Section 4.6 Mean Value Theorem

Objectives:
1. Interpret the mean value theorem geometrically.
2. Apply the mean value theorem to solve applied problems.

Section 4.2
Rolle's Theorem

Suppose \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \) then there exists a number \( c \) such that \( f'(c) = 0 \).

\[ f'(c) = 0 \]

[Note: possibly many different \( c \) values, but at least one!]

(Why?) If \( f(x) \) is constant then the theorem is trivially true. Any \( c \) will work!

\[ f'(c) = 0 \] for all \( c \) between \( a \) and \( b \).
So we can assume $f(x)$ is not constant. Since $f(x)$ is cont. on $[a,b]$ it has an absolute min and absolute max.

Since $f(a)=f(b)$ one extreme point does not occur at the endpoints.

That is there is an absolute max or absolute min at some point $a \leq c \leq b$.

Fermat's Thm $\Rightarrow f'(c)=0.$

(Section 4.2)

Example

Verify the hypotheses and conclusion of Rolle's theorem for

$$f(x) = x^3 - x^2 - 6x + 2$$

on $[0,3]$

1. $f(x)$ is cont. on $[0,3]$.
2. $f(x)$ is diff. on $(0,3)$.
3. $f(0)=0^3-0^2-6\cdot0+2=2$
   $f(3)=3^3-3^2-6\cdot3+2=2$
**Conclusion**

\[ f'(x) = 3x^2 - 2x - 6 \]

\[ 0 = 3x^2 - 2x - 6 \]

\[ x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} \]

\[ x = \frac{2 \pm \sqrt{4 + 72}}{6} \]

\[ x = \frac{2 \pm \sqrt{76}}{6} \]

\[ c = \frac{2 + \sqrt{76}}{6} \approx 1.79 \]

between 0 & 2

\[ f'(c) = 0 \]

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**The Mean Value Theorem**

Assume that \( f(x) \) is continuous on \( [a, b] \) and differentiable on \( (a, b) \). Then there exists at least one \( c \) between \( a \) & \( b \) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
There is a point $c$ between $a$ & $b$ such that the average rate of change between $a$ & $b$ is equal to the instantaneous rate of change at $c$.

**Example:** Let $f(x) = x + \frac{1}{x}$ on $[1, 3]$. Find the $c$ guaranteed by the M.V.T.

**Assumptions of M.V.T.**

1. $f(x)$ cont. on $[1, 3]$
2. $f'(x) = 1 + (x^{-2}) = 0 \iff x^2 = 1$ R. diff. on $[1, 3]$

**Conclusion:**

$$\frac{f(3) - f(1)}{3 - 1} = \frac{(3 + \frac{1}{3}) - (1 + \frac{1}{1})}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$
\[ f'(x) = \frac{2}{3} \]

\[ 1 - \frac{1}{x^2} = \frac{2}{3} \]

\[ 1 - \frac{2}{3} = \frac{1}{x^2} \]

\[ \frac{1}{3} = \frac{1}{x^2} \]

\[ x^2 = 3 \]

\[ x = \pm \sqrt{3} \]

\[ \Rightarrow c = \sqrt{3} \text{ is between } 1 \text{ and } 3. \]

\[ f'(\sqrt{3}) = 1 - \frac{1}{(\sqrt{3})^2} = 1 - \frac{1}{3} = \frac{2}{3} \sqrt{3} \]

**Graphical Examples**

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
(II) \[f(a) = f(b)\]

\[f'(c) = 0 = \frac{f(b) - f(a)}{b - a}\]

(Rolle's Thm? special case)

(III) \[\Delta\]

M.V.T. does not apply. f(x) is not diff on (a, b).

\[f(b) - f(a) = 0\]

Proof of MVT: (Idea use Rolle's thm.)

\[y - y_i = m(x - x_i)\]

\[y - f(a) = m(x - a)\]

\[l(x) = f(a) + m(x - a)\]

Let \[g(x) = f(x) - l(x)\]

then

1. \(g(x)\) is cont. on \([a, b]\)
2. diff on \((a, b)\).
\[ g(a) = f(a) - l(a) \]
\[ = f(a) - (f(a) + \frac{f(b) - f(a)}{b-a} (a-a)) \]
\[ = f(a) - (f(a) + 0) \]
\[ = 0 \checkmark \]

\[ g(b) = f(b) - l(b) \]
\[ = f(b) - (f(a) + \frac{f(b) - f(a)}{b-a} (b-a)) \]
\[ = f(b) - (f(a) + f(b) - f(a)) \]
\[ = f(b) - f(b) = 0 \]

\[ g(a) = g(b) = 0. \]

By Rolle's thm.,
\[ g'(c) = 0 \text{ for some } c \text{ between } a \& b. \]

\[ g'(x) = f'(x) - l'(x) \]
\[ g'(x) = f'(x) + m \]

\[ g'(c) = f'(c) - m \]
\[ 0 = f'(c) - m \]
\[ m = f'(c). \]
Applications of MVT:

Suppose \( I \) is an interval.

1. If \( f'(x) = 0 \) for all \( x \) in \( I \) then \( f(x) \) is constant on \( I \).

2. If \( f'(x) > 0 \) for all \( x \) in \( I \) then \( f(x) \) is increasing on \( I \).

3. If \( f'(x) < 0 \) for all \( x \) in \( I \) then \( f(x) \) is decreasing on \( I \).

Proof of (2):

Suppose \( f'(x) > 0 \) for all \( x \) in \( I \). Let \( a, b \) be in \( I \) such that \( a < b \). Then there is a \( c \) between \( a \) and \( b \) such that

\[
    f'(c) = \frac{f(b) - f(a)}{b - a}
\]

positive \( b/c \) in the interval \( I \).

Thus, \( f(b) - f(a) > 0 \). That is, \( f(b) > f(a) \).

So if \( a < b \) then \( f(a) < f(b) \). \( \Box \)