Activity 13 - Kinematics

Learning outcome: Solve initial value problems.

0. (Warm-up) Suppose the velocity of a rocket is given by \( v(t) = t^2 - 2t + 1 \) as \( 0 \leq t \leq 2 \). Also suppose that the rocket is initially located on the ground.

(a) Set up an initial value problem for the position of the rocket \( x(t) \). You need two things:
   (i) a differential equation involving \( x'(t) \), and
   (ii) an initial condition \( x(t_0) = x_0 \).

\[
\begin{align*}
    x'(t) &= t^2 - 2t + 1 \\
    x(0) &= 0
\end{align*}
\]

(b) In part (a), you shouldn't have actually solved anything. You just wrote the math problem to solve. Now, let's solve it! Integrate the differential equation to find \( x(t) \).

Use the initial condition to solve for \( C \).

\[
x(t) = \frac{t^3}{3} - 2\frac{t^2}{2} + t + C
\]

\[
0 = x(0) = 0^3 - 2\frac{0^2}{2} + 0 + C \quad \Rightarrow \quad C = 0
\]

(c) What have you done? Using the derivative and an initial value, you found the position of the rocket at any point in time \( t \). In the rest of the activity, we'll derive a very useful equation in physics called a kinematic equation.

If you've taken a physics class, you've probably seen some equations like \( x(t) = -\frac{1}{2}gt^2 + v_0t + x_0 \), where \( g \) is the acceleration due to gravity (\( g = 9.8 \text{ m/s}^2 \approx 32.2 \text{ ft/s}^2 \)), \( v_0 \) is the initial velocity, and \( x_0 \) is the initial height of the object above the ground. These equations don't come from physics, they come from mathematics (recall, mathematics > physics)!

We start with Newton's second law. (Newton, by the way, was a mathematician. He is credited for co-founding calculus with Gottfried Leibniz.)

\[
F = ma
\]

That is, force is equal to mass times acceleration.

1. Newton's second law is a differential equation involving position \( x(t) \). Rewrite Newton's second law so that you see the derivative. The only variables in your equation should be \( F \), \( m \), and \( x(t) \).

\[
F = m x''(t)
\]
2. For this activity, we'll consider an object under the force of gravity. Like a rock, falling down a well of unknown height. Or like Evel Knievel after he's launched out of a circus cannon. These are important problems: I want to know how deep that well is, and we need to know where on the ground to put a pillow so we can save Evel Knievel a trip to the hospital.

Use the fact that the force due to gravity is \( F = -m \cdot g \). Simplify the differential equation from the previous problem so that the only variables you see are \( x(t) \) and \( g \).

\[-mg = mx''(t)\]
\[-g = x''(t)\]

3. So, we have a differential equation. Now we need some initial conditions. What are initial values for? (See problem 0b for help.) Since the equation above uses two derivatives, how many initial conditions do we need?

Two initial cond.

Solve for the constant c

4. Suppose at time \( t_0 = 0 \), the position of the object is given by \( x_0 \) and the initial velocity is \( v_0 \).

Write an initial value problem including the differential equation and the initial conditions.

The differential equation and both initial conditions should be expressed in terms of \( x(t) \).

\[
\begin{cases}
  x''(t) = -g \\
  x'(t) = v_0 \\
  x(t_0) = x_0 \\
\end{cases}
\]

5. Solve the initial value problem by integrating twice. You should get \( x(t) = -\frac{1}{2}gt^2 + v_0t + x_0 \).

\[
x''(t) = -g \\
x'(t) = \int -g \, dt \\
x'(t) = -gt + c \\
x_0 = x'(0) = -g \cdot 0 + c \\
v_0 = c \\
x'(t) = -gt + v_0 \\
x(t) = \int -gt + v_0 \, dt = -\frac{1}{2}gt^2 + v_0t + c \\
\]

\( c = x_0 \)
6. Remember the well from Activity 11? We said that \( s(t) = h - 16t^2 \). Where did that equation come from? Using the fact that \( g = 32 \text{ ft/s}^2 \), explain how we came up with that equation.

\[
\begin{align*}
  v_0 &= 0 \\
  g &= 32 \\
  x_0 &= h \\
\end{align*}
\Rightarrow
\begin{align*}
  x(t) &= \frac{-32}{2} t^2 + v_0 t + x_0 \\
  x(t) &= h - 16t^2 \\
\end{align*}
\]

7. Evel Knievel is shot out a cannon located on the ground. The cannon is positioned at some angle, so Evel travels up and to the right (2D motion!). We want to find out where to put the pillow so that he lands safely. To do this, we'll split up the dimensions into two problems.

(a) Let's first just look at his vertical motion, ignoring his motion left and right. We can use the equation we found in step (5) since gravity is influencing Evel’s vertical position. Suppose the upward component of his velocity out of the cannon is 60 ft/s. Use the kinematic equation to find out when Evel hits the ground.

\[
x(t) = -\frac{32}{2} t^2 + 60t + x_0 \\
x_0 = 0 \\
v_0 = 60 \\
\begin{align*}
  x(t) &= -16t^2 + 60t \\
  0 &= -16t^2 + 60t \\
  0 &= t(-16t + 60) \\
  t &= \frac{60}{16} = 3.75 \text{ sec}
\end{align*}
\]

(b) Now let's just look at Evel’s horizontal motion, ignoring his motion up and down. We can’t use the same equation as we derived in step (5) because gravity is not influencing Evel’s horizontal position. Horizontally, there are no forces acting on Evel (we're neglecting air resistance). Repeat steps (1-5) using \( F = 0 \) instead of \( F = -mg \) to derive an equation for Evel’s horizontal position \( x(t) \).

\[
\begin{align*}
  x''(t) &= 0 \\
  x'(0) &= v_0 \quad \text{horiz. direction} \\
  x(0) &= 0 \\
\end{align*}
\Rightarrow
\begin{align*}
  x(t) &= v_0 t + x_0 \\
  x(t) &= v_0 t
\end{align*}
\]

(c) Suppose the rightward component of his velocity out of the cannon is 20 ft/s. Equipped with the time Evel hits the ground and your new kinematic equation for his horizontal position, find out how far away from the cannon you should put the pillow. He's counting on you; don't let him down.

\[
x(3.75) = 20 \cdot 3.75 = 75 \text{ ft}
\]