Section 4.5  Linear approximation.

**Question:** How are derivatives used as approximation tools?

**Objectives:**

1. Interpret tangent line as an approximation to $f(x)$ near $a$.
2. Use linear approximation to approximate values of $f(x)$ near $a$.
3. Use concavity to determine if approx. is an overestimate or underestimate.

**Idea:**

For values near $a$, the tangent line is a good approx. to $f(x)$.

**Equation of tangent line:**

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

**Linear approx. of $f(x)$ at $a$:**

$$L(x) = f(a) + f'(a)(x-a)$$
Theorem: For values of $x$ near $a$, \( f(x) \approx L(x) \).

That is, \( f(x) \approx f(a) + f'(a)(x-a) \).

Example: Let \( f(x) = \sqrt{x} \). Find the linear approx. of \( f(x) \) at \( a = 4 \).

\[
\begin{align*}
f(x) &= x^{\frac{1}{2}} \\
f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\end{align*}
\]

Point:
\( f(a) = \sqrt{4} = 2 \)
\( (4, 2) \)

Slope:
\[
m = f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}
\]

Linear approx:
\( L(x) = 2 + \frac{1}{4}(x - 4) \)

\( L(x) = \frac{1}{4}x + 2 \)
Ex: Estimate $\sqrt{4.1}$ using the linearization.

$$\sqrt{4.1} \approx L(4.1) = \frac{1}{4}(4.1) + 1 = 1.025 + 1 = 2.025$$

Actual value $\sqrt{4.1} = 2.24845$

The error

$$|\sqrt{4.1} - L(4.1)| = 0.00154327$$

Percentage error

$$\left| \frac{\sqrt{4.1} - L(4.1)}{\sqrt{4.1}} \right| \times 100 = 0.0762177\%$$

Ex: Estimate $\sqrt{2}$ using the linearization.

$$\sqrt{2} \approx \frac{1}{4}(2) + 1 = 1.5$$

Actual value $\sqrt{2} = 1.41235$

% error

$$\left| \frac{\sqrt{2} - L(2)}{\sqrt{2}} \right| \times 100 = 6.07\%$$
Find the linear approx. \[ f(x) = \ln(x+3) \quad a = -2. \]

\[ f'(x) = \frac{1}{x+3} \]

\[ f'(a) = \frac{1}{-2+3} = \frac{1}{1} = 1 \]

Slope:

Point:
\[ f(a) = \ln(-2+3) = \ln(1) = 0 \]

\[ (-2, 0) \]

\[ L(x) = 0 + 1(x+2) \]

\[ L(x) = x+2 \]

\[ a \text{ linear approx. at } a = -2. \]

Approximate \( \ln 2 \) with the linear approx.

\[ f(x) \approx L(x) \]

\[ \ln(2) = \ln(-1+3) = f(-1) \approx -1+2 = 1 \]
Ex: Let \( f(x) = e^{-x^{3/2}} \). Find the linear approximation of \( f(x) \) at \( a = 1 \).

\[
\begin{align*}
  f'(x) &= e^{-x^{3/2}} \cdot \left( -\frac{3x}{2} \right) \\
  f'(x) &= -x e^{-x^{3/2}}
\end{align*}
\]

Slope: \( m = f'(a) = \\
= -1 \cdot 1^{-3/2} = -e^{-1/2} \)

Point: \( f(a) = e^{-1/2} = e^{-1/2} = \frac{1}{\sqrt{e}} \)

\((1, \frac{1}{\sqrt{e}})\)

\[
\begin{align*}
  L(x) &= \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}} (x - 1) \\
  L(x) &= \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}} x + \frac{1}{\sqrt{e}} \\
  L(x) &= -\frac{1}{\sqrt{e}} x + \frac{2}{\sqrt{e}}
\end{align*}
\]

Linear approx. at \( a = 1 \).
Ex:
Use your approximation to estimate $e^{-1}$.

$$e^{-x^2} \approx L(x) = -\frac{1}{\sqrt{e}} x + \frac{2}{\sqrt{e}}$$

$$e^{-1} = e^{-\left(\frac{\sqrt{2}}{2}\right)^2} \approx L\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{\sqrt{e}} + \frac{2}{\sqrt{e}}$$

$$= \frac{2 - \sqrt{2}}{\sqrt{e}} \approx 0.3583$$

Actual value $e^{-1} = 0.3678795...$

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**Estimate**

- $f''(a) > 0$ concave up
  - under-estimate

- $f''(a) < 0$ concave down
  - over-estimate