Observations on Load and Strength Factors in Bearing Capacity Analysis

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Abstract: The widely differing factors of safety obtained by strength reduction and load increase in the ultimate bearing capacity of foundations have been reexamined analytically. In most cases, factors of safety based on load increase need to be higher than those based on strength reduction to achieve the same level of safety in design. The greater sensitivity of foundation systems to strength reduction over load increase may impact the choice of factors in ultimate limit state design. **DOI: 10.1061/(ASCE)GT.1943-5606.0001316.** © *2015 American Society of Civil Engineers.*

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Introduction

It is well known that slope stability and bearing capacity analyses for design typically target factors of safety of around 1.5 and 3.0, respectively (e.g., Terzaghi et al. 1996; Salgado 2008). Both applications involve ultimate limit state conditions, so an impression might be given that bearing capacity design needs to be *twice as safe* as that required for a slope. Clearly this is not the case, given that the two factors of safety are based on completely different premises. The factor of safety used in slopes is a type of resistance factor, i.e., the factor by which the shear strength must be reduced to reach failure, whereas that for bearing capacity is a load factor, i.e., the factor by which the load must be increased to reach failure. A general coverage of the difference between factors of safety based on resistance and load for slope and bearing capacity problems, and the disparity between them, was noted by Duncan and Wright (2005), and there are other important references in which the use of load and resistance factors have been discussed in detail (e.g., Becker 1996). The purpose of this note is to present a direct comparison between load increase and strength reduction in ultimate bearing capacity. An analytical approach has been facilitated by the introduction of a novel identity for the passive earth pressure coefficient K_P.

Bearing Capacity Factors

Consider the bearing capacity problem shown in Fig. 1, involving a rough strip footing of width B supported on a uniform $c' - \phi'$ soil of unit weight γ with a surface surcharge of q.

The ultimate bearing capacity q_{ult} of the footing can be given by Terzaghi's bearing capacity equation

$$q_{\mathrm{ult}} = c'N_c + qN_q + \frac{\gamma B}{2}N_\gamma \tag{1}$$

where N_c, N_q and N_γ = bearing capacity factors.

It is recognized that direct computational methods are also available for obtaining q_{ult}, that avoid the superposition implied by Eq. (1) which is not theoretically valid unless $\gamma = 0$. Notable among these are the finite-element method (e.g., Griffiths 1982; Smith et al. 2014), which is particularly powerful for nonuniform ground conditions, and the method of characteristics (e.g., Martin 2005) which can lead to exact numerical solutions. For the purposes of this note however, the classical Eq. (1) offers a convenient vehicle for direct comparison of load and strength factors.

The factor of safety in bearing capacity analysis is typically based on loads, and given by

$$FS_q = \frac{q_{\mathrm{ult}}}{q_{\mathrm{all}}} \tag{2}$$

where q_{all} = allowable design pressure that the footing can safely support. A target factor of safety against bearing failure of about $FS_q \approx 3$ (e.g., Salgado 2008) would typically be required in design. It is recognized that serviceability limit states regularly govern design in practice, especially for less dense soils and wider footings; however, the focus of this note is on the ultimate limit state.

For comparison purposes, the factor of safety against bearing failure based on strength reduction is now considered. In this case, the strength reduction factor of safety given by $FS_{c',\tan\phi'}$, is the factor by which c' and $\tan\phi'$ must be reduced in Eq. (1) to cause bearing failure, i.e., $q_{\mathrm{ult}} = q_{\mathrm{all}}$.

To compare FS_q and $FS_{c',\tan\phi'}$ analytically, the following expressions for the bearing capacity factors will be used

$$N_c = (K_p e^{\pi \tan\phi'} - 1)\cot\phi' \tag{3}$$

$$N_q = K_p e^{\pi \tan\phi'} \tag{4}$$

$$N_\gamma = 1.5(K_p e^{\pi \tan\phi'} - 1)\tan\phi' \tag{5}$$

Eqs. (3) and (4) for the N_c and N_q-terms are mathematically rigorous, and attributable to Prandtl (1921). The N_γ term has proved more challenging and has no direct analytical solution. Martin (2005) has produced exact numerical solutions using the

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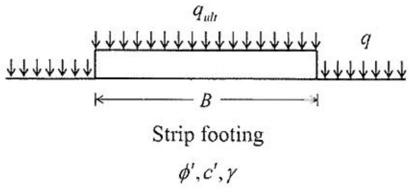

Fig. 1. Bearing capacity of a strip footing

method of characteristics (assuming an associated flow rule), and these results are closely approximated by Eq. (5) from Brinch Hansen (1970).

Eqs. (3)–(5) all include the passive earth pressure coefficient K_p which can be commonly expressed in several different ways, e.g

$$K_p = \tan^2\left(45° + \frac{\phi'}{2}\right) = \frac{1+\sin\phi'}{1-\sin\phi'} = \left(\frac{\cos\phi'}{1-\sin\phi'}\right)^2 \tag{6}$$

In this work, a less familiar trigonometric identity for K_p is used as given by Eq. (7). This version, which expresses K_p purely in terms of $\tan\phi'$, has been discussed previously by Griffiths et al. (2002), and is convenient for analytical consideration of strength reduction:

$$K_P = \left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 \tag{7}$$

With reference to Eq. (1), each of the three bearing capacity factors given in Eqs. (3)–(5) will be considered separately, leading to analytical expressions relating the conventional factor of safety based on load increase (FS_q), to that based on strength reduction ($FS_{c',\tan\phi'}$). After this, an example will be presented with all terms included.

N_c Term [Eq. (3)]

In this case it is assumed that $q = 0$ and $\gamma = 0$, hence the ultimate bearing capacity is given by

$$\begin{aligned} q_{\text{ult}} &= c'N_c = c'(K_p e^{\pi\tan\phi'} - 1)\cot\phi' \\ &= c'\left[\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} - 1\right]\frac{1}{\tan\phi'} \end{aligned} \tag{8}$$

and the allowable value based on strength reduction by

$$\begin{aligned} q_{\text{all}} = \frac{c'}{FS_{c'\tan\phi'}} &\left\{\left[\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right]^2\right. \\ &\left.\times e^{\pi(\tan\phi'/FS_{c'\tan\phi'})} - 1\right\}\frac{FS_{c'\tan\phi'}}{\tan\phi'} \end{aligned} \tag{9}$$

Substitution of Eqs. (8) and (9) into Eq. (2) gives the function

$$FS_q = \frac{(\tan\phi' + \sqrt{1+\tan^2\phi'})^2 e^{\pi\tan\phi'} - 1}{\left[\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan\phi'/FS_{c'\tan\phi'})} - 1} \tag{10}$$

N_q Term [Eq. (4)]

In this case, it is assumed that $c' = 0$ and $\gamma = 0$, hence the ultimate bearing capacity is given by

$$q_{\text{ult}} = qN_q = qK_p e^{\pi\tan\phi'} = q\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} \tag{11}$$

and the allowable value based on strength reduction by

$$q_{\text{all}} = q\left[\frac{\tan\phi'}{FS_{c',\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c',\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan\phi'/FS_{c',\tan\phi'})} \tag{12}$$

Substitution of Eqs. (11) and (12) into Eq. (2) gives the function

$$FS_q = \frac{(\tan\phi' + \sqrt{1+\tan^2\phi'})^2 e^{\pi\tan\phi'}}{\left[\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan\phi'/FS_{c'\tan\phi'})}} \tag{13}$$

N_γ Term [Eq. (5)]

In this case, it is assumed that $c' = 0$ and $q = 0$, hence the ultimate bearing capacity is given by

$$\begin{aligned} q_{\text{ult}} &= \frac{\gamma B}{2}N_\gamma = \frac{\gamma B}{2}[1.5(K_p e^{\pi\tan\phi'} - 1)\tan\phi'] \\ &= \frac{3\gamma B}{4}\left[\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} - 1\right]\tan\phi' \end{aligned} \tag{14}$$

and the allowable value based on strength reduction by

$$\begin{aligned} q_{\text{all}} = \frac{3\gamma B}{4} &\left\{\left[\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right]^2\right. \\ &\left.\times e^{\pi(\tan\phi'/FS_{c'\tan\phi'})} - 1\right\}\frac{\tan\phi'}{FS_{c'\tan\phi'}} \end{aligned} \tag{15}$$

Substitution of Eqs. (14) and (15) into Eq. (2) gives the function

$$FS_q = \frac{[(\tan\phi' + \sqrt{1+\tan^2\phi'})^2 e^{\pi\tan\phi'} - 1]FS_{c'\tan\phi'}}{\left[\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan\phi'/FS_{c'\tan\phi'})} - 1} \tag{16}$$

Eqs. (10), (13), and (16) are shown plotted in Figs. 2–4, respectively for a range of ϕ'-values. It may be noted from Figs. 2 and 4 for the N_c and N_γ terms, respectively, that $FS_q > FS_{c',\tan\phi'}$ for all $\phi' > 0$. On the other hand, analysis of Eq. (13) for N_q reveals that $FS_q < FS_{c',\tan\phi'}$ for all $\phi' < 11.08°$, and $FS_q < FS_{c',\tan\phi'}$ can also occur for $\phi' > 11.08°$ if $FS_{c',\tan\phi'}$ exceeds some critical value. This effect is demonstrated in Fig. 3, where, for example, the line for $\phi' = 14°$ crosses the dashed line ($FS_{c',\tan\phi'} = FS_q$) at about $FS_{c',\tan\phi'} = 1.65$. Table 1 summarizes for a range of ϕ' values, the transition point at which $FS_q < FS_{c',\tan\phi'}$ for the N_q bearing capacity factor.

In summary, $FS_q < FS_{c',\tan\phi'}$ is only observed in the N_q term for practical purposes when ϕ' is small (e.g., $\phi' < 15°$). Higher friction angles can also lead to $FS_q < FS_{c',\tan\phi'}$, but only when $FS_{c',\tan\phi'}$ is unrealistically high (e.g., for $\phi' = 25°$ when $FS_{c',\tan\phi'} > 7.86$).

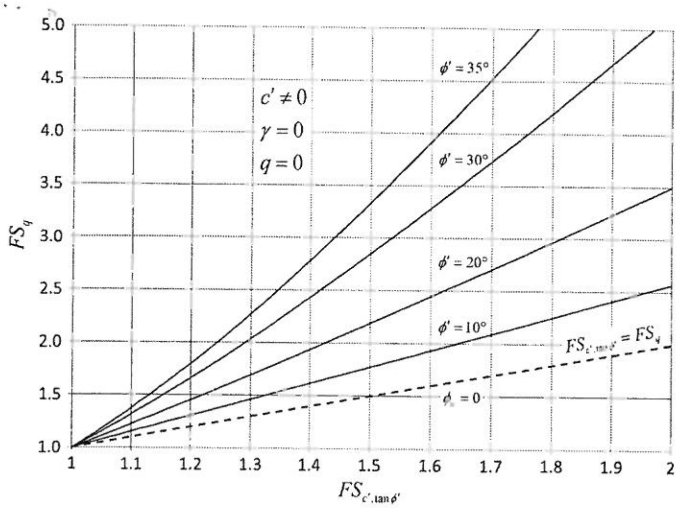

Fig. 2. $FS_{c',\tan\phi'}$ versus FS_q for the N_c term [Eq. (10)]

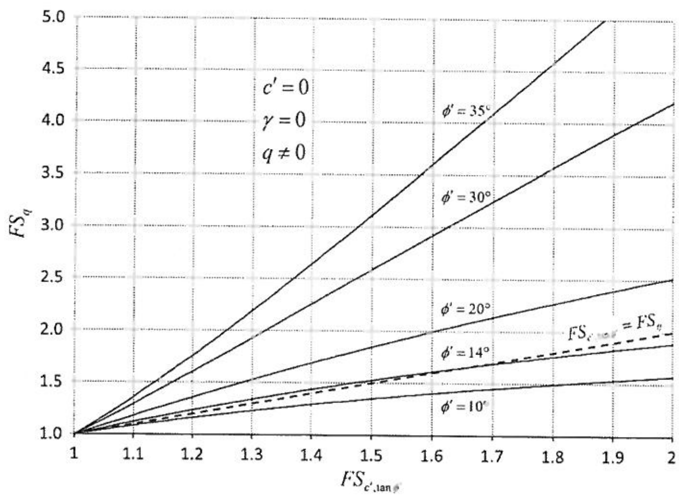

Fig. 3. $FS_{c',\tan\phi'}$ versus FS_q for the N_q term [Eq. (13)]

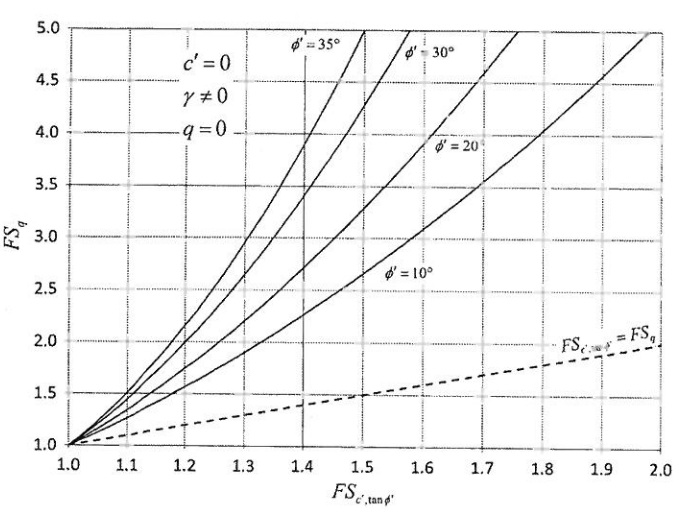

Fig. 4. $FS_{c',\tan\phi'}$ versus FS_q for the N_γ term [Eq. (16)]

Table 1. Range of $FS_{c',\tan\phi'}$ for Which $FS_q < FS_{c',\tan\phi'}$ in the N_q-Term from Eq. (13)

ϕ' (degrees)	$FS_{c',\tan\phi'}$
<11.08	>1.00
12	>1.18
14	>1.65
16	>2.26
18	>3.03
20	>4.02
25	>7.86

Bearing Capacity Example

To include all the terms of the bearing capacity equation, an example from Salençon and Matar (1982) is now considered where, with reference to Fig. 1, $\phi' = 30°$, $c' = 16$ kPa, $\gamma = 18$ kN/m^3, $B = 4$ m, and $q = 18$ kPa.

Fig. 5 shows plots of $FS_{c',\tan\phi'}$ vs. FS_q for each of the three bearing capacity factors corresponding to $\phi' = 30°$. From Eqs. (3)–(5), the bearing capacity factors are $N_c = 30.1$, $N_q = 18.4$, and $N_\gamma = 15.1$, and the bearing capacity from Eq. (1) is given by

$$q_{\text{ult}} = 16 \times 30.1 + 18 \times 18.4 + \frac{18 \times 4}{2} \times 15.1 = 1{,}356 \text{ kPa} \tag{17}$$

Load Increase

If a typical load-based factor of safety against bearing failure of $FS_q = 3$ is used, the allowable bearing pressure is given by

$$q_{\text{all}} = \frac{1{,}356}{3} = 452 \text{ kPa} \tag{18}$$

Strength Reduction

From a strength reduction perspective, the value of $FS_{c',\tan\phi'}$ that would be needed to reduce the bearing capacity given in Eq. (17) from $q_{\text{ult}} = 1356$ kPa to $q_{\text{ult}} = q_{\text{all}} = 452$ kPa is given by the nonlinear equation

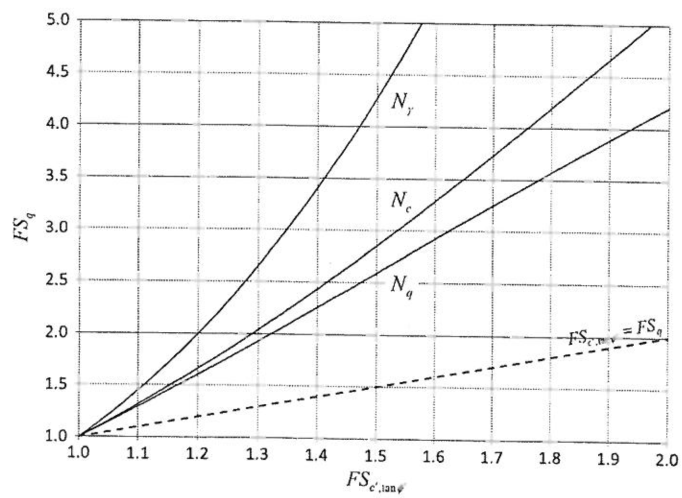

Fig. 5. $FS_{c',\tan\phi'}$ versus FS_q for the three bearing capacity factors corresponding to $\phi' = 30°$

$$452 = 16\left[\left[\frac{\tan 30°}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan 30°}{FS_{c'\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan 30°/FS_{c'\tan\phi'})} - 1\right]\frac{1}{\tan 30°}$$

$$+ 18\left[\frac{\tan 30°}{FS_{c',\tan\phi'}} + \sqrt{1+\left(\frac{\tan 30°}{FS_{c',\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan 30°/FS_{c'\tan\phi'})}$$

$$+ \frac{3\times 18\times 4}{4}\left\{\left[\frac{\tan 30°}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan 30°}{FS_{c'\tan\phi'}}\right)^2}\right]^2 e^{\pi(\tan 30°/FS_{c'\tan\phi'})} - 1\right\}\frac{\tan 30°}{FS_{c'\tan\phi'}} \quad (19)$$

which after solution gives

$$FS_{c',\tan\phi'} = 1.60 \quad (20)$$

This example gives $FS_q \approx 2FS_{c',\tan\phi'}$, which might be expected from Fig. 5 for bearing capacity on a soil with $\phi' = 30°$. It is clear from the examples considered, that geotechnical design is more sensitive to strength reduction than load increase, i.e., an allowable bearing capacity based on $FS_{c',\tan\phi'} = 1.5$ (say) will be considerably lower (more conservative) than one based on $FS_q = 1.5$.

Concluding Remarks

The note has examined the difference between the factor of safety defined by strength reduction ($FS_{c',\tan\phi'}$) and that defined by load increase (FS_q) in an ultimate bearing capacity problem. A novel identity for K_p led to the development of closed-form expressions directly relating FS_q to $FS_{c',\tan\phi'}$ for each of the three bearing capacity factors. It was observed that when $\phi' > 0$, $FS_q > FS_{c',\tan\phi'}$ for nearly all cases, except for certain combinations of parameters in the N_q term. A bearing capacity example involving a soil with $\phi' = 30°$ and all terms of the bearing capacity equation, led to $FS_q \approx 2FS_{c',\tan\phi'}$. The difference would be greater with higher friction angles; however, the results show that for a typical friction angle, the level of safety in design based on a strength reduction factor of safety of approximately $FS_{c',\tan\phi'} \approx 1.5$ is not significantly different to that based on a load increase factor of safety of $FS_q \approx 3.0$. The greater sensitivity of foundation systems to strength reduction over load increase may impact the choice of factors in ultimate limit state design.

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