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THE PROPERTIES OF ANISOTROPIC CONICAL FAILURE SURFACES IN RELATION TO THE MOHR–COULOMB CRITERION

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SUMMARY

An earlier publication[1] considered the properties of circular conical failure surfaces whose axes coincide with the space diagonal in principal stress space. The present work uses a similar approach to analyse conical surfaces that are offset from the space diagonal. It is shown that cones fitted to the Mohr–Coulomb surface in triaxial compression contain a potential singularity. The occurrence and location of the singularity depends on the Mohr–Coulomb friction angle to which the surface is fitted in triaxial extension. It is shown that for a cone fitted to the same friction angle in both triaxial extension and compression, singular conditions occur when that angle reaches $\sin^{-1}(\sqrt{7}-2)$ $(=40{\cdot}22^\circ)$. Even cones fitted to smaller friction angles give significant overestimations of material strength for certain stress paths.

1. INTRODUCTION

Use of the hexagonal conical Mohr–Coulomb surface is known to yield a reliable and conservative estimate of the strength of soil in principal stress space. However, because of the inconvenient corners on the Mohr–Coulomb surface, circular conical surfaces have frequently been proposed as suitable alternatives. In an earlier publication,[1] two such circular cones were considered which lay just outside and just inside the Mohr–Coulomb surface. The internal cone was shown to be conservative and generally acceptable, whereas the external cone exhibited the well-known[2] singularity when the friction angle in triaxial compression became equal to 36·87°. Even for smaller friction angles, the external cone could seriously overestimate the material strength for certain stress paths.

The present work extends these arguments to deal with circular conical surfaces not centred on the space diagonal. The investigation was prompted by the overestimates of strength obtained using such a model at a recent workshop.[3] The general case shown in Figure 1 is of such a cone viewed in a deviatoric plane of principal stress space. The cone appears as a circle in this projection, and the stress space has been rendered dimensionless by dividing all coordinates by the mean stress s, where s represents the perpendicular distance of the deviatoric plane from the origin of stress space. The cone is assumed to have its apex at the origin of stress space, and so the present treatment is confined to cohesionless materials. In the deviatoric plane, the circle is symmetrically placed about the vertical axis, but its centre does not necessarily coincide with the space diagonal.

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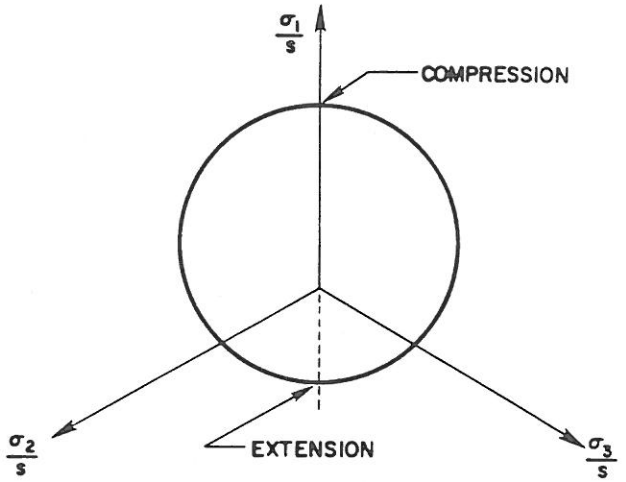

Figure 1. Anisotropic cone as viewed in a deviatoric plane

Criteria of this type are said to exhibit cross-anisotropy, and are typical of failure surfaces used in some kinematic plasticity models (see e.g. Reference 4). The surface is usually anchored at two locations on its circumference corresponding to triaxial compression and triaxial extension conditions. With these two locations forming a diameter, the rest of the circle follows automatically. Of particular interest in the present investigation is the equivalent Mohr–Coulomb friction angle implied by such a surface on its circumference at intermediate locations between the fixed triaxial values.

2. REPRESENTATION OF STRESS IN PRINCIPAL STRESS SPACE

A compression-negative sign convention is used throughout, and a stress point in principal stress space is defined using invariants:

$$(s, t, \theta) \tag{1}$$

where

$$\begin{aligned} s &= (\sigma_1+\sigma_2+\sigma_3)/\sqrt{3} \\ t &= \{\tfrac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]\}^{1/2} \\ \theta &= \tfrac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{6}J_3}{t^3}\right) \end{aligned} \tag{2}$$

The third deviatoric stress invariant is given by:

$$J_3 = s_1 s_2 s_3 \tag{3}$$

where

$$s_1 = \tfrac{1}{3}(2\sigma_1-\sigma_2-\sigma_3) \tag{4}$$

with similar expressions for s_2 and s_3. In this notation, $-s$ gives the perpendicular distance of the deviatoric plane from the origin and (t, θ) act as polar coordinates within that plane. The Lode angle θ varies in the range $\pm 30°$ provided $\sigma_1 \leq \sigma_2 \leq \sigma_3$

3. SINGULARITIES

Here we define the conditions that must be fulfilled if the cone is to become singular. A singularity in this context occurs when the cone becomes tangential to the Mohr–Coulomb surface, corresponding to $\phi_{MC}=90°$, as shown in Figure 2. A friction angle of 90° implies an infinite stress ratio at failure, and hence infinite strength.

The distance of the Mohr–Coulomb surface from the origin of stress space in a deviatoric plane varies with θ according to the relation[1]

$$L=\frac{\sqrt{2}\sin\phi_{MC}}{\sqrt{3}\cos\theta-\sin\theta\sin\phi_{MC}} \tag{5}$$

Hence, if ϕ'_c and ϕ'_e are the friction angles in triaxial compression and extension respectively, we obtain

$$\theta=30° \text{ (compression)} \quad L_c=\frac{2\sqrt{2}\sin\phi'_c}{3-\sin\phi_c} \tag{6}$$

$$\theta=-30° \text{ (extension)} \quad L_e=\frac{2\sqrt{2}\sin\phi'_e}{3+\sin\phi_e} \tag{7}$$

Assuming the conical surface to be anchored at these two locations on the isotropic axis, its radius in the deviatoric plane is given by

$$R=\sqrt{2}\left(\frac{\sin\phi'_c}{3-\sin\phi'_c}+\frac{\sin\phi'_e}{3+\sin\phi'_e}\right) \tag{8}$$

and the distance of its centre from the origin by

$$b=\sqrt{2}\left(\frac{\sin\phi'_c}{3-\sin\phi'_c}-\frac{\sin\phi'_e}{3+\sin\phi_e}\right) \tag{9}$$

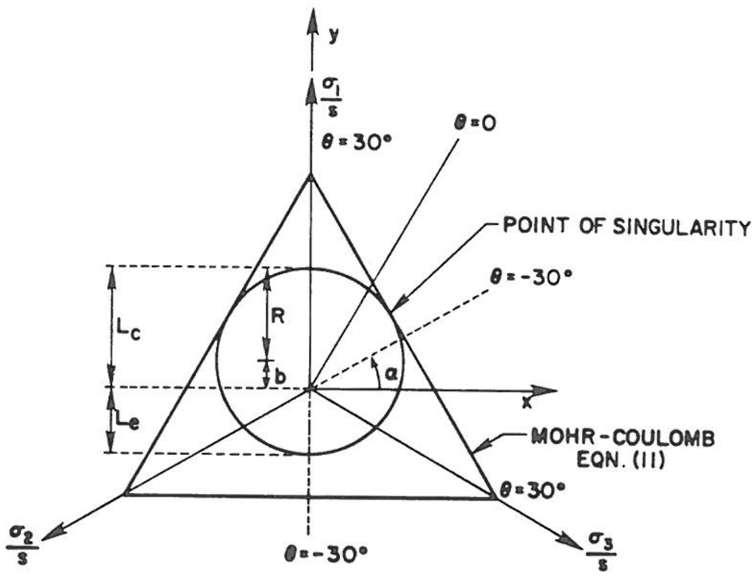

Figure 2. Limiting circle shown tangential to Mohr–Coulomb surface with $\phi_{MC}=90°$

We now consider a Cartesian representation of the Mohr–Coulomb surface in the range $-30° \leq \theta \leq 30°$ which has the equation

$$y = -\sqrt{3}\left(\frac{1+\sin\phi_{MC}}{3-\sin\phi_{MC}}\right)x + \left(\frac{2\sqrt{2}\sin\phi_{MC}}{3-\sin\phi_{MC}}\right) \tag{10}$$

Of particular interest is the line corresponding to $\phi_{MC} = 90°$, for which equation (10) becomes

$$y = -\sqrt{3}x + \sqrt{2} \tag{11}$$

We seek the values of ϕ'_c and ϕ'_e that will cause the circular projection of the conical surface in the deviatoric plane to become tangential to the line given by equation (11), as shown in Figure 2. Referring to Figure 3, we consider a general line $y = mx + c$. The perpendicular at the point (x_c, y_c) intersects the y-axis at $(0, b)$ and is of length R. We also introduce an angle α which measures the angular position of any point in the xy plane, where in terms of principal stresses, the dimensionless coordinates are given by

$$x = \frac{1}{\sqrt{2}s}(\sigma_2 - \sigma_3) \tag{12}$$

$$y = \frac{-1}{\sqrt{6}s}(2\sigma_1 - \sigma_2 - \sigma_3) \tag{13}$$

hence

$$\tan\alpha = y/x \tag{14}$$

It is easily shown that:

$$x_c = \frac{m(b-c)}{m^2+1} \tag{15}$$

$$y_c = \frac{m^2 b + c}{m^2+1} \tag{16}$$

$$R = \frac{c-b}{(m^2+1)^{1/2}} \tag{17}$$

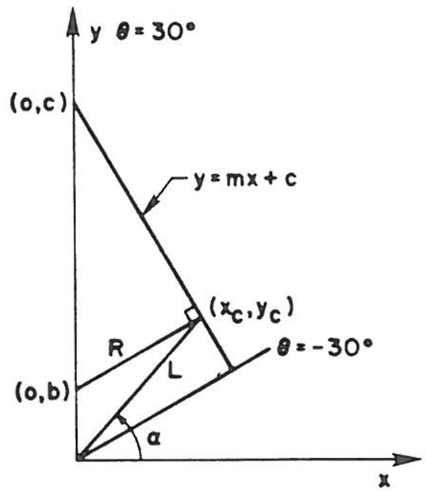

Figure 3. Cartesian representation of stress space to find the point of singularity.

Noting from equation (11) that $m=-\sqrt{3}$ and $c=\sqrt{2}$, and substituting for R and b from equations (8) and (9), we obtain after some rearrangement

$$\sin \phi'_c = \frac{3}{4+\sin \phi'_e} \tag{18}$$

This expression gives the relation between ϕ'_c and ϕ'_e that will cause a singularity to occur at the point (x_c, y_c). The angular position at which this singularity occurs is given by eqauation (14), which after substitutions can be written as

$$\tan \alpha = \frac{\sqrt{3}(7 \sin \phi'_c - 3)}{9(1-\sin \phi'_c)} \tag{19}$$

Some singular cones in the range $\phi'_c \leq \phi'_e \leq 90°$ are summarized in Table I, together with the angular location of the point of tangency. The top entry in Table I represents the familiar singularity[1,2] corresponding to the external cone fitted to the Mohr–Coulomb surface in triaxial compression at $\phi'_c = 36{\cdot}87°$; the bottom entry represents the singularity corresponding to $\phi'_c = \phi'_e$ which occurs at $\alpha = 39{\cdot}55°$, when $\phi'_c = \sin^{-1}(\sqrt{7}-2) = 40{\cdot}22°$, from equation (18).

Table I indicates the limiting values of ϕ'_c and ϕ'_e, but even non-singular failure surfaces may predict excessive material strength for certain values of α. This is examined further in Section 4 for the special case $\phi'_c = \phi'_e$.

4. EQUIVALENT MOHR–COULOMB ANGLE ($\phi'_c = \phi'_e$)

In the absence of more detailed information on the shear strength in extension and compression, a reasonable assumption would be to make the friction angle corresponding to both these limits the same. Thus, for $\phi'_c = \phi'_e = \phi'$ it was shown in Section 3 that the resulting conical surface becomes singular when $\phi' = 40{\cdot}22°$. However, even when $\phi' < 40{\cdot}22°$ a considerable variation in the equivalent Mohr–Coulomb friction angle at intermediate positions on the surface of the cone can occur.

From Figure 2, the equation of the circle is

$$x^2 + (y-b)^2 = R^2 \tag{20}$$

and by making the substitution

$$x = L \cos \alpha$$

$$y = L \sin \alpha \tag{21}$$

where L is the distance of any point on the circumference from the origin of stress space, we obtain

$$L^2 - 2bL \sin \alpha - (R^2 - b^2) = 0 \tag{22}$$

Table I. Singular combinations of ϕ'_c and ϕ'_e

ϕ'_c	ϕ'_e	α	θ
36·87°	90·00°	30·00°	−30·00°
37·00°	80·04°	30·38°	−29·62°
38·00°	60·79°	33·26°	−26·74°
39·00°	50·09°	36·11°	−23·89°
40·00°	41·85°	38·93°	−21·07°
40·22°	40·22°	39·55°	−20·45°

which after solution and taking the positive root gives

$$L = b\sin\alpha + (R^2 - b^2\cos^2\alpha)^{1/2} \tag{23}$$

Equation (5) can be rearranged to give

$$\sin\phi_{MC} = \frac{\sqrt{3}\,L\cos\theta}{\sqrt{2} + L\sin\theta} \tag{24}$$

and, in order to compute ϕ_{MC} as a function of α, we must replace θ by α:

$$\begin{aligned} \theta &= \alpha + 60^\circ, & -90^\circ \leq \alpha < -30^\circ \\ \theta &= -\alpha, & -30^\circ \leq \alpha < +30^\circ \\ \theta &= \alpha - 60^\circ, & +30^\circ \leq \alpha \leq +90^\circ \end{aligned} \tag{25}$$

After substituting for R and b from equations (8) and (9), and noting that $\phi'_c = \phi'_e = \phi'$, we obtain equation (24) in the form

$$\phi_{MC} = f(\phi', \alpha) \tag{26}$$

which is shown plotted for $\phi' = 20^\circ$, 30° and 40° in Figure 4. As expected, for $\alpha = \pm 90^\circ$, $\phi_{MC} = \phi'$, reflecting the locations at which the cone is anchored. Between these limits there is considerable variation in ϕ_{MC}. At $\alpha = -30^\circ$ a minimum is reached where $\phi_{MC} < \phi'$, but as α increases ϕ_{MC} reaches a maximum around $\alpha \approx 40^\circ$.The singularity corresponding to $\phi' = 40{\cdot}22^\circ$ is strongly indicated by the high peak of the curve when $\phi' = 40^\circ$.

The maximum value of ϕ_{MC} that can be achieved for a given cone anchored at ϕ' in both extension and compression is obtained from differentiation of equation (26):

$$d\phi_{MC}/d\alpha = 0 \tag{27}$$

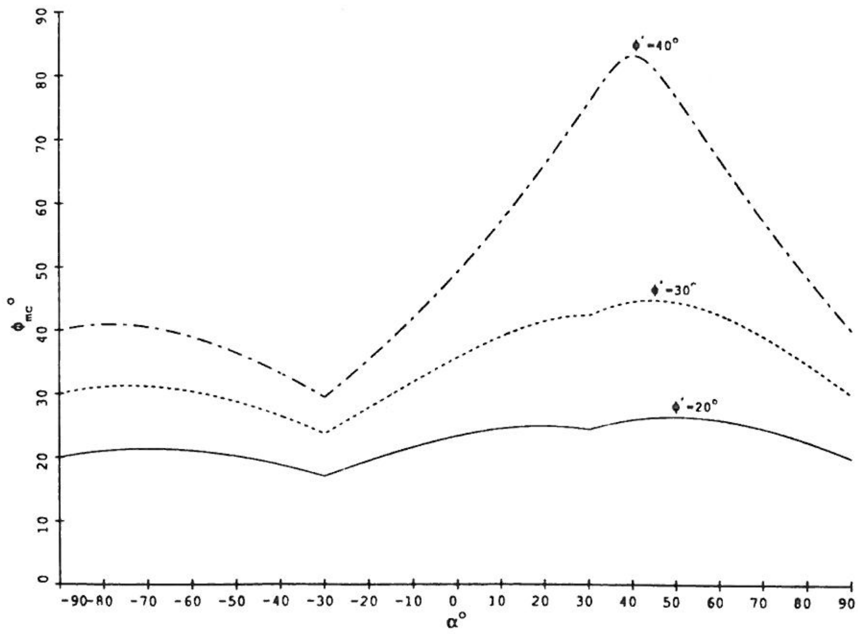

Figure 4. Equivalent Mohr–Coulomb friction angle ϕ_{MC}, as a function of α, for cones anchored at $\phi' = 20^\circ$, 30° and 40°

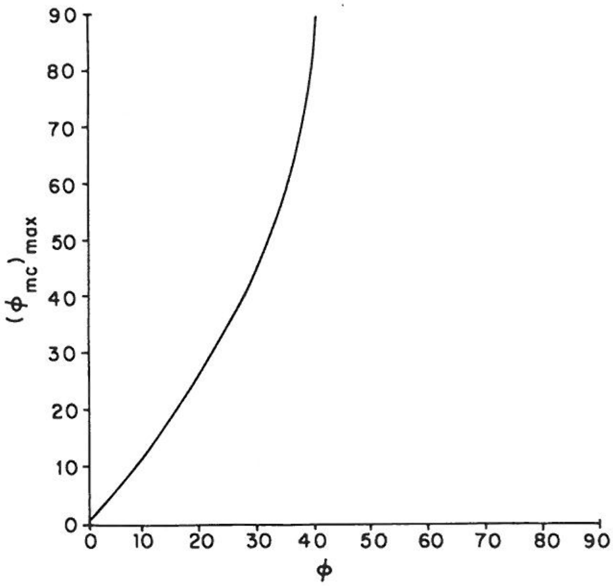

Figure 5. Maximum Mohr–Coulomb friction angle ϕ_{MC} that can be reached for cones anchored in the range $0<\phi'<40{\cdot}22°$

Table II. Location and magnitude of maximum ϕ_{MC} for cones anchored with $\phi'_c=\phi'_e=\phi'$

ϕ'	$(\phi_{MC})_{max}$	α_{max}	θ_{max}
19·97°	26·50°	49·77°	−10·23°
30·01°	45·00°	44·61°	−15·39°
40·01°	83·50°	39·66°	−20·34°
40·22°	90·00°	39·55°	−20·45°

which can be solved to give

$$\sin\phi'=(\sin^2\phi_{MC}+3\sin\phi_{MC}+3)^{1/2}-(\sin^2\phi_{MC}+3)^{1/2} \quad (28)$$

Figure 5 shows this relationship, and demonstrates how the material strength can be seriously overestimated for certain stress paths. The angular position at which the maxima of equation (28) occurs can be easily deduced, and some typical values are summarised in Table II. It is shown that a cone anchored at $\phi'\approx 40°$, for example, could give predictions as high as $\phi_{MC}=83{\cdot}5°$ if a stress path with the direction $\alpha=39{\cdot}7°$ were followed.

5. CONCLUSIONS

Circular conical failure surfaces offset from the space diagonal to account for anisotropy have been examined in relation to the well-known Mohr–Coulomb criterion. It is shown that conical surfaces of this type may contain a singularity where infinite shear strength is predicted for certain stress paths. If the cone is fitted to the Mohr–Coulomb surface in both triaxial extension and compression, the singularity occurs when $\phi'=\sin^{-1}(\sqrt{7}-2)$. It is also shown that, even when cones are generated with a friction angle less than the critical value, excessive material strength can be predicted. No real problems are encountered for small friction angles, or even for large friction angles provided a 'safe' stress path is followed. In general, however, circular conical surfaces of the type described are not recommended for use in complex boundary value problems.

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