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AN ITERATIVE METHOD FOR PLASTIC ANALYSIS OF FRAMES

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Abstract—A numerical approach for the computation of yield and plastic collapse of framed structures is described. Based on the 'initial stress' method in which the global matrix is formed once only, the algorithm differs from other computer approaches in that moments in excess of their yield values are redistributed using 'constant stiffness' iterations. Under a given set of external loads, convergence of the numerical process gives a solution in which equilibrium is satisfied without any violation of yield at the joints. Collapse of a structure is indicated by a sudden increase in the number of iterations required for convergence, and correspondingly large nodal displacements. Examples are presented in which both loading and unloading cycles are followed, leading to demonstrations of elastic shake down and incremental collapse. Comparisons with solutions obtained by more traditional methods suggest that the proposed algorithm represents an accurate and versatile approach to the analysis of yield in framed structures.

NOTATION

Scalars

EA	axial rigidity
EI	bending rigidity
GJ	torsional rigidity
M_p, M_p^1, M_p^2	plastic moments
M_1, M_2	end moments
U_{max}	maximum displacement
t	tolerance
C	couple
L	element length
θ	element inclination
λ, λ_1, λ_2	load factors
W	load
W_{ult}	collapse load
δ_c	movement of point C
α	plastic hinge rotation angle
i, e, l	counters
NEL	number of elements
$IMAX$	iteration limit

Arrays and vectors

$\mathbf{K}$	global stiffness matrix
$\mathbf{k}_e$	element stiffness matrix
$\mathbf{U}^l$	displacements accumulator
$\Delta\mathbf{P}_a^l$	applied load increments
$\mathbf{r}_e^l$	element actions accumulator
$\Delta\mathbf{U}^i$	displacement increments
$\Delta\mathbf{F}^i$	load increments
$\Delta\mathbf{P}^i$	correction load increments
$\boldsymbol{\delta}_e^i$	element displacements
$\mathbf{p}_e^i$	element actions
$\mathbf{q}_e^i$	element correction vector

Operators

$\mathbf{A}$	element assembly operator

1. INTRODUCTION

The paper describes a computer oriented technique for monitoring the onset of plasticity and the collapse of framed structures subjected to various combinations of external loading.

Much has been written on the subject of yield and plastic collapse of framed structures (e.g. [1–3]) so only a brief introduction will be given here. Traditional methods for determining the plastic collapse loads of fairly simple structures have tended to invoke a combination of upper and lower bound limit theorems (e.g. [4]). The lower bound method involved finding bending moment distributions in equilibrium with applied loads and nowhere violating the yield criterion, whereas the upper bound method involved postulating failure mechanisms in which the work done by external loads was equal to energy dissipated internally at the plastic hinges. The lower bound method for more complicated structures can be expressed in the form of a linear programming problem [5] amenable to computer solution. Although systematic, this approach has not proved as popular as the upper bound 'method of combined mechanisms' in which a trial and error search is made for the minimum loads consistent with a kinematically admissible failure mechanism. A summary of various methods of tackling the plastic collapse problem is to be found in [2].

Although intuition can play a large part in the selection of mechanisms for obtaining 'quick' solutions by the upper bound method, great care must be exercised in more complex problems where a more rigorous approach may be necessary. An interesting example of a structure in which intuition could mislead is the case of a simply supported Vierendeel girder described by Horne [4] and shown in Fig. 1. Analysis shows that the weakest point of the beam (ignoring the effects of axial forces on plastic moments) for plastic collapse due to a single load does not lie on the centreline!

With the advent of digital computers, algorithms have been devised which automate the process of obtaining the collapse load of beams and frames (e.g. [6–8]). A popular method is to use a step-wise elastic

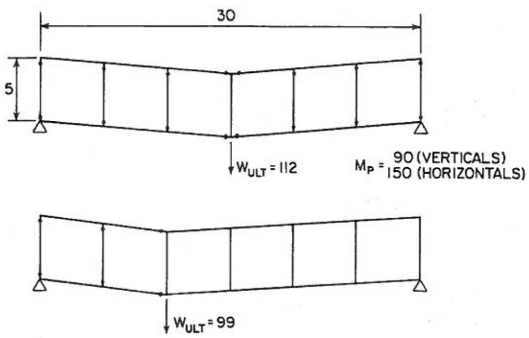

Fig. 1. Collapse of Vierendeel girder.

analysis in which moments obtained elastically are linearly extrapolated such that the most critical joint reaches its plastic moment M_p. This joint is then replaced by a pin and the process repeated noting that a running total is kept of moments at all joints from one step to the next. Eventually, sufficient joints go plastic such that a mechanism develops signalled by a sudden increase in nodal displacements.

As discussed by Wang [6], the formation of a pin can be dealt with either by the inclusion of an extra rotational degree of freedom, or by modifying the element stiffness matrix. The latter course has the advantage that the global stiffness matrix remains unchanged for every stage of the elastic analysis. This is also a feature of the method described later in this paper.

Although based on the stiffness method, some computer solutions incorporate a modified approach in which the effects of axial deformations of the members are ignored. From the point of view of automation of the method, however, the inclusion of axial deformations of the members leads to a more general formulation and has been retained in the present work. If required, the effects of axial deformations can be virtually eliminated by the incorporation of large relative values of the axial rigidity EA. Alternatively, axial effects can be completely eliminated by the use of *tied freedoms* as will be described in a subsequent section.

2. GENERAL DESCRIPTION OF THE METHOD

The algorithm described in this paper is based on the standard matrix stiffness method, and is obtained by making some simple modifications to an ordinary elastic program for the analysis of frames. The present approach does not account for changes in element stiffness due to axial forces, but modifications to the stiffness matrices to include stability functions [9] could be easily made if required. The method differs from traditional solution techniques in that equilibrium between applied *loads* (forces and moments), and internal element *actions* without violation of yield, is achieved by iteratively modifying the global loads vector. This modification continues repeatedly until the moments at all joints are less than or equal to their respective plastic moment limits within certain tolerances. The global stiffness matrix remains unchanged throughout the process. The algorithm is designed for incrementally applied loads, although loads can be applied in a single increment if desired without any loss of accuracy. Failure of a structure, and the formation of a mechanism, is signalled by a sudden increase in the nodal *displacements* (translations and rotations) and failure of the iterative process to reach a converged state. An iteration limit $IMAX$ is included in the algorithm which, if reached, would normally indicate failure conditions.

The analysis of cyclic loading of a structure following initial yielding is easily dealt with by the application of negative load increments. The effects of elastic shakedown can also be examined this way. The directness of the method, whereby the actual loads of interest are applied to the structure and the corresponding displacements computed, enables phenomena such as incremental collapse to be reproduced numerically by the application of two alternating loading functions.

The next two sections deal with the stiffness matrix formulation and the method of moment redistribution. For clarity, the examples are in two dimensions and only one plastic moment value per node per element needs to be checked. The method is easily extended to three dimensions, in which case three plastic 'moment' values (two bending, one torsion) must be checked against their limiting values. In the three-dimensional case, it also will be necessary to incorporate some form of yield criterion to account for the various interactions between the components of bending [10].

It is assumed throughout this work that the greater stresses occurring in the frames are due to bending, and that the effects of shear and axial forces on the ultimate moments are negligible. The usual assumption regarding small changes in geometry having no effect on the equilibrium equations is adhered to, although it is recognised that the assumption may be less justifiable if large deflections beyond the elastic limit occur.

3. STIFFNESS MATRIX FORMULATION

The framed structure to be analysed is discretised in terms of beam-column finite elements, ensuring that nodes are located at any points of loading and also at any locations where plastic hinges may occur. In two dimensions, each node has three degrees of freedom (two translations, one rotation). Structures in which axial deformations can be neglected are easily dealt with as special cases of the general

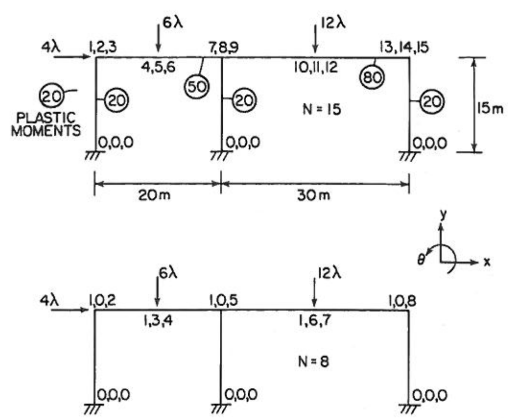

Fig. 2. Freedom numbering including (a), and excluding (b) axial deformations.

stiffness formulation. Consider the example given in Fig. 2 [4], which will also be used later to demonstrate the computation of plastic collapse. Figure 2(a) allows for the possibility of axial deformations in both the horizontal and vertical members and requires the solution of 15 simultaneous equations for the elastic case. In Fig. 2(b), the freedoms have been renumbered in a way which eliminates axial deformations by imposing a constant sway deflection in the horizontal direction, and a zero axial movement of vertical members. In the latter case, the number of equations to be solved has been reduced to eight. In both the figures, the three numbers written against each joint correspond to x-translation, y-translation, and rotation respectively, and refer to the subscripts of the corresponding global displacement vector $\mathbf{U}$. A zero implies a completely restrained freedom which has a known value (0), and therefore does not appear in the equilibrium equations.

Provided the assumption is justified, the use of *tied freedoms* as described above can lead to a substantial reduction in the amount of computational effort but the method does require some intuition on the part of the analyst in recognising the sway mechanisms. For more complex structures involving several storeys, or structures incorporating inclined members, the more general approach is recommended. Even if the general freedom numbering scheme outlined in Fig. 2(a), is adopted for a particular analysis, the axial deformations can be virtually eliminated by making the axial rigidity EA several orders of magnitude greater than the bending rigidity EI for each member.

Once the freedom numbering has been defined, the global stiffness matrix $\mathbf{K}$ can be formed by assembly of the individual element stiffness matrices $\mathbf{k}_e$. The resulting stiffness matrix is symmetrical and banded, and strategies to reduce the storage requirements can be introduced at this stage if desired. The global stiffness matrix is formed once only and immediately factorised using a suitable Gaussian elimination technique. The iterative process now begins, and is based on repeated solutions of the equilibrium equations

$$\mathbf{K}\Delta\mathbf{U}^i = \Delta\mathbf{F}^i, \tag{1}$$

where $\Delta\mathbf{F}^i$ and $\Delta\mathbf{U}^i$ are the applied load increments and the resulting elastic nodal displacement increments respectively at the ith iteration. As the global stiffness matrix has been previously factorised, the solution vector $\Delta\mathbf{U}^i$ is economically obtained by forward and backward substitution.

4. MOMENT REDISTRIBUTION STRATEGY

During the moment redistribution phase of the algorithm, all joints of the structure are monitored until a plastic moment value M_p is exceeded. When this occurs, the excess moment is distributed to other joints of the structure which still have reserves of strength.

Following the structure chart of Fig. 3, the algorithm starts with the application of the first ($l = 1$) set of load increments $\Delta\mathbf{P}_a^1$. The first iteration ($i = 1$) is always an ordinary elastic solution to the loaded structure, resulting in the set of global displacement increments $\Delta\mathbf{U}^1$. For each element in turn ($e = 1, 2, \ldots, NEL$), the nodal displacement vector $\boldsymbol{\delta}_e^i$ is extracted from the global vector, multiplied by the element stiffness matrix $\mathbf{k}_e$ and added to the element 'actions' $\mathbf{r}_e^{l-1}$ remaining from the last load step. If all the moments computed at this stage are less than their respective plastic moment values, the frame remains elastic, no further iterations are required and the algorithm moves on to the next load step. If a plastic moment value is exceeded, a plastic hinge will develop and further iterations will be needed. At the ith iteration within a typical load step l, it is convenient to think of the global applied loads $\Delta\mathbf{F}^i$ as comprising of two components, thus

$$\Delta\mathbf{F}^i = \Delta\mathbf{P}_a^l + \Delta\mathbf{P}^i, \tag{2}$$

where

$\Delta\mathbf{P}_a^l$ = external global loads increment

and

$\Delta\mathbf{P}^i$ = internal global correction loads vector.

The external loads vector $\Delta\mathbf{P}_a^l$ is constant from one iteration to the next, whereas the correction loads vector $\Delta\mathbf{P}^i$ is continually being adjusted. As the global stiffness matrix $\mathbf{K}$ remains unchanged, the method amounts to a modified Newton–Raphson approach.

At the element level, if one or both nodes have a moment in excess of the plastic moment value $\pm M_p$, a self-equilibrating correction load vector $\mathbf{q}_e^i$ is generated. This vector involves a moment or moments equal and opposite to the amount by which yield has

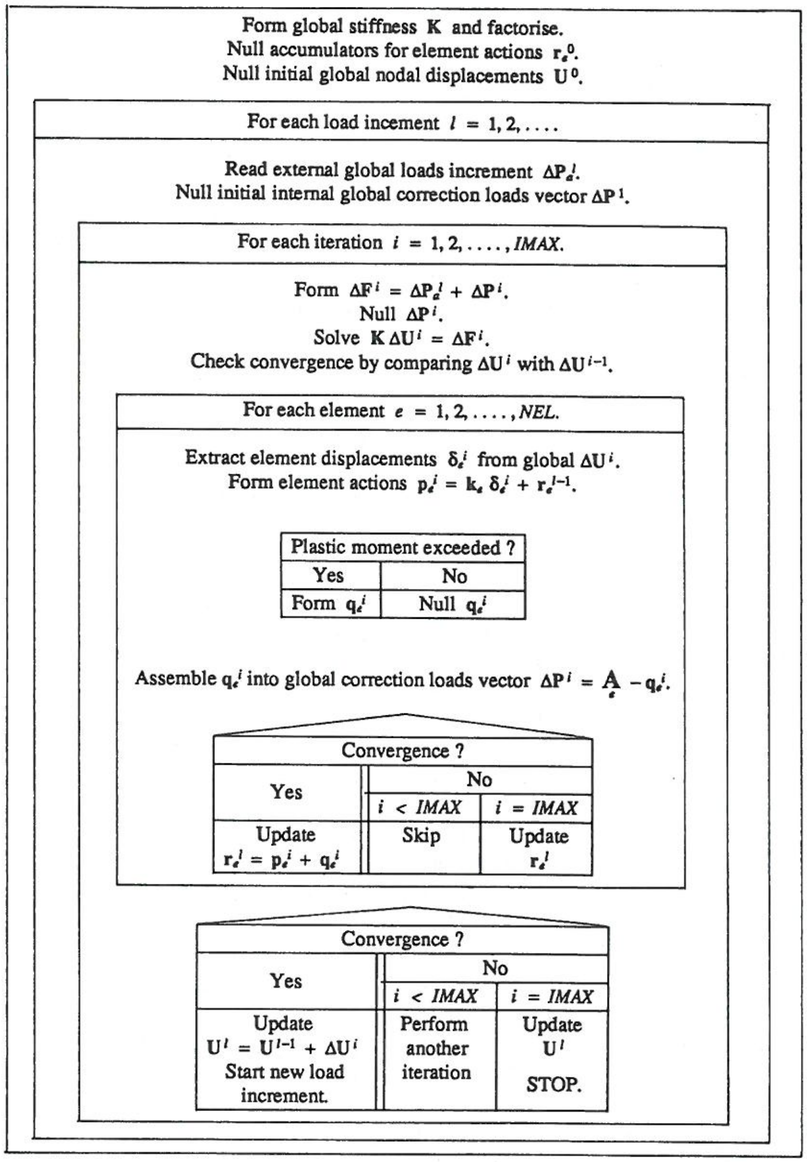

Fig. 3. Structure chart for moment redistribution algorithm.

been exceeded, and a couple to maintain equilibrium. The couple is transformed into the global coordinate directions and assembled into the global correction vector thus

$$\Delta\mathbf{P}^i = \underset{e=1}{\overset{NEL}{\mathbf{A}}} - \mathbf{q}_e^i, \tag{3}$$

where NEL = the number of elements.

For elements in which no yield is occurring, the terms of $\mathbf{q}_e^i$ are set equal to zero.

The formation of $\mathbf{q}_e^i$ for a typical element in which yield is occurring is shown in Fig. 4. If plastic hinges are forming at both ends as shown in Fig. 4(a), the correction that must be applied involves two moments and a couple [Fig. 4(b)], which must be transformed into global coordinate directions [Fig. 4(c)]. The sign convention adopted here takes

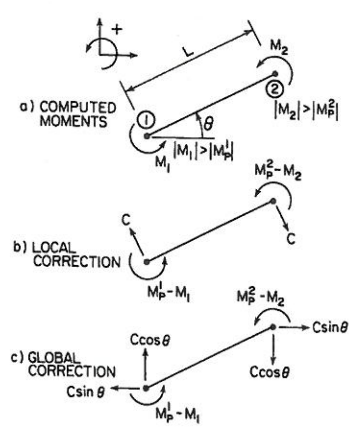

Fig. 4. Correction 'forces' for element with two plastic hinges.

anti-clockwise moments and rotations to be positive, but it should be noted that the plastic limit values M_p^1, M_p^2 always have the same sense as the moments with which they are being compared. A typical correction vector will be of the form

$$\mathbf{q}_e = \begin{bmatrix} -C \sin\theta \\ C \cos\theta \\ M_p^1 - M_1 \\ C \sin\theta \\ -C\cos\theta \\ M_p^2 - M_2 \end{bmatrix}, \tag{4}$$

where

$$C = \frac{M_p^1 + M_p^2 - M_1 - M_2}{L} \tag{5}$$

L, θ = element length and inclination

M_p^1, M_p^2 = plastic moments at ends 1 and 2.

The self-equilibrating nature of the vectors $\mathbf{q}_e^i$ implies their presence in the global loads vector will have no effect on the net loading on the system. Once all the element correction vectors have been collected by the global vector $\mathbf{\Delta P}^i$, and assuming convergence has not yet occurred, the applied loads vector is modified once more according to eqn (2) and the cycle repeated.

5. CONVERGENCE

A dimensionless convergence criterion is used which decides when sufficient accuracy has been achieved, and thus when no further iterations are necessary. The state of convergence is judged by comparing global displacement vectors $\mathbf{U}^i$ at successive iterations. It is thus necessary to provide extra storage to hold the displacements at the $(i-1)$th iteration.

After the displacement vector $\mathbf{U}^i$ has been computed, the maximum value $U_{\max}$ is found. Convergence is said to have occurred if for all the N freedoms

$$\frac{|U_j^i - U_j^{i-1}|}{U_{\max}} < t \quad j = 1, 2, \ldots, N, \tag{6}$$

where t is a user-prescribed tolerance. The value of t depends on the accuracy required, but a typical value might be 0.001. In all the examples described later in this paper, a rather small tolerance of 0.0001 has been adopted.

6. EXAMPLE 1: COLLAPSE OF A TWO-BAY PORTAL FRAME

The frame shown in Fig. 2 was subjected to proportional loading by gradually increasing the load factor λ. The analysis was performed ignoring the effects of axial deformations of the members, thus the freedom numbering scheme shown in Fig. 2(b) was used. The resulting load/displacement behaviour is shown in Fig. 5, together with the number of iterations to reach convergence at each load step. With the load factor given by $\lambda = 1.37$, convergence was achieved after 65 iterations, but the small increase to $\lambda = 1.38$ failed to converge after 200 iterations with a sudden increase in the horizontal translation. Thus the computed collapse load of the frame is in very close agreement with $\lambda = 1.375$ as given by the method of combined mechanisms. To show the insensitivity of the method to load increment size, the result for a single increment of $\lambda = 1.37$ was also obtained. At convergence, the displacement corresponding to this single load increment was indistinguishable from that obtained incrementally and took 91 iterations with the same tolerance. By observing the nodal translations and the values of the moments at the joints when $\lambda = 1.38$ and after 200 iterations, the expected failure mechanism is retrieved as shown in Fig. 6.

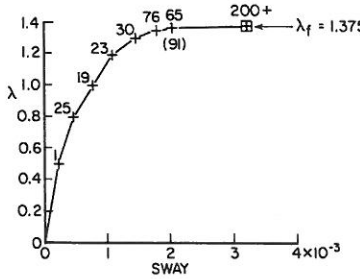

Fig. 5. Computed load factor λ vs sway displacements.

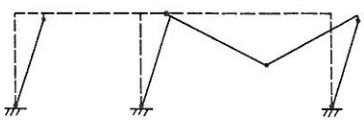

Fig. 6. Hinge locations and collapse mechanism.

7. EXAMPLE 2: CYCLIC LOADING

This example demonstrates a full cycle of loading, unloading and reloading on the same portal frame considered in the previous example. The loading is assumed to remain proportional throughout, so the only parameter that needs to be varied is the load factor λ. Figure 7 shows the response for a loading cycle in which λ is increased from 0 to 1.37, reduced from 1.37 to -1.37, and increased again to 1.37. An elastic-perfectly plastic moment/curvature response has been assumed to operate in both loading and unloading. The Bauschinger effect observed in practice has been ignored in this example, although it would be possible to refine the plastic moments under

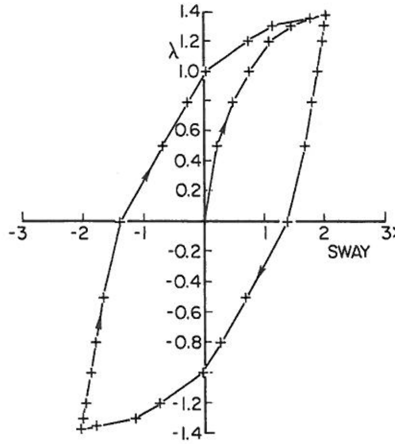

Fig. 7. Responses of two bay frame to complete cycle of loading/unloading.

cyclic loading if sufficiently detailed experimental data was available.

8. EXAMPLE 3: INCREMENTAL COLLAPSE OF A SINGLE-BAY FRAME

The previous examples used a proportional loading system governed by the single parameter λ. A more general loading system could be applied in which the loads on a structure are varied quite independently of each other. The simplest extension would be to have two loading functions governed by the parameters λ_1 and λ_2 as shown in Fig. 8. In this example the load functions are alternatively applied and removed as indicated in Table 1.

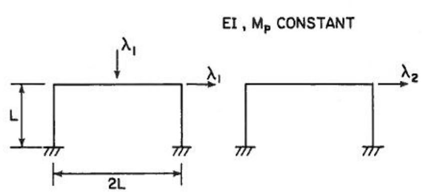

Fig. 8. Independent loading functions.

Table 1. Alternating loading

Load step	λ_1	λ_2
1	W	0
2	0	0
3	0	W
4	0	0
5	W	0
6	0	0
etc.		

It can be shown [2] in the case $\lambda_1 = \lambda_2 = W$, that for the range

$$2.737 < \frac{WL}{M_p} < 2.857 \tag{7}$$

cyclic loading causes incremental plastic deformations which tend asymptotically to a definite limit as the frame shakes down. When this occurs, all the loads are eventually borne by wholly elastic action. However, for the range

$$2.857 < \frac{WL}{M_p} < 3.000 \tag{8}$$

incremental plastic collapse occurs in which each positive load increment causes additional plastic deformation. Figure 9 shows the side-sway at each load step for the cases of WL/M_p equal to 2.8 and 2.9 respectively. For the less intense loading case, the

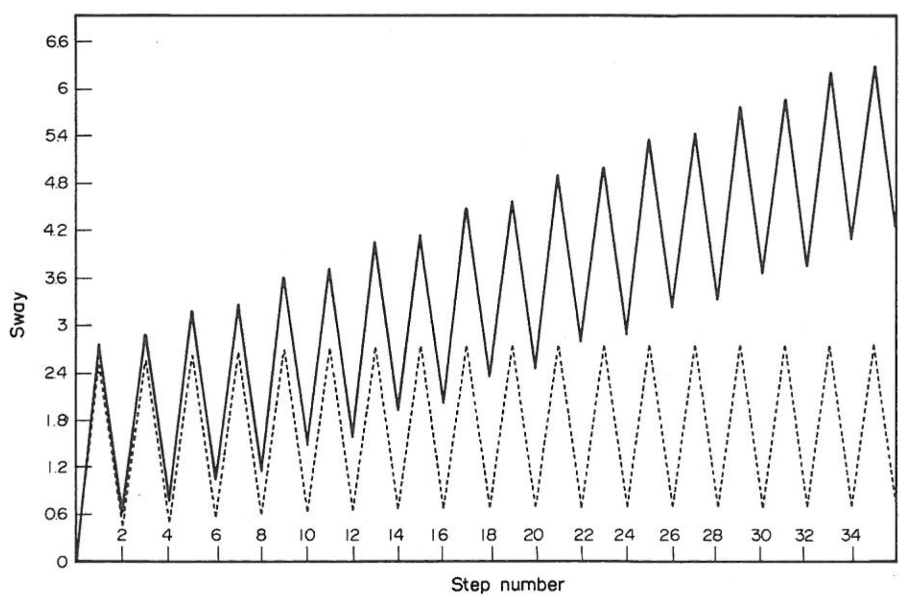

Fig. 9. Accumulation of displacements for two levels of loading.

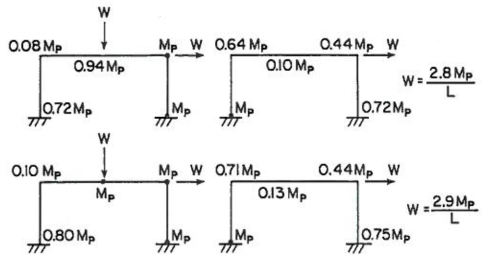

Fig. 10. Plastic hinge locations after several cycles.

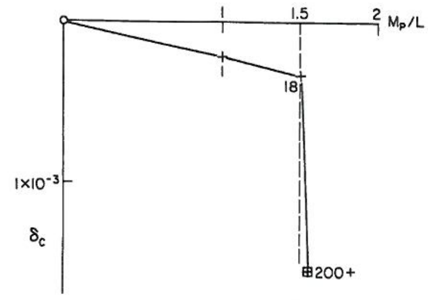

Fig. 12. Computed load vs deflection at C.

displacements (hatched lines) increase slightly with initial yielding but soon reach a steady state after a few cycles. For the more intense loading case, the displacements (solid lines) continue to increase with each cycle at a constant rate. An essential requirement of incremental plastic collapse is that a genuine mechanism of failure has been reached. The plastic hinge locations and moment values for both cases after 36 steps are shown in Fig. 10. It is seen that the case where $WL/M_p = 2.9$ has an extra plastic hinge which completes the failure mechanism.

It may be noticed that in this example, the collapse loads for each loading case on its own would be $\lambda_1 = 3M_p/L$ and $\lambda_2 = 4M_p/L$ respectively. In order to generate incremental collapse in this case, therefore, the first loading function must be within 95% of its own collapse load.

9. EXAMPLE 4: TRANSVERSE LOADING OF A GRID

The method is readily extended to three-dimensions [11] in which case there will be six degrees of freedom per node, and three plastic 'moments' (two bending, one torsion) to be checked. The element correction vector $\mathbf{q}_e^i$ will in general contain two sets of moments and couples, and a self-equilibrating torque. The transformation to the global system which was obtained explicitly in the two-dimensional case (Fig. 4), must now be performed using conventional three-dimensional matrix transformation techniques (e.g. [12]).

The example shown in Fig. 11 represents a grid of rigidly connected, equally spaced beams of equal cross-section rigidly fixed along AB. A vertical concentrated transverse load is to be applied at C. Due to the symmetries that exist in the problem, and ignoring axial deformations, only 12 independent freedoms need be considered. The resistance of the beams to twisting has also been ignored in the present case by assigning to them a relatively small torsional rigidity GJ. The computed load/deflection response of point C is shown in Fig. 12, indicating a collapse load of $1.5M_p/L$. Figure 13 gives the computed locations of the plastic hinges at collapse, and a simple energy check on the work dissipated at these hinges due to rotations of α and $\alpha/2$ leads to a collapse load W_{ult} in close agreement with the calculated value.

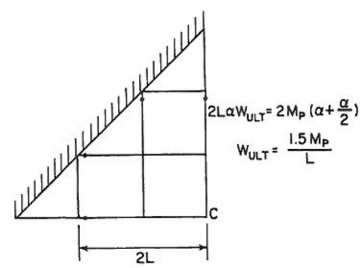

Fig. 13. Hinge locations and energy check at failure of grid.

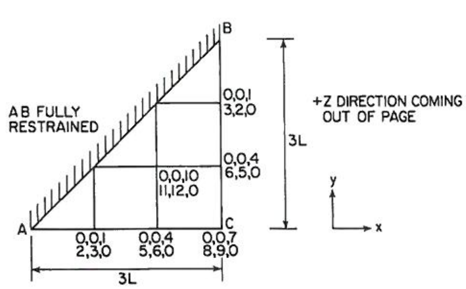

Fig. 11. Transversely loaded grid.

10. CONCLUSIONS

A numerical algorithm for analysing yield and plastic collapse of framed structures has been described. The program is quite general, and is based on a stiffness formulation in which plastic moments in excess of yield are redistributed using repeated elastic solutions. The global stiffness matrix is formed once only, and convergence is achieved by iteratively modifying the applied loads vector, thus the method amounts to a modified Newton–Raphson approach. Frames in which axial deformation of the members can be ignored, are treated as a special case in the general formulation. The algorithm is designed for incrementally applied loads, although the converged solution is shown to be insensitive to the size of the load increment. Both loading and unloading paths have been followed, and the phenomena of elastic shakedown and incremental collapse reproduced.

The method has been applied to several examples of plastic yield in both two† and three dimensions, and close agreement obtained with solutions obtained by more traditional methods. It is suggested that the method represents a versatile and accurate approach to the analysis of yield in framed structures.

† A full FORTRAN listing of the program used to obtain the two-dimensional results described in this paper can be found in the text by Smith and Griffiths [13].

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