Activity 8 - Related Rates

Learning outcome: Given $f(x)$ with a physical meaning, interpret $f'(x)$ as a rate of change of $f$ with respect to $x$ with correct units.

Learning outcome: Compute the derivative of an implicitly defined function.

It is common in mathematics, science, and engineering to have interest in related quantities that are changing in time. Sometimes, we can use known (or measurable) rates of change in order to find out about rates that are harder to measure.

1. Consider the equation for kinetic energy: $KE = \frac{1}{2} mv^2 = \frac{1}{2} \cdot m \cdot v^2$. If I ask you to take the derivative of kinetic energy, you should ask “the derivative with respect to what?”

   (a) Compute the derivative of $KE$ with respect to $v$, $\frac{d(KE)}{dv}$.

   \[
   \frac{d(KE)}{dv} = \frac{d}{dv} \left( \frac{1}{2} mv^2 \right) = \frac{1}{2} \cdot m \cdot 2v \cdot \frac{dv}{dv} = mv.
   \]

   (b) Who takes derivatives with respect to velocity? No one. Except you, just now. Sorry. (It is just coincidence that the derivative of $KE$ is momentum; there is no real physical significance to this.)

   But, the rate of change of energy with respect to time is something meaningful: Power. Now, consider velocity $v$ to be a function of time, $v(t)$. We will rewrite $KE$ showing this time dependence: $KE(t) = \frac{1}{2} \cdot m \cdot v(t)^2$. Show that the derivative of $KE$ with respect to $t$ is the force times the velocity, $Fv$. That is, compute $\frac{d(KE)}{dt}$. \((Hint: use Newton’s second law, $F = ma$, to simplify.)\)

   \[
   \frac{d(KE)}{dt} = \frac{d}{dt} \left( \frac{1}{2} mv^2 \right) = \frac{1}{2} \cdot m \cdot 2v \cdot \frac{dv}{dt} = \underline{mv\frac{dv}{dt}}.
   \]

   (c) In the computation above, we assumed $m$ was constant, and $v$ was changing in time.

   Think of a physical situation in which both $m$ and $v$ are varying in time.

   Rocket burning fuel.

   (d) Compute the Power when both mass and velocity are changing in time. (First rewrite $KE(t)$ showing the time dependence, then compute $\frac{d(KE)}{dt}$.)

   \[
   \frac{d(KE)}{dt} = \frac{d}{dt} \left( \frac{1}{2} mv^2 \right) = \frac{1}{2} \frac{dm}{dt} v^2 + \frac{1}{2} m \cdot 2v \cdot \frac{dv}{dt} = \underline{\frac{1}{2} \frac{dm}{dt} v^2 + m v \frac{dv}{dt}}.
   \]

In this and all of the following problems, we differentiate with respect to $t$, but $t$ is not always explicitly present in the equation! This is implicit differentiation.
2. Ohm's law gives the relationship between voltage, resistance, and current in a simple circuit: 
\[ V = IR. \]

(a) Find \( \frac{dV}{dt} \) in terms of \( I, R, \) \( \frac{dI}{dt} \) and \( \frac{dR}{dt} \). Hint: this is a lot like problem #1(d). 

\[
\frac{dV}{dt} = \frac{d}{dt}(IR) = \left( \frac{dI}{dt} R + I \frac{dR}{dt} \right)
\]

(b) We have a circuit with a 500 \( \Omega \) resistor, and we ramp up the voltage. How quickly is the current changing when we are increasing the voltage by 10 V/s? It will help to remember that \( t \) is the independent variable here. Include units, note that 1 \( \Omega = 1V/A \).

\[
\frac{dV}{dt} = \frac{dI}{dt} R + I \frac{dR}{dt}
\]

\[
10 \text{ V/s} = \frac{dI}{dt} 500 \Omega + I \cdot 0
\]

\[
\frac{10 \text{ V/s}}{500 \text{ V/A}} = \frac{dI}{dt}
\]

\[
\frac{1}{50} \text{ A/s} = \frac{dI}{dt}
\]

(c) Is the current increasing or decreasing? How can you tell?

3. When gas expands adiabatically (without gaining or losing heat) in a gasoline engine, the pressure and volume of the gas are related: 
\[ PV^{7/3} = 1.58 \times 10^9. \]

(a) Take the derivative of the equation above to find a formula relating \( P, V, \frac{dP}{dt}, \) and \( \frac{dV}{dt} \).

\[
\frac{d}{dt}(PV^{7/3}) = \frac{d}{dt}(1.58 \times 10^9)
\]

\[
\frac{dP}{dt} V^{7/3} + P \cdot \frac{7}{3} V^{4/3} \frac{dV}{dt} = 0
\]

(b) Suppose the volume of gas is increasing at a rate of 0.2 \( \text{cm}^3/\text{s} \). When the volume is 300\( \text{cm}^3 \), how quickly is the pressure changing? Hint: You'll need to compute \( P \).

\[
P = \frac{1.58 \times 10^9}{(300)^{7/3}} = 2622.451 \text{ units of pressure.}
\]

\[
\frac{dP}{dt} (300)^{7/3} + (2622.451) \cdot \frac{7}{3} (300)^{4/3} \cdot 0.2 \text{ cm}^3/\text{s} = 0
\]

Solve for \( \frac{dP}{dt} = -4.07937 \text{ units of pressure per second.} \)