Parametrizing by the Ellentuck space

Timothy Trujillo

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Overview

• Topological Ramsey spaces and the abstract Ellentuck theorem

• The Ellentuck and Milliken spaces

• Parametrizing by a topological Ramsey space

• The main result and its corollaries

• Application: $n$-dimensional parametrized perfect set theorem
### Definition

Consider a triple \((\mathcal{R}, \leq, r)\) where

- \(\mathcal{R}\) is non-empty.

For each \(a\) in the range of \(r\) and each \(B \in \mathcal{R}\), let \([a, B] = \{A \in \mathcal{R} : A \leq B \land (\exists n) r(A, n) = a\}\).

The Ellentuck topology on \(\mathcal{R}\) is the topology generated by the sets \([a, B]\) where \(a\) is in the range of \(r\) and \(B \in \mathcal{R}\).
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- \(r\) is a function with domain \(\mathcal{R} \times \mathbb{\omega}\).

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The **Ellentuck topology** on \(R\) is the topology generated by the sets \([a, B]\) where \(a\) is in the range of \(r\) and \(B \in R\).
Definition

A subset $\mathcal{X}$ of $\mathcal{R}$ is **Ramsey** if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. 
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- A subset $\mathcal{X}$ of $\mathcal{R}$ is **Ramsey null** if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.
Topological Ramsey Spaces

Definition

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Definition

A triple $(\mathcal{R}, \leq, r)$ with its Ellentuck topology is a **topological Ramsey space** if

- Every subset of $\mathcal{R}$ with the Baire property (with respect to the Ellentuck topology) is Ramsey.

- Every meager subset of $\mathcal{R}$ (with respect to the Ellentuck topology) is Ramsey null.
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The Ellentuck Space

Definition

- \([\omega]^\omega\) is the collection of the infinite subsets of \(\omega\).
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- For $n < \omega$ and $X = \{x_0, x_1, \ldots\}$ in $[\omega]^\omega$ enumerated in increasing order $r(X, n) = \{x_0, \ldots, x_{n-1}\}$.
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• \([\omega]^{\omega}\) is the collection of the infinite subsets of \(\omega\).

• For \(n < \omega\) and \(X = \{x_0, x_1, \ldots\}\) in \([\omega]^{\omega}\) enumerated in increasing order \(r(X, n) = \{x_0, \ldots, x_{n-1}\}\).

• The Ellentuck space is the topological Ramsey space associated to the triple \(([\omega]^{\omega}, \subseteq, r)\).
Definition

A **block sequence** is a sequence \((x_i)\) of nonempty subsets of \(\omega\) such that

\[\forall i, \max(x_i) < \min(x_{i+1}).\]
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Notation

- \( \text{FIN} \) is the collection of all nonempty finite subsets of \( \omega \).
- \( \text{FIN}^n \) is the collection of all block sequences of length \( n \).
- \( \text{FIN}^\infty \) is the collection of all infinite block sequences.
- If \( X \) is a block sequence then \( \text{FU}(X) \) denotes the subset of \( \text{FIN} \) consisting of all finite unions of the elements of the sequence \( X \).
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Definition
For block sequences $X = (x_i)$ and $Y = (y_i)$ we write $X \leq Y$ if

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Definition

For a block sequence $X = (x_i)$ in $\text{FIN}^\infty$ and a natural number $n$ we let

$$r(X, n) = (x_0, x_1, \ldots, x_{n-1}).$$
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**Theorem (Milliken, [2])**

*The triple $(\text{FIN}^{\infty}, \leq, r)$ is a topological Ramsey space.*
The abstract Ellentuck theorem

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Theorem (Todorcevic, [1])

If \((R, \leq, r)\) is a closed subspace of \((\text{ran}(r))\_\omega\) and satisfies A.1-A.4 then \((R, \leq, r)\) is a topological Ramsey space.
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Axioms

- **A.1** *(Sequencing)*
- **A.2** *(Finitization)*

Theorem (Todorcevic, [1])

If \((R, \leq, r)\) is a closed subspace of \((\text{ran}(r), \omega)\) and satisfies A.1-A.4 then \((R, \leq, r)\) is a topological Ramsey space.

Trujillo

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The abstract Ellentuck theorem

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- **A.1** (Sequencing)
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- **A.3** (Amalgamation)

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Axioms

- **A.1** *(Sequencing)*
- **A.2** *(Finitization)*
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- **A.4** *(The pigeonhole principle for $\mathcal{R}$)*

If $(\mathcal{R}, \leq, r) \subseteq \text{ran}(r) \omega$ and satisfies A.1-A.4 then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.
### Axioms

- **A.1** (*Sequencing*)
- **A.2** (*Finitization*)
- **A.3** (*Amalgamation*)
- **A.4** (*The pigeonhole principle for $\mathcal{R}$*)

### Theorem (Todorcevic, [1])

If $(\mathcal{R}, \leq, r)$ is a closed subspace of $(\text{ran}(r))^{\omega}$ and satisfies **A.1-A.4** then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.
Definition

Suppose that \((\mathcal{R}, \leq, r)\) and \((\mathcal{S}, \leq, r)\) satisfy \textbf{A.1-A.4}. For \(n < \omega\) and \((X, Y) \in \mathcal{R} \times \mathcal{S}\), define \(r\) with domain \(\mathcal{R} \times \mathcal{S} \times \omega\) by the formula:

\[
\begin{align*}
  r(X, Y, n) &= \{\emptyset\} & \text{if } n = 0 \\
  &= (r(X, n), r(Y, n)) & \text{otherwise.}
\end{align*}
\]

We say that \(\mathcal{R}\) parametrizes \(\mathcal{S}\) if there is a quasi-order \(\leq\) on \(\mathcal{R} \times \mathcal{S}\) such that \((\mathcal{R} \times \mathcal{S}, \leq, r)\) is a topological Ramsey space.
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Suppose that \((\mathcal{R}, \leq, r)\) and \((\mathcal{S}, \leq, r)\) satisfy **A.1-A.4**. For \(n < \omega\) and \((X, Y) \in \mathcal{R} \times \mathcal{S}\), define \(r\) with domain \(\mathcal{R} \times \mathcal{S} \times \omega\) by the formula

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Main Result

**Theorem (T. [3])**

Suppose \((R, \leq, r)\) and \((S, \leq, r)\) are closed subspaces of \(AR^\omega\) and \(AS^\omega\), respectively, and satisfy A.1-A.4. Let \(\leq\) be a quasi-order on \(R \times S\) such that \((R \times S, \leq, r)\) satisfies A.1-A.3. If \(R\) 'diagonalizes' \(S\) with respect to the quasi-order \(\leq\) on \(R \times S\) then \(R\) is parametrized by \(S\). In particular, \((R \times S, \leq, r)\) is a topological Ramsey space that satisfies A.1-A.4.
Corollary (T. [3])

Let \((\mathcal{R}, \leq, r)\) be a closed subspace of \(\text{ran}(r)\omega\) satisfying \(\text{A.1-A.4}\). If for all \(s, t\) in the range of \(r\), \(s \sqsubseteq t \implies s \leq_{\text{fin}} t\), then the topological Ramsey space \((\mathcal{R}, \leq, r)\) parametrizes the Ellentuck space.
Corollary (T. [3])

Let \((\mathcal{R}, \leq, r)\) be a closed subspace of \(\text{ran}(r)^\omega\) satisfying A.1-A.4. If for all \(s, t\) in the range of \(r\), \(s \sqsubseteq t \implies s \leq_{\text{fin}} t\), then the topological Ramsey space \((\mathcal{R}, \leq, r)\) parametrizes the Ellentuck space.

Corollary (T. [3])

The Ellentuck space parametrizes itself. That is, there is a quasi-order \(\ll\) on \([\omega]^\omega \times [\omega]^\omega\) such that \(([\omega]^\omega \times [\omega]^\omega, \ll, r)\) is a topological Ramsey space.
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Corollary (T. [3])

The Milliken space parametrizes the Ellentuck space. That is, there is a quasi-order \(\ll\) on \(\text{FIN}^\infty \times [\omega]^\omega\) such that \((\text{FIN}^\infty \times [\omega]^\omega, \ll, r)\) is a topological Ramsey space.
Remark

Axiom A.2 (Finitization) requires the existence of a quasi-order \( \leq_{\text{fin}} \) on \( \text{ran}(r) \). If A.2 holds, \( X \in \mathcal{R} \) and \( a \in \text{ran}(r) \) then we let

\[
\text{depth}_X a = \begin{cases} 
\min \{ n \in \omega : a \leq_{\text{fin}} r(X, n) \} & \text{if } \exists n, a \leq_{\text{fin}} r(X, n) \\
\infty & \text{otherwise.}
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Axiom A.2 (Finitization) requires the existence of a quasi-order \( \leq_{\text{fin}} \) on \( \text{ran}(r) \). If \( \text{A.2} \) holds, \( X \in \mathcal{R} \) and \( a \in \text{ran}(r) \) then we let

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Definition

For \((X, Y), (S, T) \in \mathcal{R} \times \mathcal{S}\), we write \((X, Y) \ll (S, T)\) if \(X \leq S, Y \leq T\) and \(\forall n, \text{depth}_S r(X, n) \leq \text{depth}_T r(Y, n)\).
Axiom A.2 (Finitization) requires the existence of a quasi-order \( \leq_{\text{fin}} \) on \( \text{ran}(r) \). If A.2 holds, \( X \in \mathcal{R} \) and \( a \in \text{ran}(r) \) then we let

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Remark

The Ellentuck space does not parametrize the Milliken space via the quasi-order \( \ll \).
Theorem (parametrized perfect set theorem, [1])

For every Souslin-measurable coloring of \( \mathbb{R} \times [\omega]^{\omega} \) with finitely many colors there exists a perfect set \( P \subseteq \mathbb{R} \) and infinite set \( Y \subseteq \omega \) such that the coloring is monochromatic on \( P \times [Y]^{\omega} \).
Theorem (parametrized perfect set theorem, [1])

For every Souslin-measurable coloring of $\mathbb{R} \times [\omega]^\omega$ with finitely many colors there exists a perfect set $P \subseteq \mathbb{R}$ and infinite set $Y \subseteq \omega$ such that the coloring is monochromatic on $P \times [Y]^\omega$.

Theorem ($n$-dimensional parametrized perfect set theorem, [1])

Let $0 < n < \omega$. For every Souslin-measurable coloring of $\mathbb{R}^n \times [\omega]^\omega$ with finitely many colors there exists a sequence $(P_i)_{i < n}$ of perfect sets subsets of $\mathbb{R}$ and an infinite set $Y \subseteq \omega$ such that the coloring is monochromatic on $\prod_{i=0}^{n-1} P_i \times [Y]^\omega$. 
Proof Sketch

Since we are constructing a perfect set on the first coordinate we can work with the subset of irrationals which is homeomorphic to $[\omega]^\omega$ with its metric topology. Without loss of generality it is enough to prove the statement for Souslin-measurable colorings of $[\omega]^\omega \times [\omega]^\omega$ with respect to the product topology where each factor is taken with its metric topology.

Let $c : [\omega]^\omega \times [\omega]^\omega \to k$ be a Souslin-measurable coloring with $k \in \omega$. For each $i < k$, $c^{-1}\{i\}$ is an analytic subset of $[\omega]^\omega \times [\omega]^\omega$. Since the collection of sets with the Baire property with respect to Ellentuck topology on $[\omega]^\omega \times [\omega]^\omega$ is closed under the Souslin operation each $c^{-1}\{i\}$ has the Baire property with respect to the Ellentuck topology. Since $([\omega]^\omega \times [\omega]^\omega, \leq, r)$ is a topological Ramsey space each $c^{-1}\{i\}$ is Ramsey. Thus there exists $(X, Y) \in [(\emptyset, \emptyset), (\omega, \omega)]$ such that for each $i < k$, either $[(\emptyset, \emptyset), (X, Y)] \subseteq c^{-1}\{i\}$ or $[(\emptyset, \emptyset), (X, Y)] \cap c^{-1}\{i\} = \emptyset$. That is, $c$ is monochromatic on $[(\emptyset, \emptyset), (X, Y)]$. 
Thank You!
