Applications of Integrals

Obj.

1. Compute the average value of a function over an interval.

2. Interpret the definite integral of a rate of change as a net change.

The Net Change Theorem

Suppose \( Q \) changes over time at the rate \( Q' \). Then the net change in \( Q \) between \( t=a \) and \( t=b \) is

\[
Q(b) - Q(a) = \int_a^b Q'(t) \, dt
\]

Moreover, given \( Q(0) \) the future value of \( Q \) at time \( t \geq 0 \) is

\[
Q(t) = Q(0) + \int_0^t Q'(x) \, dx
\]
The owners of an oil reserve begin extracting oil at \( t=0 \). The projected extraction rate is given by

\[ Q'(t) = \sqrt{2 + \sqrt{t}} \text{ millions of barrels per year.} \]

How many barrels are extracted in the first ten years?

We want the net change of \( Q \) from \( t=0 \) to \( t=10 \). So, by the net change theorem,

\[
Q(10) - Q(0) = \int_0^{10} Q'(t) \, dt
\]

\[
= \int_0^{10} \sqrt{2 + \sqrt{t}} \, dt
\]

\[
U(0) = 2 + \sqrt{0} = 2
\]

\[
U(10) = 2 + \sqrt{10}
\]

\[
U = 2 + \sqrt{t} \quad \Rightarrow \quad \sqrt{t} = u - 2
\]

\[
du = \frac{1}{2 \sqrt{t}} \, dt
\]

\[
du = \frac{1}{2 (u - 2)} \, dt
\]

\[
(2u - 4) \, du = dt
\]
\[
Q(10) - Q(0) = \int_{u(0)}^{u(10)} \sqrt{u} (2u - 4) \, du
\]
\[
= \int_{2}^{2 + \sqrt{10}} u^{1/2} (2u - 4) \, du
\]
\[
= \int_{2}^{2 + \sqrt{10}} 2u^{3/2} - 4u^{1/2} \, du
\]
\[
= \left. \frac{2u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2} \right|_{2}^{2 + \sqrt{10}}
\]
\[
= \frac{4u^{5/2}}{5} - \frac{8u^{3/2}}{3}
\]
\[
= \left( \frac{4(2 + \sqrt{10})^{5/2}}{5} - \frac{8(2 + \sqrt{10})^{3/2}}{3} \right) - \left( \frac{4(2)^{5/2}}{5} - \frac{8(2)^{3/2}}{3} \right)
\]
\[
\approx 20,178 \text{ million of barrels of oil}
\]
Avg. value of $f(x)$ on $[a,b]$

$\Delta x = \frac{b-a}{n}$

$n = \frac{b-a}{\Delta x}$

Approx to avg. value:

\[ \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \]

\[ \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{b-a/\Delta x} \]

\[ \approx \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x \]

Thus

Avg. value of $f(x)$ on $[a,b]$

\[ = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ = \frac{1}{b-a} \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \]
Find the avg. value of 
\[ y = \frac{1}{2 + x^2} \] on \([0, \pi/4]\).

\[
\frac{1}{\pi/4 - 0} \int_0^{\pi/4} \frac{1}{2 + x^2} \, dx
\]

\[
\frac{4}{\pi} \int_0^{\pi/4} \frac{1}{2(1 + \frac{x^2}{2})} \, dx
\]

\[
\frac{4}{\pi} \int_0^{\pi/4} \frac{1}{2(1 + \frac{x}{\sqrt{2}})^2} \, dx
\]

\[
\frac{2}{\pi} \int_0^{\pi/4} \frac{1}{1 + \left(\frac{x}{\sqrt{2}}\right)^2} \, dx
\]

\[
u = \frac{x}{\sqrt{2}} \quad u(0) = 0
\]

\[
du = \frac{1}{\sqrt{2}} \, dx \quad u(\pi/4) = \frac{\pi}{(1/2)(1/4)}
\]

\[
\frac{2}{\pi} \int_0^{1/(\sqrt{2})} \frac{u(\pi/4)}{1 + u^2} \, du
\]

\[
\frac{2\sqrt{2}}{\pi} \int_0^{\pi/4} \frac{1}{1 + u^2} \, du
\]
\[
\frac{2\sqrt{2}}{\pi} \tan^{-1}(u) \bigg|_{0}^{\frac{\pi}{4}\sqrt{2}}
\]

\[
\frac{2\sqrt{2}}{\pi} \tan^{-1}\left(\frac{\pi}{4\sqrt{2}}\right) - \frac{2\sqrt{2}}{\pi} \tan^{-1}(0)
\]

\[
\frac{2\sqrt{2}}{\pi} \tan^{-1}\left(\frac{\pi}{4\sqrt{2}}\right) + \frac{2\sqrt{2}}{\pi} \cdot 0
\]

\[
= 0.35847
\]
Position, Velocity and Distance

1. Displacement from \( t = a \) to \( t = b \):
\[
S(b) - S(a) = \int_a^b v(t) \, dt
\]

2. Distance traveled from \( t = a \) to \( t = b \):
\[
\int_a^b |v(t)| \, dt
\]

Ex.

Suppose \( v(t) = \frac{1}{t+1} \) on \([0, 8] \). Find the total distance traveled.

\[
\int_0^8 |v(t)| \, dt = \int_0^8 \frac{1}{t+1} \, dt
\]

Let \( u = t + 1 \) then \( du = dt \):

\[
\int_0^9 \frac{1}{u} \, du
\]

\[
= \ln |u| \bigg|_1^9 = \ln 9 - \ln 1 = \ln 9 = \ln(9) \text{ miles}
\]
Find the total distance traveled

\[ v(t) = (t-1)(t-3) \text{ mph} \]

from \( t=0 \) to \( t=5 \).

\[
\int_0^5 |v(t)| \, dt = \int_0^5 |(t-1)(t-3)| \, dt
\]

\[
= \int_0^5 |t^2 - 4t + 3| \, dt
\]

\[
= \int_0^1 (t^2 - 4t + 3) \, dt + \int_1^3 -(t^2 - 4t + 3) \, dt + \int_3^5 (t^2 - 4t + 3) \, dt
\]

\[
= \left( \frac{28}{3} \right) \text{ miles}
\]