Special cases of the three body problem

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Three-body problem deals with the question of how three objects, with different masses, different initial positions and velocities, will move under one of the physical forces, such as gravity, Coulomb force and elasticity. The most simple and typical sample of the three-body problem is the motions of the Sun, Earth and Moon in the solar system.

In the giant solar system, the sizes of the celestial bodies are so small to be neglect. So they can be considered as mass points. Without counting the influence of other celestial bodies, the motions of the Sun, Earth and Moon under universal gravitational forces is a three-body problem.

Generally, the three mass points, which have random masses, initial positions and velocities, moving under the gravitational interactions, will build-up eighteen first order ordinary differential equations. Since the motion of every celestial body under the gravitational forces of the other two celestial bodies can be determined by three second order ordinary differential equations or six first order ordinary differential equations. However, by now what we have is only ten independent algebraic first integrals: three for the center of mass, three for the linear momentum, three for the angular momentum and one for the energy for the general cases of the three-body problem. [1]

Since we cannot strictly solve the three-body problem, the only way to get the solutions is making approximations for certain special cases. There are many ways to study the three-body problem, and generally all the ways are contain in the catalogs listed below:
1) Analytical Method: Expend the positions and velocities of the celestial bodies by time or other small parameters, to get the approximate expressions. And then discuss the motions of the celestial bodies’ dependent of time.

2) Qualitative Method: Use the qualitative theory of the differential equations to study the properties in large space and long time.

3) Numerical Method: Calculate the differential equations directly to obtain the positions and velocities in certain time.

All the three methods have its advantages and disadvantages separately. Improving the methods and exploring the new integrals are the important issues in the study the three-body problem. [2]

The restricted three-body problem assumes that the mass of one of the bodies is negligible, as well as the interaction forces between it and the other two bodies. So the problem is simplified as a two-body problem, and their orbitals are conic sections with the center of mass on its one of their focuses. Since there are four types of conic sections, we have four kinds of restricted three-body problem respectively: the circular restricted three-body problem, the elliptical restricted three-body problem, the parabolic restricted three-body problem, and the hyperbolic restricted three-body problem. [2]

The circular restricted three-body problem is the special case in which two of the bodies are in circular orbits around their common center of mass, and the third mass is small and moves in the same plane (approximated by the Sun-Earth-Moon system and many others). The restricted problem (both circular and elliptical) was worked on by many famous mathematicians and physicists. In the circular problem, with respect to a rotating reference frame, the two co-orbiting bodies are stationary, and the third have five equilibrium points. Three of them are collinear with the masses (in the rotating frame), are unstable. The other two are located on the third vertex of both equilateral triangles of which the two bodies are the first and second vertices. For sufficiently small mass ratio, the triangular equilibrium points are stable. The five equilibrium
points are known as the Lagrange points in the circular restricted three-body problem. In addition, it can be quite helpful to consider the effective potential. [1]

To solve the restricted three-body problem, we should first build up the coordinate system:

![Fig1: The planar, circular restricted three-body problem, on an inertial coordinate system (x, y) and on a rotating coordinate system (x', y'). Note that on the rotating coordinate system the positions of the two massive bodies are fixed. [3]](image)

The three-body problem is related to nine coupled second order differential equations:

\[
\frac{d^2 r_1}{dt^2} = -Gm_2 \frac{r_1 - r_2}{|r_1 - r_2|^3} - Gm_3 \frac{r_1 - r_3}{|r_1 - r_3|^3}
\]

\[
\frac{d^2 r_2}{dt^2} = -Gm_3 \frac{r_2 - r_3}{|r_2 - r_3|^3} - Gm_1 \frac{r_2 - r_1}{|r_2 - r_1|^3}
\]

\[
\frac{d^2 r_3}{dt^2} = -Gm_1 \frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_2 \frac{r_3 - r_2}{|r_3 - r_2|^3}
\]

For the restricted three-body problem, the exact three-body equations can be simplified by setting \(m_3 = 0\) (the efficient small body compare to the other two).
The simplification decouples the third vector equation from the other two, which are the two-body Kepler equations. Choose a special solution of the two-body Kepler equations with a circular orbit in the center of mass frame. To further simplify the equations, set the lengths in units of the radius of the circular orbit, and the time is measured in units of the inverse angular speed of circular motion. And write all the constants contained in the above equations in terms of a single parameter.

\[
\frac{d^2 r_1}{dt^2} = -Gm_2 \frac{r_1 - r_2}{|r_1 - r_2|^3}
\]
\[
\frac{d^2 r_2}{dt^2} = -Gm_1 \frac{r_2 - r_1}{|r_2 - r_1|^3}
\]
\[
\frac{d^2 r_3}{dt^2} = -Gm_1 \frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_2 \frac{r_3 - r_2}{|r_3 - r_2|^3}
\]

As we consider the circular restricted three-body problem, the orbit is circular, and the radius is a constant, which equals to a half of the major axis. And if the initial position and velocity of the lightest body is on the orbital plane of the two massive bodies’, the lightest body will move forever on the plane. So we assume the lightest body is moving on the orbital plane of the heavy ones, and the problem will be simplified even further. \[\text{[4]}\]

In the inertial frame, the two heavy bodies rotate as a rigid dumbbell. But in the rotating frame, which is fixed to the bodies, the Newton’s second law does not valid. The rotating frame is an accelerating frame, and the heavy bodies are not moving in this frame, though they have the gravitational forces working on them.

Assume the inertial frame and the rotating frame are the same at t = 0, then the coordinates of the lightest mass in the frames have the relationships as below:

\[
x(t) = x'(t) \cos t - y'(t) \sin t
\]
\[
y(t) = x'(t) \sin t + y'(t) \cos t
\]
Take the second derivative of the above equations, and get the following ones:

\[
\ddot{x} = \dot{x}' \cos t - 2\dot{x}' \sin t - x' \cos t - \dot{y}' \sin t - 2\dot{y}' \cos t + y' \sin t
\]

\[
\ddot{y} = \dot{x}' \sin t + 2\dot{x}' \cos t - x' \sin t + \dot{y}' \cos t - 2\dot{y}' \sin t - y' \cos t
\]

The equations above can be solved for the accelerations in the rotating frame:

\[
\ddot{x} \cos t + \ddot{y} \sin t = \dot{x}' - x' - 2\dot{y}'
\]

\[
-\ddot{x} \sin t + \ddot{y} \cos t = \dot{y}' - y' + 2\dot{x}'
\]

The motions of the lightest mass in the rotating frame can be obtained by taking place of the gravitational values of the accelerations in the inertial frame.

\[
\dot{x}' = -\frac{(1 - \alpha)(x' - \alpha)}{((x' - \alpha)^2 + y'^2)^{3/2}} - \frac{\alpha(x' + 1 - \alpha)}{((x' + 1 - \alpha)^2 + y'^2)^{3/2}} + x' + 2\dot{y}'
\]

\[
\dot{y}' = -\frac{(1 - \alpha)y'}{((x' - \alpha)^2 + y'^2)^{3/2}} - \frac{\alpha y'}{((x' + 1 - \alpha)^2 + y'^2)^{3/2}} + y' - 2\dot{x}'
\]

The first two terms on the right hand sides are the gravitational accelerations caused by the two heavy bodies. The third terms are the centripetal accelerations, which is proportional to the distance between the lightest body and the original point. And the fourth terms are the Coriolis accelerations, which is linear dependent of velocity.

(All the calculations above are taken in Mathematica, and the file is attached.)

To solve the problem, we would better consider the effective potential. And we know that for this circular restricted three-body problem there are five Lagrangian points which are the five particular solutions for this problem. Lagrange wanted to make the problem simpler, so he came out a simple hypothesis: The trajectory of an object is determined by finding a path that minimizes the action over time. This is the basis of Lagrangian mechanics. With this new system of calculations, Lagrange solved the way in which the lightest body would orbit around the two heavy bodies in a circular orbit. In the rotating frame, he found the five specific points where no net force is working on the lightest body. The Lagrangian mechanics concerns the quantity subtracting the potential energy from the kinetic energy. When Lagrange studied the Sun-Earth-Moon system, he used the rotating frame and an effective potential energy in this frame which contained the centripetal effects to get the five Lagrangian
The negative gradient of the effective potential could give the gravitational and centripetal forces, which relate to the first three terms of the motion equations. For the Coriolis terms, they depend on the velocities, which could not be obtained from the potential function that depends only on the positions. When the gradient of the effective potential goes to zero, there are five extremum points, at which the gravitational and centripetal accelerations sum to zero. Three of them (L1, L2, L3) are saddle points, and the other two (L4, L5) are maxima.\[1\]

\[
V(x', y') = -\frac{1 - \alpha}{\sqrt{(x' - \alpha)^2 + y'^2}} - \frac{\alpha}{\sqrt{(x' + 1 - \alpha)^2 + y'^2}} - \frac{x'^2 + y'^2}{2}
\]

Fig2: The five Lagrangian points. Three of them (L1, L2, L3) are saddle points, and the other two (L4, L5) are maxima.\[4\]

Use Mathematica to plot the effective potential. From the plot, it might seem that the five Lagrangian points are all unstable, and every orbit will either crash onto one of the heavy bodies or wander off to infinity. However, when the lightest body starts to move, the Coriolis terms are useful. Solve the motion equations by numerical methods, we will get many orbits which localized near each of the Lagrangian points, and the orbits that wander between the points, or go to infinity.\[4\]

Since the lightest body is confined in the two-dimensional orbit plane of the two
heavy bodies’, this problem is just like the Kepler problem in the center of mass frame. There are two conserved quantities in the Kepler problem. One is the energy, because the force is conservative. The other one is the angular momentum, because the force is central. For this two-dimensional circular restricted three-body problem, the energy of the lightest body is not conserved because of the velocity-dependent of the Coriolis acceleration. Also the angular momentum of the lightest body is not conserved because the force is not central. However there is one conserved quantity known as the Jacobi Integral.

\[ C = x'^2 + y'^2 + \frac{2(1 - \alpha)}{\sqrt{(x' - \alpha)^2 + y'^2}} + \frac{2\alpha}{\sqrt{(x' + 1 - \alpha)^2 + y'^2}} - x'^2 - y'^2 \]

Since there are no independent conserved quantities other than the Jacobi Integral, the problem is not integrable. The orbits depend on the initial conditions quite a lot. To find the condition for the Jacobi Integral, we use the fact that the kinetic energy is always positive. For any given conserved constant C, which is due to the initial conditions, the lightest body cannot go to the region where the following condition is broken.

\[ x'^2 + y'^2 + \frac{2(1 - \alpha)}{\sqrt{(x' - \alpha)^2 + y'^2}} + \frac{2\alpha}{\sqrt{(x' + 1 - \alpha)^2 + y'^2}} - C = x'^2 + y'^2 \geq 0 \]

Now the problem left is solving the ordinary differential equations. The Euler method is a first-order numerical procedure to solve the ordinary differential equations with a given initial value. This method is quite useful in mathematics and computational science. It is the most basic kind of explicit method for numerical integration of the ordinary differential equations and it is the simplest kind of Runge-Kutta method. \[^{[1]}\] A typical application is to systems of particles: the positions and velocities of the particles are functions of one variable, the time. And the motion equations of N particles are a set of 6N coupled first order ordinary differential equations. \[^{[4]}\]

A simple example of a single first order ordinary differential equation is the exponential decay equation:
The Euler method for solving the decay equation approximately is using the first two terms in a Taylor series expansion to relate the solution at two different time values:

\[ \frac{dx}{dt} = -x(t), \quad x(t) = x(0)e^{-t}, \]

\[ x(t_2) = x(t_1 + \delta t) \approx x(t_1) + \frac{dx}{dt} \bigg|_{t=t_1} \delta t = x(t_1) - x(t_1)\delta t \]

The simplest way to set the solution numerically is starting with the initial time at \( t = 0 \), and generates a sequence of values at successive equally spaced times. \(^4\)

Here are some solutions of the examples of three-body problems. The orbitals are shown below: \(^5\)

1) Satellite orbiting a moon that circularly orbits a planet:

2) Satellite orbiting a moon that elliptically orbits a planet:
3) Periodic orbit of a retrograde satellite in the double planet system:

![Restricted Three-body Problem](image)

Solve for the motion equations of the lightest body:

\[ \mathbf{r}(t) = (x(t) \cos(t) - y(t) \sin(t), x(t) \sin(t) + y(t) \cos(t)) \]
\[ \mathbf{v}(t) = D[\mathbf{r}(t), t] \]
\[ \mathbf{a}(t) = D[\mathbf{v}(t), t] \]
\[ g = (g_x \cos(t) - g_y \sin(t), g_x \sin(t) + g_y \cos(t)) \]
\[ \text{Solve}[\mathbf{a}(t) = g, \{x''(t), y''(t)\}] \]
\[ \text{Simplify}[%] \]
\[ \mathbf{a}_{p} = (x''(t), y''(t)) \]

Out[1] = \{Cos[t] \cdot x'[t] - Sin[t] \cdot y'[t], Sin[t] \cdot x'[t] + Cos[t] \cdot y'[t]\}
Out[2] = \{-Sin[t] \cdot x'[t] - Cos[t] \cdot y'[t] + Cos[t] \cdot x''[t] - Sin[t] \cdot y''[t],
                    Cos[t] \cdot x''[t] - Sin[t] \cdot y'[t] + Sin[t] \cdot x'[t] + Cos[t] \cdot y'[t]\}
Out[3] = \{-Cos[t] \cdot x'[t] + Sin[t] \cdot y'[t] - 2 Sin[t] \cdot x''[t] - 2 Cos[t] \cdot y''[t] +
                    Cos[t] \cdot x'''[t] - Sin[t] \cdot y''[t],
                    -Sin[t] \cdot x''[t] - Cos[t] \cdot y'[t] +
                    2 Cos[t] \cdot x''[t] - 2 Sin[t] \cdot y''[t] + Sin[t] \cdot x'''[t] + Cos[t] \cdot y'''[t]\}
Out[4] = \{g_x \cos(t) - g_y \sin(t), g_y \cos(t) + g_x \sin(t)\}
Out[5] = \{\{x''[t] \to g_x + x'[t] + 2 y'[t], y''[t] \to g_y + y'[t] - 2 x'[t]\}\}
Out[6] = \{\{x''[t] \to g_x + x'[t] + 2 y'[t], y''[t] \to g_y + y'[t] - 2 x'[t]\}\}
Out[7] = \{g_x + x'[t] + 2 y'[t], g_y + y'[t] - 2 x'[t]\}
Show the effective potential:\(^4\):

\[
V[x, y, a] := -(1 - a) / \text{Sqrt}[(x - a)^2 + y^2] - a / \text{Sqrt}[(x + 1 - a)^2 + y^2] - (x^2 + y^2) / 2;
\]

Manipulate[Plot[V[x, 0, a], {x, -1.5, 1.5}], {a, 0, 0.5}]
Manipulate[Plot3D[V[x, y, a], {x, -1.5, 1.5}, {y, -1.5, 1.5}], {a, 0, 0.5}]
Manipulate[ContourPlot[V[x, y, a], {x, -1.5, 1.5}, {y, -1.5, 1.5}], {a, 0, 0.5}]
Show there are no other independent conserved quantities except the Jacobi Integral[^4]:

```mathematica
In[11]:= \[ P[x_, y_, a_, C_] := If[lhs = -2 V[x, y, a] - C; lhs \geq 0, lhs, 0]; 

Manipulate[
  Plot3D[P[x, y, a, C], {x, -1.5, 1.5}, {y, -1.5, 1.5}],
  
  forbidden = ContourPlot[2 V[x, y, a] + C = 0, {x, -1.5, 1.5}, {y, -1.5, 1.5}];
  heavies = Graphics[Point[{(a, 0), (a - 1, 0)}]];
  Show[forbidden, heavies], {a, 0, 0.5}, {C, 0, 4, Appearance -> "Labeled"}]
```

[^4]: Partially inspired by [Abramovitz and Stegun, 1972](https://dlmf.nist.gov/).
Use Euler’s Method to solve the ordinary differential equation:\[{d^4x\over dt^4} = H\]

In[35]:= (*Get input from user*)\[\texttt{dt} = \text{Input}["Enter time step \texttt{dt}: "]\];\[\texttt{tMax} = \text{Input}["Enter maximum time: "]\];

(*Initial values*)\[\texttt{t0} = 0;\]
\[\texttt{x0} = 10;\]

(*Initialize variables and output list*)\[\texttt{t} = \texttt{t0};\]
\[\texttt{x} = \texttt{x0};\]
\[\texttt{solution} = \{(t, x)\};\]

(*Calculate results and store in list*)\[\text{While}[\texttt{t} \texttt{\textless} \texttt{tMax}, \texttt{dx} = -\texttt{x} \texttt{dt};\]
\[\texttt{t} = \texttt{t} + \texttt{dt};\]
\[\texttt{x} = \texttt{x} + \texttt{dx};\]
\[\texttt{solution} = \text{Append}[\texttt{solution}, \{(t, x)\}];\]

(*Plot output and exact solution*)\[\texttt{euler} = \text{ListPlot}[\texttt{solution}];\]
\[\texttt{exact} = \text{Plot}[\texttt{x0 Exp[-(t - \texttt{t0})]}, \texttt{\{t, 0, 5\}}];\]
\[\text{Show}[\texttt{euler}, \texttt{exact}]\]