Section 3.2 Working with derivatives

Question: What can we say about $f'(x)$?

Objectives:

1. Interpret $f'(x)$ as a function whose output at any point is the slope of the tangent line of $f(x)$ at that point.

2. Recall that differentiability implies continuity.

3. Identify points where a function is not differentiable.

Example

Sketch the graph of $f'(x)$.
Graph of $f'(x)$

What is happening at $x = -3, -1, 3$?

\[
\lim_{h \to 0^+} \frac{f(-3 + h) - f(-3)}{h} = -1 \quad \text{all secant lines with } h > 0 \text{ have slope of -1.}
\]

\[
\lim_{h \to 0^-} \frac{f(-3 + h) - f(-3)}{h} = 1 \quad \text{all secant lines with } h < 0 \text{ have slope of 1}
\]

So \[
\lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h} \text{ DNE. That is } f \text{ is not differentiable at } x = -3.
\]
**Defn.** If \( \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \) and \( \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \) both exist and
\[
\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \neq \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}
\]
then \( f(x) \) is said to have a **corner** at \( x = a \).

**Example.** \( f(x) \) from last example has corners at \( x = -3, -1, 3 \).

**Thm.** If \( f(x) \) has a corner at \( x = a \) then \( f'(x) \) does not exist.

**Defn.** If \( \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \pm \infty \) and \( \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \pm \infty \) then we say \( f(x) \) has a **vertical tangent** line at \( x = a \).
Sketch the graph of $f'(x)$.

**Example**

Horizontal tangents at $x = -1.5, -0.5, 0.5, 1.5$

Vertical tangents at $x = -1, 0, 1$

\[
\begin{align*}
\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} &= -\infty \\
\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} &= \infty \\
\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} &= -\infty \\
\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \infty \\
\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} &= -\infty \\
\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} &= \infty
\end{align*}
\]
Corners at $x = -2 \ & 2$

\[
\begin{align*}
  x = 2 & \quad \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \infty \\
  & \quad \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = 1
\end{align*}
\]

\[
\begin{align*}
  x = -2 & \quad \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} = -1 \\
  & \quad \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} = \infty
\end{align*}
\]

Thus, $f(x)$ is not differentiable at $x = -2, -1, 0, 1, 2$.

Graph of $f'(x)$
Q. When is a function not continuous?

1. \( f \) is not continuous at \( x = a \).
2. \( f \) has a corner at \( x = a \).
3. \( f \) has a vertical tangent line.

Example

Find the values of \( y(x) \) at which \( g(x) \) is not differentiable.

\[
\begin{align*}
  x = -2 & \quad \text{corner} \\
  x = -1 & \quad \text{Not continuous} \\
  x = 0 & \quad \text{Not continuous} \\
  x = 1 & \quad \text{Vertical tangent} \\
  x = 2 & \quad \text{Vertical tangent (cusp)}
\end{align*}
\]
Example
Sketch the graph of $g'(x)$.
Example

Sketch the graph of $h(x)$.

Graph of $h(x)$.

Graph of $h'(x)$. 