynamical system model is given by

$$a_{n+1} = 1.01a_n, \quad n = 0, 1, 2, 3, \ldots$$  
$$a_0 = 1000$$
2 Difference Equations

2.1 Linear Difference Equations \( a_{n+1} = ra_n \)

A Difference Equation is a process of generating an infinite sequence of numbers by giving a rule for calculating each number.

2.1.1 Fibonacci Sequence

\[
y_n = y_{n-1} + y_{n-2}, \quad y_0 = 0, y_1 = 1
\]

Then, the Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13, . . .

A difference equation is of first order if the value \( y_n \) depends only on \( y_{n-1} \), \( n \) and constants.

2.1.2

\[
y_n = 3y_{n-1} - n^2 + 2, \quad y_0 = 0.1
\]

is a first order difference equation. The Fibonacci sequence is a second order difference equation.

A difference equation is said to be autonomous if its calculation does not use \( n \). For example, the Fibonacci equation is autonomous but the last example is not. Note that this is a property of the equation and not a property of the actual sequence.

Consider,

\[
y_n = n \text{ versus } y_n = y_{n-1} + 1, \quad y_0 = 0
\]

are both the same sequence yet the second is autonomous while the first is not.

A difference equation of order \( k + 1 \) is said to be linear if it is of the form,

\[
y_n = f_n(n)y_{n-1} + f_{n-1}(n)y_{n-2} + \cdots + f_{n-k}y_{n-k}
\]

where the \( k + 1 \) functions are any functions of \( n \). It is said to be affine if it has the form,

\[
y_n = f_n(n)y_{n-1} + f_{n-1}(n)y_{n-2} + \cdots + f_{n-k}(n)y_{n-k} + g(n)
\]

Thus, an autonomous first order affine difference equation must look like

\[
y_n = ay_n + b
\]

for some constants \( a \) and \( b \).

Consider the first order linear autonomous equation,

\[
a_n = ra_{n-1}
\]

Applying this equation recursively results in the following

\[
a_{n+1} = ra_n = r(ra_{n-1}) = r[r(ra_{n-2})] = \cdots = r^{n+1}a_0
\]

Then this difference equation has the solution

\[
a_n = r^n a_0
\]

Thus, the solution for a simple linear difference equation involves an expression of the form \( r^n \) where \( n \) is the generation number.

Note: To check, we see that for \( a_n = r^n a_0 \) then \( a_{n-1} = r^{n-1}a_0 \) and

\[
\frac{a_n}{a_{n-1}} = r \Rightarrow a_n = ra_{n-1}
\]
2.2 Affine Linear Difference Equations $a_{n+1} = ra_n + b$

A number $a$ is called an equilibrium value or fixed point of the dynamical system if $a_k = a$ for all $k = 1, 2, 3, \ldots$ when $a_0 = a$. That is, $a_k = a$ is a constant solution to the dynamical system.

Then, for $a_{n+1} = ra_n + b$ we see that the equilibrium value is

$$a = \frac{b}{1 - r}$$

If $r = 1$ and $b = 0$, every number is an equilibrium value. If $r = 1$ and $b \neq 0$, no equilibrium value exists.

The solution of this dynamical system for $r \neq 1$ is

$$a_k = r^k c + \frac{b}{1 - r}$$

for constant $c$ dependent upon the initial condition.

2.2.1 Annuities

An annuity is a savings account that pays interests on the amount present and allows the investor to withdraw a fixed amount each month until the account is depleted.

Consider an annuity with 1% monthly interest rate and a monthly withdrawal of $1000. This gives the dynamical system

$$a_{n+1} = 1.01a_n - 1000$$

Investigate the long-term behavior given the initial investments of

- $a_0 = 90,000$
- $a_0 = 100,000$
- $a_0 = 110,000$