Technical Note

Analytical and Numerical Observations on the Hetényi Solution for Buckling of Beams on Elastic Foundations

D. V. Griffiths, F.ASCE[1]; and G. Bee, S.M.ASCE[2]

Abstract: This paper considers the minimum buckling load and mode shape of a simply supported beam on an elastic foundation. Solutions are obtained by solving the eigenvalue problem delivered by a finite-element formulation and by using the analytical solutions involving (1) trials and (2) rounding of real numbers to integers as described by Hetényi [Hetényi, M. (1946). *Beams on Elastic Foundations*, University of Michigan Press, Ann Arbor, MI]. The comparison shows that the solution by rounding can lead to overestimation of the buckling load close to the transition zone between mode shapes. The paper explains the reason for the overestimation and offers a simple direct algorithm that always leads to the correct mode shape and minimum buckling load. **DOI: 10.1061/(ASCE)EM.1943-7889.0000827.**

Author keywords: Beam; Elastic foundation; Buckling; Finite elements; Analytical solution.

Introduction

During a recent research study involving numerical prediction of thermal buckling of pipes embedded in variable seafloor soils (e.g., Li and Batra 2007; Bee 2013), a program was developed for modeling buckling of a beam on a spatially random elastic foundation. Code validation was performed for uniform foundations by comparing solutions of the eigenvalue problem delivered by a finite-element (FE) formulation, with analytical solutions described in Hetényi (1946). The comparison highlighted an ambiguity in the analytical solution for calculating the buckling load close to the region in which one mode shape transitions to the next. The problem lies in the calculation of an integer representing the mode shape number used in the buckling load formula. It will be shown that the rounding strategy recommended in the analytical solution can lead to an overestimation of the buckling load. This paper investigates the reason for this anomaly and suggests an improved algorithm for always finding the correct mode number.

Review of FE Formulation

The governing differential equation for a beam of length l and stiffness EI, resting on a foundation of stiffness k under the action of a transverse distributed load q and an axial compressive force N, is given by

$$EI\frac{d^4y}{dx^4} + ky + N\frac{d^2y}{dx^2} = q \tag{1}$$

[1]Professor, Dept. of Civil and Environmental Engineering, Colorado School of Mines, Golden, CO 80401; and Australian Research Council Centre of Excellence for Geotechnical Science and Engineering, Univ. of Newcastle, Callaghan, NSW 2308, Australia (corresponding author). E-mail: d.v.griffiths@mines.edu

[2]Graduate Student, Dept. of Civil and Environmental Engineering, Colorado School of Mines, Golden, CO 80401.

Note. This manuscript was submitted on May 5, 2013; approved on May 22, 2014; published online on July 1, 2014. Discussion period open until December 1, 2014; separate discussions must be submitted for individual papers. This technical note is part of the ***Journal of Engineering Mechanics***, © ASCE, ISSN 0733-9399/06014014(5)/$25.00.

A typical configuration is shown in Fig. 1 with discretization by five beam elements.

A typical two-node beam element of length L is shown in Fig. 2, in which x is a spatial coordinate along the axis of the beam, y is the transverse deflection, and $\theta = dy/dx$ is the (small) rotation.

In the absence of transverse loading ($q = 0$) and after discretization in space (see Smith and Griffiths 2004), the various terms of Eq. (1) lead to the matrix form shown in Eq. (2)

$$[[k_m] + [m_m]]\{w\} = N[g_m]\{w\} \tag{2}$$

where $\{w\} = [y_1 \quad \theta_1 \quad y_2 \quad \theta_2]^T$; and y_i and $\theta_i = (dy/dx)_i$ are the translation and rotation, respectively, at node $i = 1, 2$. The various matrix terms from Eq. (2) and their counterparts from the differential Eq. (1) are given in the Appendix.

The beam and foundation stiffness matrices can be combined as

$$[k'_m] = [k_m] + [m_m] \tag{3}$$

and, after assembly and introduction of boundary conditions, the global generalized eigenvalue equation (where global matrices and vectors are denoted by uppercase symbols) can be written after rearrangement as

$$[G_m]\{W\} = \frac{1}{N_{cr}}[K'_m]\{W\} \tag{4}$$

Solution of Eq. (4) by vector iteration (see Griffiths and Smith 2006) will lead at convergence to the largest eigenvalue $1/N_{cr}$ and eigenvector $\{W\}$, where N_{cr} is the critical (lowest) buckling load of the beam on an elastic foundation and $\{W\}$ is the corresponding buckled mode shape.

Analytical Solution

The analytical solution for the critical buckling load for beams on elastic foundations with hinged (simply supported) ends is now summarized from Chapter VII of Hetényi (1946). It is first shown that the critical buckling load of an infinitely long beam of flexural stiffness EI on an elastic foundation of modulus k is given by

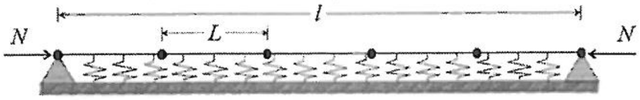

Fig. 1. Simply supported beam on an elastic foundation discretized with five elements

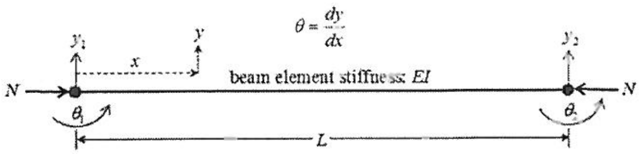

Fig. 2. Typical beam element used for buckling analysis

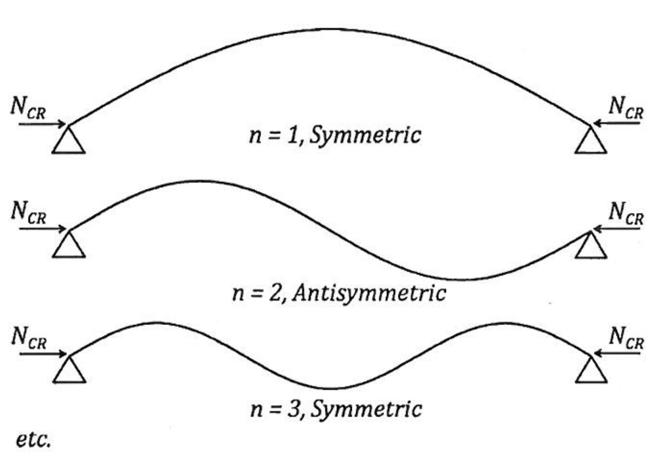

Fig. 3. First three mode shapes of a buckled beam on an elastic foundation

$$N_{cr,\infty} = 2\sqrt{kEI} \tag{5}$$

and it is further noted that because of the additional restraint, the critical buckling load of a simply supported beam of finite length l, with all other conditions the same, must be greater than or equal to this value (see De Angelis and Cancellara 2012)

$$N_{cr} \geq 2\sqrt{kEI} \tag{6}$$

Thus for the case of a simply supported beam, Hetényi gives Eq. (7) for the critical load N_{cr}

$$N_{cr} = \left(\frac{n\pi}{l}\right)^2 EI + \left(\frac{l}{n\pi}\right)^2 k \tag{7}$$

where the integer n should be chosen by trials so as to make the value of N_{cr} a minimum (e.g., Brush and Almroth 1975). Later in the paper, two examples will compare the method using trials with a direct algorithm proposed by the authors.

By differentiation and letting $dN_{cr}/dn = 0$, the following expression is obtained

$$n = \frac{l}{\pi}\sqrt[4]{\frac{k}{EI}} \tag{8}$$

Clearly the value of n from Eq. (8) is a real number; however, it cannot be substituted directly into Eq. (7) because l would cancel, and the expression would simply return the infinitely long beam solution given in Eq. (5). As explained by Hetényi in relation to his Fig. 118, "Taking for n an integer number which is the nearest to the value determined from the equation above [Eq. (8)] and substituting that number in (126) [Eq. (7)], we get the value of the critical load." (equations in brackets refer to the current paper).

It can also be noted that odd and even values of integer n imply, respectively, symmetrical and antisymmetrical modes of buckling where n is the number of waves (or maxima and minima) in the buckled shape as shown in Fig. 3 for the first three mode shapes given by $n = 1, 2,$ and 3.

Comparison of FE and Analytical Solutions

In the following, it can be assumed that dimensional quantities are provided in a consistent system of units. Fig. 4 shows three plots for a simply supported beam with stiffness $EI = 100$ on an elastic foundation with stiffness $k = 50$. The first two plots are of beam length (l) versus buckling load (N_{cr}) as computed by the Hetényi analytical solution with rounding as explained previously, and by FEs with beam element lengths of $L = 0.01$ (i.e., 200 beam elements would be used to model a total beam length of $l = 2$, and so on). The third plot is of beam length (l) versus number of iterations for convergence of the vector iteration method used to solve the eigenvalue problem from Eq. (4). A beam length range of $2 \leq l \leq 6$ was chosen in Fig. 4 because it captures the transition between the first ($n = 1$) and second ($n = 2$) mode shape as shown in Fig. 3.

It is clear from the FE solutions that the number of iterations required for convergence grows rapidly as the transition point between the first ($n = 1$) and second ($n = 2$) mode shapes is approached. Indeed, the algorithm would fail to converge entirely if an analysis was attempted at the exact transition point.

The analytical and FE solutions for the buckling load are almost indistinguishable for most beam lengths, falling to a minimum of $N_{cr} = 141.4$ at $l = 3.74$ given by Eq. (5) for the infinitely long case; however, the two solutions diverge in the range $5.28 < l < 5.60$. In this range, the analytical solution continues to rise to about $N_{cr} = 190.3$ at $l = 5.60$, before falling suddenly to rejoin the FE curve at $N_{cr} = 165.3$. It can be noted that $l = 5.28$ and $l = 5.60$ are, respectively, the lengths at which the FE solution and the analytical solution transition to the $n = 2$ mode. The reason for the divergence, however, is that the analytical solution does not round to $n = 2$ until the real value of n given by Eq. (7) reaches $n = 1.5$, which does not occur until $l = 5.60$.

Although not presented in this paper, similar discontinuities were observed at all transition points corresponding to higher values of n, but the one between $n = 1$ and $n = 2$ highlighted here is the most pronounced. It can be concluded that in all transition zones, the analytical solution based on simple rounding will always overestimate the true buckling load. Simitses and Hodges (2006) showed a similar result and noted that the approximate solution based on an infinite beam on an elastic foundation becomes increasingly accurate as the mode number increases.

Adjustment to Hetényi Solution

In the following, the authors have taken the liberty of modifying Hetényi's original notation to avoid confusion between integers and real numbers. The symbol m is strictly an integer, and represents the

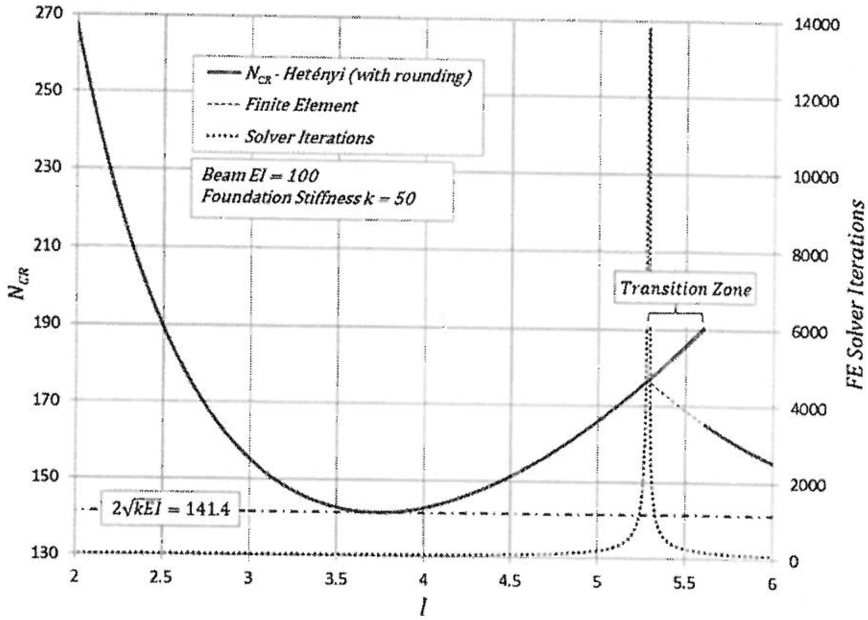

Fig. 4. Beam length l versus buckling load N_{cr} and FE solver iteration count

mode number previously called n in Fig. 3. The symbol n is strictly a real number as computed by Eq. (8).

At any transition point, the buckling load must be the same for any two consecutive modes with integer values m and $m+1$, so from Eq. (7)

$$N_{cr} = m^2 \frac{\pi^2 EI}{l^2} + \frac{1}{m^2}\frac{kl^2}{\pi^2} = (m+1)^2 \frac{\pi^2 EI}{l^2} + \frac{1}{(m+1)^2}\frac{kl^2}{\pi^2} \quad (9)$$

After rearrangement, Eq. (9) can be written as

$$\sqrt{m(m+1)} = \frac{l}{\pi}\sqrt[4]{\frac{k}{EI}} = n \quad (10)$$

which equals the value of the real number n provided by Eq. (8). Using Eq. (10), Table 1 presents the actual value of (real) n at which the change in mode occurs. Clearly, the mode change happens before n reaches $(2m+1)/2$, which is where a number would normally be rounded up.

Although the rounded value of n is always an overestimate as shown in Table 1, the error tends to zero as m increases as shown by

$$\sqrt{m(m+1)} \to \frac{1}{2}(2m+1) \quad \text{as} \quad m \to \infty \quad (11)$$

Direct Algorithm to Find the Critical Load and Mode Number

A direct algorithm for finding the mode number (m) and the critical (lowest) buckling load (N_{cr}) for a simply supported beam of length l and stiffness EI resting on an elastic foundation of stiffness k is

1. Compute

$$n = \frac{l}{\pi}\sqrt[4]{\frac{k}{EI}} \quad (12)$$

Table 1. Summary of (Real) n Values for the First Three Transition Points

Mode transition m:$(m+1)$	Correct value of n at transition		Incorrect value of n at transition (based on rounding)
1:2	$\sqrt{2}$	1.4142	1.5
2:3	$\sqrt{6}$	2.4495	2.5
3:4	$\sqrt{12}$	3.4641	3.5

2. Choose an integer j such that

$$j \le n \le j+1 \quad (13)$$

3. Select the correct mode number m

$$\text{If} \quad j \le n \le \sqrt{j(j+1)}, \quad \text{let} \quad m = j \quad (14a)$$

$$\text{If} \quad \sqrt{j(j+1)} \le n \le j+1, \quad \text{let} \quad m = j+1 \quad (14b)$$

4. Compute the buckling load from

$$N_{cr} = \left(\frac{m\pi}{l}\right)^2 EI + \left(\frac{l}{m\pi}\right)^2 k \quad (15)$$

Note that if $n = \sqrt{j(j+1)}$, an exact transition point has been reached and m can be set equal to either j or $j+1$, because both values of m will give the same value of N_{cr} from Step 4.

Worked Examples

Two examples are presented here that use both the direct algorithm proposed by the authors and the method using trials described by Hetényi and demonstrated by Brush and Almroth (1975).

Working in consistent units, consider a simply supported beam of length $l = 10$ resting on a foundation of stiffness $k = 100$.

Case 1: EI = 75

Direct Algorithm

1. $n = 3.4205$ [from Eq. (12)]
2. $j = 3$ [from Eq. (13)]
3. $3 \leq 3.4205 \leq \sqrt{12}$, so $m = 3$ [from Eq. (14*a*)]
4. $N_{cr} = 179.20$ [from Eq. (15)]

Trial Approach

1. Compute N_{cr} from Eq. (15) as

$$N_{cr} = \left(\frac{m\pi}{10}\right)^2 (75) + \left(\frac{10}{m\pi}\right)^2 (100)$$
$$= 7.4022m^2 + \frac{1,013.2118}{m^2}$$

2. Using trials, find m to give the minimum value of N_{cr}

m	1	2	**3**	4
N_{cr}	1,020.61	282.91	**179.20**	181.76

Case 2: EI = 70

Direct Algorithm

1. $n = 3.4800$ [from Eq. (12)]
2. $j = 3$ [from Eq. (13)]
3. $\sqrt{12} \leq 3.4800 \leq 4$, so $m = 4$ [from Eq. (14*b*)]
4. $N_{cr} = 173.87$ [from Eq. (15)]

Trial Approach

1. Compute N_{cr} from Eq. (15) as

$$N_{cr} = \left(\frac{m\pi}{10}\right)^2 (70) + \left(\frac{10}{m\pi}\right)^2 (100)$$
$$= 6.9087m^2 + \frac{1,013.2118}{m^2}$$

2. Using trials, find m to give the minimum value of N_{cr}

m	2	3	**4**	5
N_{cr}	280.94	174.76	**173.87**	213.25

It can be noted that if n from Step 1 in the Direct Algorithm had been rounded to the nearest integer, leading to $m = 3$, an incorrect buckling load of $N_{cr} = 174.76$ would have been predicted.

Conclusions

Hetényi described a trial approach for finding the critical buckling load of a beam on an elastic foundation; however, the FE study presented in this paper has highlighted an inconsistency in his instructions for rounding of the mode number that can lead to an overestimation of the buckling load (N_{cr}) close to the transition between mode shapes. The paper has presented a direct algorithm in Eqs. (12)–(15) for correctly selecting the mode number m and hence computing the critical (lowest) buckling load N_{cr} for all cases.

Appendix. Matrix Terms from Differential Eq. (1)

Term in Eq. (1)	Matrix in Eq. (2)
$EI\frac{d^4}{dx^4}$	$[k_m] = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix}$
k	$[m_m] = \frac{kL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$
$\frac{d^2}{dx^2}$	$[g_m] = -\frac{1}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$

Acknowledgments

The authors acknowledge the support of National Science Foundation grant CMMI-0970122 on GOALI: Probabilistic Geomechanical Analysis in the Exploitation of Unconventional Resources.

Notation

The following symbols are used in this paper:

Scalars

EI = beam flexural stiffness;
i = integer counter;
j = integer used in modified N_{cr} algorithm;
k = foundation stiffness;
L = beam element length;
l = total beam length;
m = integer representing buckled mode number;
N = axial load;
N_{cr} = critical buckling load;
$N_{cr,\infty}$ = critical buckling load of an infinitely long beam;
n = integer or real number used to represent buckled mode number;
q = transverse distributed load;
x = spatial coordinate along beam axis;
y = transverse displacement of beam;
y_i = transverse displacement at ith node; and
θ_i = rotation at ith node.

Vectors and Matrices

$[G_m]$ = global geometric matrix;
$[g_m]$ = element geometric matrix;
$[K'_m]$ = global modified stiffness matrix;
$[k_m]$ = element stiffness matrix;
$[k'_m]$ = element modified stiffness matrix;
$[m_m]$ = element mass matrix;

$\{W\}$ = global nodal displacements and rotations vector; and
$\{w\}$ = element nodal displacements and rotations vector.

References

Bee, G. (2013). "Beam buckling on random elastic foundations." M.S. thesis, Colorado School of Mines, Golden, CO.

Brush, D. O., and Almroth, B. O. (1975). *Buckling of bars, plates and shells*, McGraw Hill, New York.

de Angelis, F., and Cancellara, D. (2012). "On the influence of the elastic medium stiffness in the buckling behavior of compressed beams on elastic foundation." *Appl. Mech. Mater.*, 166–169, 776–783.

Griffiths, D. V., and Smith, I. M. (2006). *Numerical methods for engineers*, 2nd Ed., Chapman & Hall/CRC Press, Boca Raton, FL.

Hetényi, M. (1946). *Beams on elastic foundations*, University of Michigan Press, Ann Arbor, MI.

Li, S.-R., and Batra, R. C. (2007). "Thermal buckling and postbuckling of Euler-Bernoulli beams supported on nonlinear elastic foundations." *AIAA J.*, 45(3), 712–720.

Simitses, G. J., and Hodges, D. H. (2006). *Fundamentals of structural stability*, Butterworth Heinemann, Oxford, U.K.

Smith, I. M., and Griffiths, D. V. (2004). *Programming the finite element method*, 4th Ed., Wiley, Chichester, U.K.