2.4 Infinite Limits

**Driving Question:** How do we describe functions that grow without bound?

**Objectives:**
1. Interpret limits which result in a nonzero numerator and a denominator tending toward zero.
2. Determine when limits tend to infinity
   - Graphically, tabular and analytically
   - Specifically distinguish infinity and DNE.

**Idea:**
An infinite limit occurs when function values increase or decrease without bound near a point.

**Def:**
\[
\lim_{{x \to a}} f(x) = \infty
\]

means
\( f(x) \) gets arbitrarily large for all \( x \) sufficiently close to \( a \).

**Remark:** In this case, by def the limit does not exist.
Example

\[ y = \frac{2}{x^2} \]

Vertical asymptote at \( x = 0 \).

\[
\lim_{x \to 0} \frac{2}{x^2} = \infty
\]

Definition

\[
\lim_{x \to a} f(x) = -\infty
\]

means

\( f(x) \) gets arbitrarily large in magnitude and \( f(x) \) is negative for all \( x \) sufficiently close to \( a \).
Ex. Let \( f(x) = \frac{3}{x^2} \)

Vertical asymptote at \( x = 0 \).

\[
\lim_{x \to 0^-} f(x) = -\infty \quad \lim_{x \to 0^+} f(x) = \infty
\]

\( \lim_{x \to 0} f(x) \) DNE.

Finding infinite limits analytically:

Suppose that

\[
\begin{align*}
  f(x) &\to L \neq 0 \quad \text{as} \quad x \to a, \\
g(x) &\to 0
\end{align*}
\]

Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \infty \quad \text{if} \quad L > 0 \quad (-\infty \quad \text{if} \quad L < 0)
\]
Ex

Analyze \( \lim_{x \to 5} \frac{2x+7}{x-5} \).

First consider

\[ \lim_{x \to 5^+} \frac{2x+7}{x-5} \text{ approaches } 17 \]
\[ \lim_{x \to 5^-} \frac{2x+7}{x-5} \text{ approaches } 0 \]

That is, \( 2x+7 \to 17 \) and \( x-5 \to 0 \) as \( x \to 5^+ \).

Note also that for \( x > 5 \), \( \frac{2x+7}{x-5} > 0 \).

Thus, \( \lim_{x \to 5^+} \frac{2x+7}{x-5} = \infty \).

On the other hand, \( 2x+7 \to 17 \) and \( x-5 \to 0 \) as \( x \to 5^- \). But, for \( x < 5 \), \( \frac{2x+7}{x-5} < 0 \).

Thus, \( \lim_{x \to 5^-} \frac{2x+7}{x-5} = -\infty \).

Since the left and right handed limits are different,

\[ \lim_{x \to 5} \frac{2x+7}{x-5} \text{ DNE} \]
Ex

Analyze \( \lim_{x \to \pi/2} \tan(x) \),

(Right-hand limit)

\[
\lim_{x \to \pi/2^+} \tan(x) = \lim_{x \to \pi/2^+} \frac{\sin(x)}{\cos(x)} \quad \text{approaches 1}
\]

that is, \( \sin(x) \to 1 \) and \( \cos(x) \to 0 \) as \( x \to \pi/2^+ \).

For \( \pi/2^+ > x > \pi/2 \), \( \frac{\sin(x)}{\cos(x)} < 0 \).

Thus, \( \lim_{x \to \pi/2^+} \tan(x) = -\infty \)

(Left-hand limit)

Also, \( \sin(x) \to 1 \) and \( \cos(x) \to 0 \) as \( x \to \pi/2^- \).

For \( 0 < x < \pi/2 \), \( \frac{\sin(x)}{\cos(x)} > 0 \).

Thus \( \lim_{x \to \pi/2^-} \tan(x) = \infty \)

So, \( \lim_{x \to \pi/2} \tan(x) \) DNE b/c the two limits don't match.