Topological Ramsey spaces, associated ultrafilters, and their applications to the Tukey theory of ultrafilters and Dedekind cuts of nonstandard arithmetic.

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Motivations and Submissions

- [2] Blass, *Ultrafilter mappings and their Dedekind cuts*
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- [9] Laflamme, *Forcing with filters and complete combinatorics*

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- [12] T., *Ramsey for $\mathcal{R}_1$ ultrafilter mappings and their Dedekind cuts*
- [5] Dobrinen, Mijares and T., *Topological Ramsey Spaces from Fraïssé Classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points*
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Ch 2 Selective & Ramsey for $\mathcal{R}$

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Ch 4 Ramsey for \( \mathcal{R}_1 \) & their Dedekind cuts

Ch 5 Canonical theory for \( \mathcal{H}^2 \)
Definition (Choquet, [4])

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\omega$. 

Definition (Booth, [3])

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\omega$. We say that $\mathcal{U}$ is selective, if for each decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$ of members of $\mathcal{U}$ there exists $A = \{a_0, a_1, \ldots\} \in \mathcal{U}$ enumerated in increasing order such that for all $i < \omega$, $A \{a_0, a_1, \ldots, a_{i-1}\} \subseteq A_i$. (1)

Theorem (Kunen, [3])

$\mathcal{U}$ is Ramsey iff $\mathcal{U}$ is selective iff $\mathcal{U}$ is minimal in the Rudin-Keisler ordering.
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A \setminus \{a_0, a_1, \ldots, a_{i-1}\} \subseteq A_i. 
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$\mathcal{U}$ is Ramsey iff $\mathcal{U}$ is selective iff $\mathcal{U}$ is minimal in the Rudin-Keisler ordering.
Definition

For $s, t \in \omega^{<\omega}$,

$s \sqsubseteq t$ if and only if for each $i < |s|$, $s_i = t_i$. 

A tree on $\omega$ is a subset of $\omega^{<\omega}$ such that $\text{cl}(T) = T$. 

$\pi_0 : \omega^{<\omega} \to \omega_1$

$\pi_0(s) = \langle s_0 \rangle$ if $s \neq \langle \rangle$

$\pi_0(s) = \langle \rangle$ otherwise.
Chapter 2

Definition

- For \( s, t \in \omega^{<\omega} \),

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- For each \( X \subseteq \omega^\omega \),

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cl(X) = \{ s \in \omega^{<\omega} : (\exists t \in X) s \sqsubseteq t \}
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- For each $X \subseteq \omega^\omega$,
  
  $$cl(X) = \{s \in \omega^{<\omega} : (\exists t \in X)s \sqsubseteq t\}$$

- A **tree on** $\omega$ is a subset of $\omega^{<\omega}$ such that $cl(T) = T$.

- $\pi_0 : \omega^{<\omega} \to \omega^1$

  $$\pi_0(s) = \begin{cases} 
  \langle s_0 \rangle & \text{if } s \neq \langle \rangle \\
  \langle \rangle & \text{otherwise.}
  \end{cases}$$
Definition (T., [13])

Let $T$ be a tree on $\omega$ such that

- $|\{s \in [T] : \pi_0(s) = \langle 0 \rangle \}| = 1$
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- $\pi''[T] = \{\langle n \rangle \in \omega : n < \omega\}$. 
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Let $\mathcal{R}(T)$ denote the set of all subtrees of $T$ isomorphic to $T$. 
Remark
If $S \in \mathbb{R}(T_0)$ then there exists a strictly increasing sequence $(k_i)_{i < \omega}$ such that $\pi''_0[S] = \{\langle k_i \rangle : i < \omega \}$.

**Figure:** Graph of $T_0$
Remark

If $S \in \mathcal{R}(T_0)$ then there exists a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi''_0[S] = \{\langle k_i \rangle : i < \omega \}$.
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If $S \in \mathbb{R}(T_1)$ then there exists a strictly increasing sequence $(k_i)_i < \omega$ such that $\pi''_0[S] = \{\langle k_i \rangle : i < \omega\}$.

**Figure:** Graph of $T_1$
Remark

If $S \in \mathcal{R}(T_1)$ then there exists a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi_0''[S] = \{\langle k_i \rangle : i < \omega \}$. 

Figure: Graph of $T_1$
Remark
If $S \in \mathbb{R}(T_{1}^{\ast})$ then there exists a strictly increasing sequence $(k_i)_{i < \omega}$ such that $\pi''_0[S] = \{\langle k_i \rangle : i < \omega \}$. 

Figure: Graph of the tree $T_{1}^{\ast}$
Remark

If $S \in \mathcal{R}(T^*_1)$ then there exists a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi''[S] = \{\langle k_i \rangle : i < \omega \}$. 

Figure: Graph of the tree $T^*_1$
Figure: Graph of $T_2$

Remark: If $S \in \mathbb{R}((T_2))$ then there exists a strictly increasing sequence $(k_i)_i < \omega$ such that $\pi''_0[S] = \{\langle k_i \rangle : i < \omega \}$. 
Remark

If \( S \in \mathcal{R}(T_2) \) then there exists a strictly increasing sequence \((k_i)_{i<\omega}\) such that \( \pi''_0[S] = \{\langle k_i \rangle : i < \omega\} \).
Remark: If \( S \in \mathbb{R}(T_1 \otimes T_1) \) then there exists a strictly increasing sequence \( (k_i)_{i < \omega} \) such that \( \pi'_{0} [S] = \{\langle k_i, k_i \rangle : i < \omega\} \).

\[\begin{align*}
(0, 0) & \rightarrow (1, 1) \\
(1, 1) & \rightarrow (2, 2) \rightarrow (3, 3) \rightarrow (4, 4) \\
(2, 2) & \rightarrow (3, 3) \rightarrow (4, 4) \\
(3, 3) & \rightarrow (4, 4) \\
(4, 4) & \rightarrow (10, 14) \rightarrow (11, 14) \rightarrow (12, 14) \rightarrow (13, 14) \rightarrow (14, 14)
\end{align*}\]

**Figure:** Graph of the tree \( T_1 \otimes T_1 \)
Remark

If $S \in \mathcal{R}(T_1 \otimes T_1)$ then there exists a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi_0''[S] = \{ \langle (k_i, k_i) \rangle : i < \omega \}$. 

Figure: Graph of the tree $T_1 \otimes T_1$
Definition (T., [13])

For each \( S \in \mathcal{R}(T) \) there is a strictly increasing sequence \((k_i)_{i<\omega}\) such that \( \pi''_0[S] = \{\langle(k_0, \ldots, k_i)\rangle : i < \omega\} \).
Definition (T., [13])

For each $S \in \mathcal{R}(T)$ there is a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi_0''[S] = \{\langle(k_i, \ldots, k_i)\rangle : i < \omega\}$. For each $i < \omega$, let:

- $S(i) = cl(\{s \in [S] : \pi_0(s) = \langle(k_i, \ldots, k_i)\rangle\})$
Definition (T., [13])

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- $S(i) = cl(\{s \in [S] : \pi_0(s) = \langle(k_i, \ldots, k_i)\rangle\})$
- $r_i(S) = \bigcup_{j<i} S(j)$
Definition (T., [13])

For each \( S \in \mathcal{R}(T) \) there is a strictly increasing sequence \((k_i)_{i<\omega}\) such that \( \pi_0''[S] = \{\langle (k_i, \ldots, k_i) \rangle : i < \omega \} \). For each \( i < \omega \), let

- \( S(i) = cl(\{s \in [S] : \pi_0(s) = \langle (k_i, \ldots, k_i) \rangle \}) \)
- \( r_i(S) = \bigcup_{j<i} S(j) \)
- \( r : \omega \times \mathcal{R}(T) \to \{r_i(S) : i < \omega \& S \in \mathcal{R}(T)\} \)

For each \( S, S' \in \mathcal{R}(T) \), \( S \leq S' \) if and only if \( S \) is subtree of \( S' \).

Definition (Mijares, [10])

The almost-reduction relation is defined as follows: for \( S, S' \in \mathcal{R}(T) \), \( S \leq^* S' \) if and only if there exists \( i < \omega \) such that \( S \setminus r_i(S) \subseteq S' \).

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The almost-reduction relation is defined as follows: for $S, S' \in \mathcal{R}(T)$, $S \leq S'^* if and only if there exists $i < \omega$ such that $S \setminus r_i(S) \subseteq S'$.
Definition (T., [13])

For each $S \in \mathcal{R}(T)$ there is a strictly increasing sequence $(k_i)_{i<\omega}$ such that $\pi''_0[S] = \{\langle(k_i, \ldots, k_i)\rangle : i < \omega\}$. For each $i < \omega$, let

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Theorem (Dobrinen and Todorcevic, [7] & [8])

For each positive integer $n$, $(\mathcal{R}(T_n), \leq, r)$ forms a topological Ramsey space.
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For each positive integer \( n \), \( (\mathcal{R}(T_n), \leq, r) \) forms a topological Ramsey space.

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Notation

\[ \mathcal{R}_n := \mathcal{R}(T_n), \mathcal{R}_n^* := \mathcal{R}(T_n^*), \mathcal{H}^2 := \mathcal{R}(T_1 \otimes T_1) \]
Definition (Mijares, [10])

Let $k$ be a positive integer and $T$ be a tree on $\omega^k$. Suppose that $(\mathcal{R}(T), \leq, r)$ forms a topological Ramsey space. Let $\mathcal{U}$ be an ultrafilter on $[T]$. 

We say that $\mathcal{U}$ is generated by $G \subseteq \mathcal{R}(T)$, if 

$\{[S] : S \in G\}$ is cofinal in $(\mathcal{U}, \supseteq)$.

An ultrafilter $\mathcal{U}$ generated by $G \subseteq \mathcal{R}(T)$ is selective for $\mathcal{R}(T)$ if and only if for each decreasing sequence $S_0 \geq S_1 \geq S_2 \geq \ldots$ of elements of $G$, there exists another $S \in G$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$.

An ultrafilter $\mathcal{U}$ generated by $G \subseteq \mathcal{R}(T)$ is Ramsey for $\mathcal{R}(T)$ if and only if for each $i < \omega$ and each partition of $\mathcal{R}(T)$ into two parts there exists $S \in G$ such that $(S \setminus r_i(T))$ lies in one part of the partition.
Definition (Mijares, [10])

Let $k$ be a positive integer and $T$ be a tree on $\omega^k$. Suppose that $(\mathcal{R}(T), \leq, r)$ forms a topological Ramsey space. Let $\mathcal{U}$ be an ultrafilter on $[T]$.

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- We say that $\mathcal{U}$ is generated by $\mathcal{G} \subseteq \mathcal{R}(T)$, if $\{[S] : S \in \mathcal{G}\}$ is cofinal in $(\mathcal{U}, \supseteq)$.

- An ultrafilter $\mathcal{U}$ generated by $\mathcal{G} \subseteq \mathcal{R}(T)$ is selective for $\mathcal{R}(T)$ if and only if for each decreasing sequence $S_0 \geq S_1 \geq S_2 \geq \ldots$ of elements of $\mathcal{G}$, there exists another $S \in \mathcal{G}$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$. 


Definition (Mijares, [10])

Let \( k \) be a positive integer and \( T \) be a tree on \( \omega^k \). Suppose that \((\mathcal{R}(T), \leq, r)\) forms a topological Ramsey space. Let \( \mathcal{U} \) be an ultrafilter on \([T]\).

- We say that \( \mathcal{U} \) is **generated by** \( \mathcal{G} \subseteq \mathcal{R}(T) \), if \( \{[S] : S \in \mathcal{G}\} \) is cofinal in \((\mathcal{U}, \supseteq)\).

- An ultrafilter \( \mathcal{U} \) generated by \( \mathcal{G} \subseteq \mathcal{R}(T) \) is **selective for** \( \mathcal{R}(T) \) if and only if for each decreasing sequence \( S_0 \geq S_1 \geq S_2 \geq \ldots \) of elements of \( \mathcal{G} \), there exists another \( S \in \mathcal{G} \) such that for all \( i < \omega \), \( S \setminus r_i(S) \subseteq S_i \).

- An ultrafilter \( \mathcal{U} \) generated by \( \mathcal{G} \subseteq \mathcal{R}(T) \) is **Ramsey for** \( \mathcal{R}(T) \) if and only if for each \( i < \omega \) and each partition of \( (r_{r_i(T)}) \) into two parts there exists \( S \in \mathcal{G} \) such that \( (r_i^S(T)) \) lies in one part of the partition.
Chapter 3

Theorem (Mijares, [10])

Ramsey for $\mathcal{R}(T) \Rightarrow$ selective for $\mathcal{R}(T)$. 

Question (Dobrinen, [13])

For any given topological Ramsey space $\mathcal{R}$, are the notions of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ equivalent?

Theorem (T., [13])

$(\mathcal{R} \star 1, \leq \ast)$ is $\sigma$-closed and there exists a map $\Gamma : \mathcal{R} \star 1 \to \mathcal{R} 1$ such that if $G$ is a generic filter for $(\mathcal{R} \star 1, \leq \ast)$ over some ground model $V$, then $\Gamma^\prime G$ generates an ultrafilter on $\left[ T 1 \right]$ that is selective for $\mathcal{R} 1$ but not Ramsey for $\mathcal{R} 1$ in $V[G]$. 

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Theorem (Mijares, [10])

Ramsey for $\mathcal{R}(T) \Rightarrow$ selective for $\mathcal{R}(T)$.

Question (Dobrinen, [13])

For any given topological Ramsey space $\mathcal{R}$, are the notions of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ equivalent?
Chapter 3

Theorem (Mijares, [10])

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Theorem (T., [13])

\( (\mathcal{R}_1^*, \leq^*) \) is \( \sigma \)-closed and there exists a map \( \Gamma : \mathcal{R}_1^* \rightarrow \mathcal{R}_1 \) such that if \( \mathcal{G} \) is a generic filter for \( (\mathcal{R}_1^*, \leq^*) \) over some ground model \( V \), then \( \Gamma'' \mathcal{G} \) generates an ultrafilter on \( [T_1] \) that is selective for \( \mathcal{R}_1 \) but not Ramsey for \( \mathcal{R}_1 \) in \( V[\mathcal{G}] \).
\[ S \in \mathbb{R}^\ast \quad \Gamma(S) \in \mathbb{R}_1 \]

**Figure**: Graph of \( S \in \mathcal{R}_1^\ast \) and \( \Gamma(S) \in \mathcal{R}_1 \).
Theorem (T., [13])

Let \( n \) be a positive integer. \((\mathcal{R}_n^*, \leq^*)\) is \(\sigma\)-closed and there exists a map \( \Gamma_n : \mathcal{R}_n^* \to \mathcal{R}_n \) such that if \( \mathcal{G} \) is a generic filter for \((\mathcal{R}_n^*, \leq^*)\) over some ground model \( V \), then \( \Gamma_n'' \mathcal{G} \) generates an ultrafilter on \([T_n]\) that is selective for \( \mathcal{R}_n \) but not Ramsey for \( \mathcal{R}_n \) in \( V[\mathcal{G}] \).
Theorem (T., [13])

Let $n$ be a positive integer. $(\mathcal{R}_n^*, \leq^*)$ is $\sigma$-closed and there exists a map $\Gamma_n : \mathcal{R}_n^* \to \mathcal{R}_n$ such that if $G$ is a generic filter for $(\mathcal{R}_n^*, \leq^*)$ over some ground model $V$, then $\Gamma_n''G$ generates an ultrafilter on $[T_n]$ that is selective for $\mathcal{R}_n$ but not Ramsey for $\mathcal{R}_n$ in $V[G]$.

Theorem (T., [13])

Suppose that $\langle S_i : i \leq n \rangle$ is a finite sequence of trees where each $S_i$ is one of the trees $T_j$ for some $j < \omega$. $(\bigotimes_{i=0}^{n} \mathcal{R}(S_i^*), \leq^*)$ is $\sigma$-closed and there exists a map $\Gamma : \bigotimes_{i=0}^{n} \mathcal{R}(S_i^*) \to \bigotimes_{i=0}^{n} \mathcal{R}(S_i)$ such that if $G$ is a generic filter for $(\bigotimes_{i=0}^{n} \mathcal{R}(S_i^*), \leq^*)$ over some ground model $V$, then $\Gamma''G$ generates an ultrafilter on $[\bigotimes_{i=0}^{n} S_i]$ that is selective for $\bigotimes_{i=0}^{n} \mathcal{R}(S_i)$ but not Ramsey for $\bigotimes_{i=0}^{n} \mathcal{R}(S_i)$ in $V[G]$.
Definition

If $\mathcal{U}$ is an ultrafilter on the base set $X$ and $\mathcal{V}$ is an ultrafilter on the base set $Y$, then we say that $\mathcal{V}$ is Rudin-Keisler reducible to $\mathcal{U}$ and write $\mathcal{V} \leq_{RK} \mathcal{U}$ if there exists a function $f : X \to Y$ such that $\mathcal{V} = f(\mathcal{U})$, where $f(\mathcal{U}) = \langle \{ f''Z : Z \in \mathcal{U} \} \rangle$. (2)

A Rudin-Keisler mapping from $\mathcal{U}$ to $\mathcal{V}$ is a function $f : X \to Y$ such that $\mathcal{V} = f(\mathcal{U})$. (4)
Definition

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Chapter 4

Definition
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A **Rudin-Keisler mapping from** $\mathcal{U}$ **to** $\mathcal{V}$ is a function $f : X \to Y$ such that $\mathcal{V} = f(\mathcal{U})$. 
Definition (Blass, [2])

- Associated to each Rudin-Keisler mapping from $\mathcal{U}$ on $X$ to $\mathcal{V}$ on $Y$ is a Dedekind cut in the ultrapower $\omega^Y/\mathcal{V}$.
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- For any $A \subseteq X$, we define the **cardinality function of $A$ relative to $p$** by
  \[ C_A(y) = |A \cap p^{-1}\{y\}| \quad \text{for} \quad y \in Y. \] (3)
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\[ C_A(y) = |A \cap p^{-1}\{y\}| \quad \text{for} \; y \in Y. \] (3)

- The set of all equivalence classes of cardinality functions of sets in $\mathcal{U}$, and all larger elements of $\omega^Y/p(\mathcal{U})$, constitute the upper part $L$ of a cut $(S, L)$ of $\omega^Y/p(\mathcal{U})$, which we call the cut associated to $p$ and $\mathcal{U}$. 
Lemma (T., [12])

Assume the Continuum Hypothesis. If $\mathcal{V}$ is selective then there exists a Ramsey for $\mathcal{R}_1$ ultrafilter $\mathcal{U}$ such that the cut associated to $\mathcal{U}$ and $\pi$ is the standard cut in $\omega^\omega / \mathcal{V}$. 
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Lemma (T., [12])

Let $\mathcal{U}$ be a Ramsey for $\mathcal{R}_1$ ultrafilter on $[T_1]$ and $p$ be a map from $[T_1]$ to $\omega$. The cut associated to $p$ and $\mathcal{U}$ is the standard cut in $\omega^\omega / p(\mathcal{U})$. 
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Theorem (T., [12])

Assume the Continuum Hypothesis. $(S, L)$ is the cut associated to some map of some Ramsey for $\mathcal{R}_1$ ultrafilter on $[T_1]$ to $\mathcal{V}$ if and only if $\mathcal{V}$ is selective and $(S, L)$ is the standard cut in $\omega^\omega/\mathcal{V}$.
**Definition**

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\omega$.

$\mathcal{U}$ is a **p-point** if for each $f : \omega \rightarrow \omega$ there exists $X \in \mathcal{U}$ such that $f \upharpoonright X$ is constant or $f \upharpoonright X$ is finite-to-one.

$\mathcal{U}$ is **weakly-Ramsey** if for each $F : [\omega]^2 \rightarrow 3$ there exists $X \in \mathcal{U}$ such that $F$ omits a value on $[X]^2$.

**Corollary (T., [12])**

Assume the Continuum Hypothesis. There is a weakly-Ramsey ultrafilter on $[T_1]$ that is not Ramsey for $\mathcal{R}_1$. 
Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on the base sets $X$ and $Y$ respectively.

- A function $f$ from $\mathcal{U}$ to $\mathcal{V}$ is **cofinal** if every cofinal subset of $(\mathcal{U}, \supseteq)$ is mapped by $f$ to a cofinal subset of $(\mathcal{V}, \supseteq)$.(In other words, $f$ maps filter bases of $\mathcal{U}$ to filter bases of $\mathcal{V}$.)
Definition (Tukey, [14])

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on the base sets $X$ and $Y$ respectively.

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- We say that $\mathcal{V}$ is **Tukey reducible to $\mathcal{U}$** and write $\mathcal{V} \leq_T \mathcal{U}$ if there exists a cofinal map $f : \mathcal{U} \to \mathcal{V}$. 

The relation $\equiv_T$ is an equivalence relation and $\leq_T$ is a partial order on its equivalence classes. The equivalence classes are also called **Tukey types**.
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- If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ then we write $\mathcal{V} \equiv_T \mathcal{U}$ and say that $\mathcal{U}$ and $\mathcal{V}$ are **Tukey equivalent**.
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Theorem (T., [12])

Suppose $\mathcal{U}$ is a Ramsey for $\mathcal{R}_1$ ultrafilter on $[T_1]$. If $(S, L)$ is the cut associated to some map from some p-point ultrafilter in the Tukey type of $\mathcal{U}$ to some ultrafilter $\mathcal{V}$, then $\mathcal{V}$ is a p-point ultrafilter and $(S, L)$ is the standard cut.
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**Corollary (T., [12])**

Assume the Continuum Hypothesis holds and $\mathcal{U}$ is a Ramsey for $\mathcal{R}_1$ ultrafilter on $[T_1]$. There exists a weakly Ramsey ultrafilter $\mathcal{V}$ such that $\mathcal{V} \not\leq_T \mathcal{U}$. 
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Corollary (T., [12])

Assume the Continuum Hypothesis holds and $\mathcal{U}$ is a Ramsey for $\mathcal{R}_1$ ultrafilter on $[T_1]$. There exists a p-point ultrafilter $\mathcal{W}$ which is not weakly Ramsey such that $\mathcal{W} >_T \mathcal{U}$. 
Definition (T., [11])

Let \( \tilde{S}(n) \) denote the tree
\[
\{\langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in (n + 1) \times (n + 1)\}.
\]
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- Let $\tilde{S}(n)$ denote the tree
  \[ \{ \langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in (n + 1) \times (n + 1) \}. \]
- Let $T_{\langle \rangle} = \{ \langle \rangle \}$
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Chapter 5

Definition (T., [11])

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  \]

- Let \( T_{\langle \rangle} = \{ \langle \rangle \} \)

- Let \( T_{\langle 0 \rangle} = \{ \langle \rangle, \langle 0 \rangle \} \).

- For \( I, J \subseteq n + 1 \) with either \( I \) or \( J \) nonempty, let \( T(I, J) = \)
  \[
  \begin{cases}
  \{ \langle \rangle, \langle 0 \rangle, \langle 0, (i, \cdot) \rangle : i \in I \} & \text{if } I \neq \emptyset \text{ and } J = \emptyset, \\
  \{ \langle \rangle, \langle 0 \rangle, \langle 0, (\cdot, j) \rangle : j \in J \} & \text{if } I = \emptyset \text{ and } J \neq \emptyset, \\
  \{ \langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in I \times J \} & \text{if } I \neq \emptyset \text{ and } J \neq \emptyset.
  \end{cases}
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Let $\tilde{S}(n)$ denote the tree
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\]

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\{ \langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in I \times J \} & \text{if } I \neq \emptyset \text{ & } J \neq \emptyset.
\end{cases}
\]

Let $T(n)$ denote the collection of trees of the form $T_{\langle \rangle}, T_{\langle 0 \rangle}$ and $T(I, J)$ with $I, J \subseteq n + 1$ and either $I$ or $J$ nonempty.
Definition (T., [11])

Let $T \in T(n)$ and $X \in \mathcal{H}^2$.
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Let $T \in \mathcal{T}(n)$ and $X \in \mathcal{H}^2$.

Suppose that $X(n) = \{ \langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L \times K \}$.
Definition (T., [11])

- Let $T \in T(n)$ and $X \in \mathcal{H}^2$.
- Suppose that $X(n) = \{\langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L \times K \}$.
- With $L = \{l_0, \ldots, l_n\}$ and $K = \{k_0, \ldots, k_n\}$.
Definition (T., [11])

- Let $T \in \mathcal{T}(n)$ and $X \in \mathcal{H}^2$.
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- With $L = \{l_0, \ldots, l_n\}$ and $K = \{k_0, \ldots, k_n\}$.
- The $T$-projection of $X(n)$ denoted by $\pi_T(X(n))$ is given by

\[
\begin{cases}
\{\langle \rangle\} & \text{if } T = T_{\langle \rangle}, \\
\{\langle \rangle, \langle m \rangle\} & \text{if } T = T_{\langle 0 \rangle}, \\
\{\langle \rangle, \langle m \rangle, \langle m, k_j \rangle : j \in J\} & \text{if } T = T(\emptyset, J), \\
\{\langle \rangle, \langle m \rangle, \langle m, l_i \rangle : i \in I\} & \text{if } T = T(I, \emptyset), \\
\{\langle \rangle, \langle m \rangle, \langle m, (l_i, k_j) \rangle : (i, j) \in I \times J\} & \text{otherwise}.
\end{cases}
\]
Figure: Graphs of various projections of an element of $\mathcal{H}^2(2)$
Definition (T., [11])

Each $T \in \mathcal{T}(n)$ induces an equivalence relation $E_T$, on $\mathcal{H}^2(n)$ by

$$X(n)E_T Y(n) \iff \pi_T(X(n)) = \pi_T(Y(n)).$$
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- Let $\mathcal{E}(n)$ denote the collection of equivalence relations $E_T$, for $T \in \mathcal{T}(n)$. 

Theorem (Erdős-Rado Theorem for $\mathcal{H}^2$; T., [11])

For each $A \in \mathcal{H}^2(n)$ and equivalence relation $E$ on $\mathcal{H}^2(n)$ $|A|$, there exists $B \leq A$ and $E_T \in \mathcal{E}(n)$ such that $E_T \upharpoonright (\mathcal{H}^2(n)|B) = E \upharpoonright (\mathcal{H}^2(n)|B)$.
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Definition

A subset $\mathcal{F}$ of $\mathcal{A}\mathcal{H}^2$ is a **front on** $\mathcal{H}^2$ if for each $X \in \mathcal{H}^2$ there exists $a \in \mathcal{F}$ and $i < \omega$ such that $a = r_i(X)$ and $b \not\sqsubseteq c$ for all $b \neq c \in \mathcal{F}$. 
Definition

A subset $F$ of $AH^2$ is a **front on** $H^2$ if for each $X \in H^2$ there exists $a \in F$ and $i < \omega$ such that $a = r_i(X)$ and $b \not\subseteq c$ for all $b \neq c \in F$.

Definition

Let $F$ be a front on $H^2$ and let $\varphi$ be a function on $F$.

- $\varphi$ is **inner** if for each $a \in F$ there exists a family of trees $\{T_i : i < |a|\}$ such that for each $i < \omega$ $T(i) \in T(i)$ and
  
  $$\varphi(a) = \bigcup_{i < |a|} \pi_{T_i}(a(i)).$$

- $\varphi$ is **Nash-Williams** if $\varphi(a) \not\subseteq \varphi(b)$, for all $a \neq b \in F$.  
  

Definition (T., [11])

Let $\mathcal{F}$ be a front and $R$ be an equivalence relation on $\mathcal{F}$. We say $R$ is canonical if and only if there is an inner Nash-Williams function $\phi$ on $\mathcal{F}$ such that

1. For all $a, b \in \mathcal{F}$, $aRb$ if and only if $\phi(a) = \phi(b)$; and
2. $\phi$ is maximal among all inner Nash-Williams functions satisfying (1). That is, for any other inner Nash-Williams function $\phi'$ on $\mathcal{F}$ satisfying (1), there is a $Y \leq X$ such that $\phi'(a) \subseteq \phi(a)$, for all $a \in \mathcal{F}|Y$.

Theorem (T., [11])

Suppose $A \in H_2$, $\mathcal{F}$ is a front and $R$ is an equivalence relation on $\mathcal{F}$. Then there is a $B \leq A$ such that $R$ is canonical on $\mathcal{F}|B$. 
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1. for all $a, b \in \mathcal{F}$, $aRb$ if and only if $\varphi(a) = \varphi(b)$; and
2. $\varphi$ is maximal among all inner Nash-Williams functions satisfying (1). That is, for any other inner Nash-Williams function $\varphi'$ on $\mathcal{F}$ satisfying (1), there is a $Y \leq X$ such that $\varphi'(a) \subseteq \varphi(a)$, for all $a \in \mathcal{F}|Y$. 

Theorem (T., [11])

Suppose $A \in H_2$, $\mathcal{F}$ is a front and $R$ is an equivalence relation on $\mathcal{F}$. Then there is a $B \leq A$ such that $R$ is canonical on $\mathcal{F}|B$. 
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Proposition (T., [11])

Assume that $\mathcal{U}$ is a Ramsey for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$ and generated by $\mathcal{C} \subseteq \mathcal{H}^2$. Suppose $\mathcal{C}$ has basic Tukey reductions and $\mathcal{V}$ is a nonprincipal ultrafilter on $\omega$ with $\mathcal{U} \geq_T \mathcal{V}$. Then there is a front $\mathcal{F}$ on $\mathcal{C}$ and a function $f : \mathcal{F} \rightarrow \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} | \mathcal{F} \rangle)$. 

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Lemma (T., [11])

If \( \mathcal{C} \subseteq \mathcal{H}^2 \) generates a selective for \( \mathcal{H}^2 \) ultrafilter on \([T_1 \otimes T_1]\) then \( \mathcal{C} \) has basic Tukey reductions.
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Lemma (T., [11])

If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a selective for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$ then $\mathcal{C}$ has basic Tukey reductions.

Theorem (T., [11])

If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$ then for any front $\mathcal{F}$ on $\mathcal{R}$ and any equivalence relation $R$ on $\mathcal{F}$, there exists a $\mathcal{C} \in \mathcal{C}$ such that $R$ is canonical on $\mathcal{F}|\mathcal{C}$. 
Definition (T., [11])

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- Suppose that $C \subseteq \mathcal{H}^2$ generates a Ramsey for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$.
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- For each $n < \omega$ and each $T(I, J) \in T(n)$, let

$$D_{I,J} = \pi_{T(I,J)}(D_{n+1}).$$

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Suppose that $\mathcal{U}$ is a Ramsey for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$ generated $\mathcal{C} \subseteq \mathcal{H}^2$. If $\mathcal{V}$ is a nonprincipal ultrafilter and $\mathcal{U} \geq_T \mathcal{V}$, then $\mathcal{V}$ is isomorphic to an ultrafilter of $\mathcal{W}$-trees, where $\hat{S} \setminus S$ is a well-founded tree, $\mathcal{W} = (\mathcal{W}_s : s \in \hat{S} \setminus S)$, and each $\mathcal{W}_s$ is isomorphic to $\mathcal{D}_0$ or $\mathcal{D}_{i,j}$ for some $(i, j) \in \omega \times \omega$ with $(i, j) \neq (0, 0)$. 
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Suppose that $\mathcal{U}$ is a Ramsey for $\mathcal{H}^2$ ultrafilter on $[T_1 \otimes T_1]$ generated $\mathcal{C} \subseteq \mathcal{H}^2$. If $\mathcal{V}$ is an ultrafilter on $\omega$ and $\mathcal{V}$ is Tukey reducible to $\mathcal{U}$ then one of the following holds,

1. $\mathcal{V} \equiv_T \mathcal{D}_{\langle \rangle}$,
2. $\mathcal{V} \equiv_T \mathcal{D}_0$,
3. $\mathcal{V} \equiv_T \mathcal{D}_{0,1}$,
4. $\mathcal{V} \equiv_T \mathcal{D}_{1,0}$ or
5. $\mathcal{V} \equiv_T \mathcal{D}_{1,1}$.
Theorem (T., [11])

*It is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.*

**Figure:** Rudin-Keisler structure of the p-point ultrafilters within the Tukey types of nonprincipal ultrafilters Tukey reducible to $\mathcal{D}_1$
Further Questions and Problems

- Is the Ellentuck space the only topological Ramsey space where Ramsey and selective are equivalent?
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- Under CH, is the standard cut the only Dedekind cut that can arise from a selective for $\mathcal{R}_1$ ultrafilter?
- Under CH, characterize the Dedekind cuts that can arise from Ramsey/selective for $\mathcal{R}_n$ ultrafilters, for $n < \omega$.
- Under CH, characterize the Dedekind cuts that can arise from p-points in the Tukey type of a Ramsey for $\mathcal{R}_1$ ultrafilter.
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The Rudin-Keisler ordering of p-points.

Ultrafilter mappings and their Dedekind cuts.

Ultrafilters on a countable set.

Deux classes remarquables d’ultrafiltres sur $n$.


Forcing with filters and complete combinatorics.

A notion of selective ultrafilter corresponding to topological ramsey spaces.

Dissertation.
2014.

Ramsey for $R_1$ ultrafilter mappings and their Dedekind cuts.
submitted.

Selective but not Ramsey.
submitted.