Section 4.4 Optimization

Objectives:
1. Solve applied optimization problems
2. Justify solutions to optimization problems (derivative tests!)

Optimization Problems:

What is the maximum (minimum) value of an **objective function** subject to the given **constraints**?

Example:

(Walking and swimming)

A man wishes to get from an initial point on the shore of a circular lake to a point on the opposite side of the lake. (Diameter of lake is 2 mi). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.
(a) If he swims at 2mi/hr and walks at 4mi/hr, what are the maximum and minimum times for the trip?

\[
\begin{align*}
\text{distance (swim)} &= 2 \cdot \sin \left( \frac{\theta}{2} \right) \\
\text{distance (walk)} &= \frac{1}{2} (\pi - \theta) = \frac{\pi}{2} - \theta \\
\text{time (swim)} &= \frac{2 \sin \left( \frac{\theta}{2} \right)}{2} = \sin \left( \frac{\theta}{2} \right) \text{ (hr)} \\
\text{time (walk)} &= \frac{\pi - \theta}{4} = \frac{\pi}{4} - \frac{\theta}{4} \text{ (hr)}
\end{align*}
\]
Objective function

\[ T(\theta) = \sin(\theta/2) + \frac{\pi}{4} - \frac{\theta}{4} \quad [0, \pi] \]

\[ T'(\theta) = 0 \]

\( \cos(\theta/2) \cdot \frac{1}{2} - \frac{1}{4} = 0 \)

\( \frac{1}{2} \cos(\theta/2) = \frac{1}{4} \)

\( \cos(\theta/2) = \frac{1}{2} \)

\( \theta/2 = \frac{\pi}{3} \)

\( \theta = 2\pi/3 \)

\( T(0) = \sin(0) + \frac{\pi}{4} - 0\% = \frac{\pi}{4} \)

\( T(\pi) = \sin(\pi/2) + \frac{\pi}{4} - \frac{\pi}{4} = 1 \)

\( T(2\pi/3) = \sin(\pi/3) + \frac{\pi}{4} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{\pi}{4} - \% \)

\( T(0) \approx 0.785 \text{ hr} \)

\( t(\pi) \approx 1 \text{ hr} \)

\( t(2\pi/3) \approx 1.128 \text{ hr} \)
(b) If he swims at 2mi/hr and walks at 1.5mi/hr, what are the min. and max. times for the trip.

Objective function

\[ T(\theta) = \sin(\theta/2) + \frac{\pi - \theta}{1.5} \]

\[ T(\theta) = \sin(\theta/2) + \frac{2\pi}{3} - \frac{2\theta}{3} \]

\[ T'(\theta) = \frac{1}{2} \cos(\theta/2) - \frac{2}{3} \]

\[ \frac{1}{2} \cos(\theta/2) - \frac{2}{3} = 0 \]

\[ \frac{1}{2} \cos(\theta/2) = \frac{2}{3} \]

\[ \cos(\theta/2) = \frac{4}{3} \]

No solution

\[ T(\theta) = \sin(0) + \frac{\pi - 0}{1.5} = \frac{2\pi}{3} \approx 2.09 \text{ hr} \]

\[ T(\pi) = \sin(\pi/2) + \frac{\pi - \pi}{1.5} = 1 \approx 1 \text{ hr} \]
Example
(Circle & Square)

A piece of wire of length 60 is cut, and the resulting two pieces are formed to make a circle and a square, where should the wire be cut to minimize and maximize the combined area?

\[ \begin{align*}
2 \pi r &= x \\
r &= \frac{x}{2\pi} \\
\text{area} &= \pi \left( \frac{x}{2\pi} \right)^2 \\
&= \frac{x^2}{4\pi} \\
\text{Objective function} &= A_1(x) = \frac{x^2}{4\pi} + \frac{(60-x)^2}{16} \quad \text{[0, 60]}.
\end{align*} \]
\[ A'(x) = \frac{x}{2\pi} + \frac{(60-x)}{8} (-1) \]

\[ A'(x) = \frac{x}{2\pi} + \frac{x-60}{8} \]

\[ A'(x) = \frac{4x}{8\pi} + \frac{\pi x - 60\pi}{8\pi} \]

\[ A'(x) = 0 \]

\[ (4+\pi)x - 60\pi = 0 \]

\[ (4+\pi)x = 60\pi \]

\[ x = \frac{60\pi}{4+\pi} \approx 26.39 \]

**2nd Derivative Test.**

\[ A''(x) = \frac{1}{2\pi} + \frac{1}{8} \]

\[ A'' \left( \frac{60\pi}{4+\pi} \right) = \frac{1}{2\pi} + \frac{1}{8} > 0 \]

\[ \text{concave up} \]

\[ x = 26.39 \text{ is the location of the abs. min. value.} \]
The smallest area is

\[ A(26.39) = \frac{(26.39)^2}{4\pi} + \frac{(60 - 26.39)^2}{16} \]

- Largest must happen at endpoints:

\[ A(0) = 0 + \frac{60^2}{16} = 225 \]
\[ A(60) = \frac{60^2}{4\pi} + \frac{0^2}{16} \approx 286.479 \]

↑ Abs. max. area,

↑ Use all wire to make a circle!
\[ D(x) = \sqrt{(x-1)^2 + ((1-x^2)-1)^2} \]
\[ D(x) = \sqrt{x^2 - 2x + 1 + x^4} \]
\[ D'(x) = \frac{1}{2} \left( x^4 + x^2 - 2x + 1 \right)^{-\frac{1}{2}} \left( 4x^3 + 2x - 2 \right) \]
\[ D'(x) = 0 \]
\[ 4x^3 + 2x - 2 = 0 \]

\[ \text{sign chart} \]

One solution at \( x \approx 0.590 \).

Abs. min at \( (0.590, 1 - 0.59^2) \) and \( (0.590, 0.652) \) at \( x = 0.590 \).

No solution.