RESEARCH STATEMENT

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Broadly speaking my research falls under the umbrellas of combinatorics, logic and the foundations of mathematics. My areas of interest are higher-dimensional Ramsey theory, nonstandard arithmetic and their applications. I have separated this document into five sections:

- Ramsey spaces,
- Canonical Ramsey theory,
- Tukey theory of ultrafilters,
- Dedekind cuts of nonstandard arithmetic, and
- Infinite ethics.

In each section, I identify a research goal, some open questions and some problems that will guide my upcoming research agenda.

I. Ramsey spaces

Research Goal 1. Identify general methods for constructing new Ramsey spaces from a collection of given Ramsey spaces with the caveat that at least some of the spaces arising from the construction have interesting applications beyond simply being a Ramsey space.

Ramsey spaces provide a systematic framework for extending finite-dimensional Ramsey theoretic results to their infinite-dimensional counterparts, see [14]. The theory of Ramsey spaces has applications to set theory, combinatorics, the theory of Banach spaces, the Tukey theory of ultrafilters, and topological dynamics. The concept of a Ramsey space was introduced by Todorcevic and forms the basic notion of study in the introductory textbook [14]. The abstract Ramsey theorem, the main result of chapter 4, provides four axioms which can be used to give examples of Ramsey spaces. It is easy to show that all Ramsey spaces must satisfy the fourth axiom in the statement of the abstract Ramsey theorem. However, it is still unknown if the first three axioms are necessary.

Question 1. Are there examples of Ramsey spaces which do not satisfy the first three axioms from the statement of the abstract Ramsey theorem?

Most examples of Ramsey spaces have been constructed with particular applications in mind. Until recently, there have been no systematic methods for constructing new Ramsey spaces from currently known Ramsey spaces. In [16], [18] and [7], notions of products have been introduced for specific topological Ramsey spaces. Additionally, in [15] a notion of parametrization among topological Ramsey spaces is studied. Here, by topological Ramsey space we mean a Ramsey space that can be endowed with a natural topology defined from its Ramsey space structure.

In [8], Dobrinen and Todorcevic have constructed a topological Ramsey space $\mathcal{R}_1$ that in its complexity comes immediately after the Ellentuck space. In [9], Dobrinen and Todorcevic use a similar construction to identify a new hierarchy of topological Ramsey spaces $\langle \mathcal{R}_\alpha : \alpha < \omega_1 \rangle$ where $\mathcal{R}_0$ is the Ellentuck space and each $\mathcal{R}_{\alpha+1}$ coming immediately after $\mathcal{R}_\alpha$ in complexity.

Problem 1. Suppose that $\mathcal{R}$ is a topological Ramsey space. Generalize the construction from [9] to obtain a sequence of spaces $\langle \mathcal{R}_\alpha : \alpha < \omega_1 \rangle$ where $\mathcal{R}_0$ is $\mathcal{R}$ and each $\mathcal{R}_{\alpha+1}$ coming immediately after $\mathcal{R}_\alpha$ in complexity.

The fourth axiom is called the pigeonhole principle for the Ramsey space and will be equivalent to some partition property. For example, in the Ellentuck space, see [14], the pigeonhole principle is equivalent to the following statement: if the natural numbers are partitioned into two parts then at least one part of the partition is infinite. That is, $\omega \to (\omega)^1$. In [15], I have shown that there are Ramsey space whose pigeonhole principles are equivalent to weakened versions of the finite-dimensional Ramsey theorem.

Question 2. To what partition properties can we associate a Ramsey space such that the fourth axiom, the pigeonhole principle for the space, is equivalent to the given partition relation?

The spaces constructed in [15] are Ramsey spaces parametrized by the Ellentuck space and other Ramsey spaces. The quasi-orders used in these constructions, although they apply to any pair of topological Ramsey spaces, sometimes fail to be restrictive enough to give rise to a topological Ramsey space.

Problem 2. Identify more restrictive quasi-orders that give rise to new Ramsey spaces that are parametrized by other Ramsey spaces. Then use these theorems to give new proofs of parametrized perfect set theorems as in [15].

II. Canonical Ramsey theory

Research Goal 2. Prove new canonical Ramsey theorems for topological Ramsey spaces.

Topological Ramsey theory guarantees the existence of monochromatic sets for colorings, with finitely many colors, satisfying certain topological conditions such as begin continuous with respect to the Ellentuck topology. Canonical Ramsey theory considers colorings that have infinitely many colors. Note that there are colorings with infinitely many colors which are not monochromatic on any set. However, for spaces like the Ellentuck space there is a list of canonical colorings such that for any coloring c there is a canonical coloring f and a set such that for all X and Y in the set, $c(X) = c(Y)$ if and only if $f(X) = f(Y)$. These more general colorings provide for more powerful theorems and a wide range of applications that can not be obtained by the topological Ramsey theory alone.

Many of the factors in the parametrized spaces constructed in [15] have a well-studied canonical Ramsey theory. The main theorem of [15] identifies a way to prove the pigeonhole principle for the parametrized space using the pigeonhole principles from each factor provided that one space diagonalizes the other space.

Problem 3. Suppose that $\mathcal{R}$ parametrizes $\mathcal{S}$ as in [15] and there are canonical Ramsey theorems for each factor. Show that if $\mathcal{R}$ diagonalizes $\mathcal{S}$, as defined in [15], then these two canonical theorems can be used to prove a canonical Ramsey theorem for the parametrized space.

III. Tukey theory of ultrafilters

Research Goal 3. Identify new initial Tukey structures for ultrafilters.

An ultrafilter $\mathcal{U}$ is Tukey-reducible to an ultrafilter $\mathcal{V}$ if there is a map from $\mathcal{U}$ to $\mathcal{V}$ such that the image of each cofinal subset of $(\mathcal{U}, \supseteq)$ under the map is a cofinal subset of $(\mathcal{V}, \supseteq)$. For example, if $f : \mathbb{N} \to \mathbb{N}$ and $\mathcal{V} = \{f^{-1}(X) : X \in \mathcal{U}\}$, then $f^{-1} : \mathcal{U} \to \mathcal{V}$ is well-defined and maps cofinal subsets of $(\mathcal{U}, \supseteq)$ to cofinal subsets of $(\mathcal{V}, \supseteq)$. In other words, if $\mathcal{U}$ is Rudin-Keisler reducible to $\mathcal{V}$ then $\mathcal{U}$ is Tukey-reducible to $\mathcal{V}$. We say that the ultrafilters are Tukey-equivalent if $\mathcal{U}$ and $\mathcal{V}$ are both Tukey reducible to one another. The equivalence class of an ultrafilter under this equivalence relation is called its Tukey type. Tukey types of ultrafilters are partially-ordered by the Tukey-reducibility relation. An initial Tukey type is any partial order that is isomorphic to the set of all Tukey-types of ultrafilters that are Tukey-reducible to some fixed ultrafilter $\mathcal{U}$ under the Tukey-reducibility relation. Rudin-Keisler reducibility has been well-studied so it is natural to try to compare the two notions of reducibility.

Problem 4. Given an ultrafilter $\mathcal{U}$, identify the Rudin-Keisler structure of the ultrafilters within the Tukey-type of $\mathcal{U}$.

Todorcevic has made a connection between canonical Ramsey theory and the Tukey theory of ultrafilters on ω (see Theorem 24 in [12]). Todorcevic has shown selective ultrafilters realize minimal Tukey types in the class of all ultrafilter on ω. More recently, Dobrinen and Todorcevic in [8] and [9] have identified generalizations of these results by proving new canonical Ramsey theorems for certain topological Ramsey spaces. Dobrinen and Todorcevic have shown that Ramsey ultrafilters for these topological Ramsey spaces realize initial Tukey structures consisting of descending chains of rapid p-points of order type $\alpha + 1$ for $\alpha < \omega_1$. Dobrinen, Mijares and I, in [7], have also identified new canonical Ramsey theorems for topological Ramsey spaces constructed from Fraïssé classes which realize new initial Tukey structures. Among the initial types identified in [7] are the collection of all finite Boolean algebras. In order to apply the canonical Ramsey theory and obtain these results it is necessary that the ultrafilters being analyzed have continuous Tukey-reductions, see [6].

Question 3. For which topological Ramsey spaces $\mathcal{R}$ do the notions of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ ultrafilters, as defined by Mijares in [11], have continuous Tukey-reductions as defined in [6]?

IV. Dedekind cuts of Nonstandard models of arithmetic

Research Goal 4. For each topological Ramsey space $\mathcal{R}$, characterize the Dedekind cuts associated to ultrafilter mappings among ultrafilters in the Tukey type of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ ultrafilters.

Associated to each ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and each map $p : \mathbb{N} \to \mathbb{N}$ is a Dedekind cut in the ultrapower $\mathbb{N}^{\mathbb{N}}/p(\mathcal{U})$. In [2], Blass has characterized, under CH, the cuts obtainable when $\mathcal{U}$ is taken to be either a p-point ultrafilter, a weakly-Ramsey ultrafilter or a Ramsey ultrafilter.

Dobrinen and Todorcevic have introduced the topological Ramsey space $\mathcal{R}_1$. In [17], I characterized, under CH, the cuts obtainable when $\mathcal{U}$ is taken to be a Ramsey for $\mathcal{R}_1$ ultrafilter and p is taken to be any map. In particular, I showed that the only cut obtainable is the standard cut, whose lower half consists of the collection of equivalence classes of constants maps.

Question 4. Under CH, is the standard cut the only Dedekind cut that can arise from a selective for $\mathcal{R}_1$ ultrafilter?

Forcing with $\mathcal{R}_1$ using almost-reduction adjoins an ultrafilter which is Ramsey for $\mathcal{R}_1$. For such ultrafilters $\mathcal{U}_1$, Dobrinen and Todorcevic have shown that the Rudin-Keisler types of the p-points within the Tukey type of $\mathcal{U}_1$ consists of a strictly increasing chain of rapid p-points of order type ω. I have shown that for any Rudin-Keisler mapping between any two p-points within the Tukey type of $\mathcal{U}_1$ the only cut obtainable is the standard cut. These results imply existence theorems for special kinds of ultrafilters. As mentioned above, Dobrinen and Todorcevic have also introduced the topological Ramsey spaces $\mathcal{R}_\alpha$ for $\alpha < \omega_1$ and proven canonical Ramsey theorems for these spaces.

Problem 5. Under CH, characterize the Dedekind cuts that can arise from Ramsey for $\mathcal{R}_\alpha$ ultrafilters, for $\alpha < \omega_1$.

Problem 6. Under CH, characterize the Dedekind cuts that can arise from p-points ultrafilters in the Tukey type of a Ramsey for $\mathcal{R}_\alpha$ ultrafilter, for $\alpha < \omega_1$.

In [2], the Dedekind cuts are characterized by applying the finite version of the Ramsey theorem for pairs. In fact, upper and lower bounds for the Ramsey numbers are used to give characterizations of Dedekind cuts arising from Ramsey ultrafilters. Recently, a small research group of three undergraduates, under my supervision, have obtained new results about upper and lower bounds for Ramsey numbers coming from the finite version of the Ramsey property for the space $\mathcal{R}_1$. These give rise to new characterizations of the Dedekind cuts arsing from Ramsey for $\mathcal{R}_1$ ultrafilters in terms of the upper and lower bounds for these Ramsey numbers. This line of reasoning motivates the next problem.

Problem 7. Identify new and known upper and lower bounds to the finite versions of the Ramsey property coming from new and known topological Ramsey spaces. Then uses those upper and lower bounds to characterize the Dedekind cuts arising from ultrafilters that are Ramsey for those spaces.

V. Infinite Ethics

Bostrom in [3] explores the 'the hyperreal approach' as a method of remedying the problem of infinitarian paralysis. This approach uses the theory of nonstandard analysis, pioneered by Robinson in the 1960s [13], to modify the value rule extending its codomain from the real numbers to the hyperreal numbers.

Research Goal 5. Construct formal aggregative consequentialist ethical systems that avoid the problem of infinitarian paralysis by carefully choosing the codomain $\mathcal{C}$, the aggregation rule $\bigsqcup$ and the summation rule $\sum$. In particular, study the following extensions:

(1) Hyperreal Extensions
   (a) In the context of the elementary axiology of the Alpha-Theory (see [1]).
   (b) In the context of iterated hyper-extensions (*e.g* reals, hyperreals, hyperhyperreals, hyperhyperhyperreals,... see [1, 4]).
   (c) In the context of Cauchy's infinitesimals principle (CIP, see [1]).

(2) Surreal Extensions (see [10])
(3) Numerosity Extensions (see [5])
(4) Non-Archimedean l-group Extensions (For example, the set of continuous functions on the open interval $(0, 1)$.)

These non-Archimedean extensions avoid the problem in the proof of the result on infinitarian paralysis since for all positive values Ω and M (finite or infinite),

$$\Omega - M < \Omega < \Omega + M.$$

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