TECHNICAL NOTE

Worst-case spatial correlation length in probabilistic slope stability analysis

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This note investigates the reliability of undrained slopes by the random finite-element method. The focus of this note is on the worst-case spatial correlation length, namely the correlation length at which the probability of slope failure reaches a maximum. It is shown that the worst-case phenomenon is most pronounced when the mean factor of safety is relatively low or the coefficient of variation is relatively high. Knowledge of the worst-case spatial correlation length is valuable, as it can be employed for conservative design in the absence of substantial soil field data.

KEYWORDS: finite-element modelling; shear strength

INTRODUCTION

Natural soil exhibits spatial variability in both the vertical and horizontal directions. Since the mid-1990s (e.g. Griffiths & Fenton, 1993), an advanced numerical tool called the random finite-element method (RFEM), which can account for soil spatial variability, has been applied to assess the reliability of a range of geotechnical systems. In several cases, it was observed that, at intermediate spatial correlation lengths – typically those close to some key reference dimension of the structure – a minimum reliability or maximum probability of failure was observed. Evidence of the 'worst-case' phenomenon was observed in the 1993 paper cited above, and on several occasions since then in relation to seepage, bearing capacity, earth pressures and settlement. Many of these are also summarised in the textbook by Fenton & Griffiths (2008). A similar phenomenon was also reported by Baecher & Ingra (1981), and more recently, for example, by Breysse *et al.* (2005) and Ching *et al.* (2017). In the framework of reliability-based design, the worst-case spatial correlation length is important (e.g. Fenton & Griffiths, 2003), because in the absence of high-quality and plentiful field data, it can be used in preliminary studies to ensure a conservative design. This note will extend the work of Griffiths *et al.* (2016), who first discussed some worst-case observations in slope stability, to achieve a more systematic understanding of the conditions under which a worst-case spatial correlation length occurs in slope stability analysis of undrained soils.

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RFEM MODEL

Figure 1 shows a typical FE mesh used in RFEM analysis of undrained slope stability. The RFEM merges elastic–plastic finite-element modelling (Smith & Griffiths, 2004) with random field theory (e.g. Vanmarcke, 1984). The random fields are generated by the local averaging subdivision (LAS) method (Fenton & Vanmarcke, 1990), which fully accounts for spatial variability and accounts for element size by way of local averaging.

This methodology performs Monte-Carlo analysis, where each repetition involves generation of a random field of undrained strength over the mesh, followed by the application of gravity loading. If the algorithm is not able to converge within 500 iterations, the slope is deemed to have failed. Non-convergence indicates no stress redistribution can be found, which simultaneously satisfies the Tresca failure criterion and global equilibrium. Preliminary studies indicated that 2000 Monte-Carlo simulations were enough to give statistically reproducible results (e.g. Griffiths *et al.*, 2009) and the probability of failure p_f is computed as the proportion of those 2000 RFEM analyses which failed.

The spatial correlation length θ is a dimensional property governing the distance over which properties are essentially similar; that is, small correlation lengths result in rapid spatial variability, whereas large correlation lengths result in slow spatial variability. In the current study, an exponential decaying correlation function is assumed as follows

$$\rho = \exp\left(\frac{-2\tau}{\theta}\right) \quad (1)$$

where ρ is the correlation coefficient between two points separated by an absolute distance τ in the random field. It should be noted that, as a result of sedimentary deposition processes, the properties of soil deposits are often more variable in the vertical direction than in the horizontal direction. However, for simplicity in this study, a dimensionless and isotropic spatial correlation length $\Theta = \theta/H$ is used, where H denotes the slope height. Studies of the effect of anisotropy on the worst-case spatial correlation length may be a topic for future investigations.

The undrained shear strength c_u is modelled as a random variable characterised by a lognormal distribution

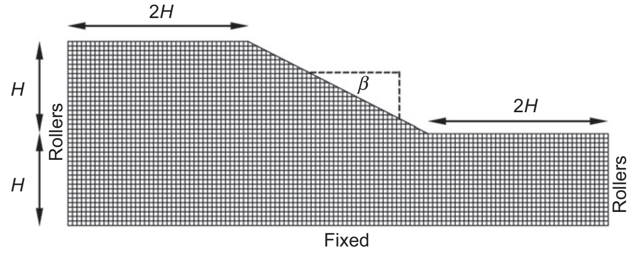

Fig. 1. Typical mesh used in RFEM analysis of undrained slope stability

throughout this study. The lognormal distribution benefits from its non-negative character and its simple transformational relationship with the classical normal distribution. The variability of c_u can be expressed in the form of coefficient of the variation v_{c_u}, given by

$$v_{c_u} = \frac{\sigma_{c_u}}{\mu_{c_u}} \tag{2}$$

where σ_{c_u} and μ_{c_u} are, respectively, the standard deviation and mean of c_u. Typical values of the coefficient of variation for c_u are thought to lie in the range 0·1–0·5 (e.g. Lee *et al.*, 1983; Cherubini, 2000). In this study, the deterministic parameters include the saturated unit weight $\gamma = 20$ kN/m^3, the undrained friction angle $\phi_u = 0$, the slope height $H = 10$ m and the slope angle β, which is varied in the parametric studies.

WORST-CASE SPATIAL CORRELATION LENGTH

Probabilistic tools for slope stability analyses can be categorised into two fundamental branches. One is the 'random field approach', where random fields with defined spatial correlation are combined with numerical discretisation schemes such as FE or finite-difference methods. The other is the simpler 'single random variable' (SRV) approach where spatial variability is ignored, which essentially amounts to an assumption of infinite spatial correlation (i.e. $\Theta \rightarrow \infty$). For undrained slopes, the SRV approach has an analytical solution as presented by Griffiths & Fenton (2004) in the form of charts, with an underlying analytical formula given by

$$p_f = \Phi\left[\frac{\ln\left(1+v_{c_u}^2\right) - 2\ln\left(\overline{\text{FS}}\right)}{2\sqrt{\ln\left(1+v_{c_u}^2\right)}}\right] \tag{3}$$

where $\Phi[.]$ is the standard normal cumulative distribution function and $\overline{\text{FS}}$ is the deterministic mean factor of safety of a uniform slope calculated using the mean undrained shear strength μ_{c_u}.

Griffiths & Fenton (2004) warned about the dangers of using the analytical approach, because it can lead to an underestimation of p_f (i.e. unconservative) when the mean factor of safety is relatively low or the coefficient of variation is relatively high. In other words, the worst-case phenomenon may be observed under these conditions. To further investigate the worst-case phenomenon, a slope angle $\beta = 26{\cdot}6°$ (2:1 slope) as shown in Fig. 1 is first considered, which was also the test slope used by Griffiths & Fenton (2000, 2004). For this simple slope, the deterministic factor of safety with uniform properties, can be computed by any traditional slope stability method, including the stability chart of Taylor (1937), which gives a factor of safety of FS = 1 when $c_u/(\gamma H) = 0{\cdot}17$. Since the factor of safety is proportional to the undrained strength, the mean factor of safety $\overline{\text{FS}}$ can be calculated as

$$\overline{\text{FS}} = \frac{\mu_{c_u}/(\gamma H)}{0{\cdot}17} \tag{4}$$

The following dimensionless spatial correlation length values were also selected for parametric analyses in this study

$$\Theta = 0{\cdot}01,\ 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}5,\ 1{\cdot}0,\ 2{\cdot}0,\ 5{\cdot}0,\ 10{\cdot}0,\ 100{\cdot}0$$

Influence of $\overline{\text{FS}}$ *and* v_{c_u}

Figure 2 shows p_f plotted against Θ for the $\beta = 26{\cdot}6°$ slope with different mean factors of safety when $v_{c_u} = 0{\cdot}5$ (an upper end of the suggested range from Lee *et al.* (1983)). When the mean factor of safety is $\overline{\text{FS}} = 1{\cdot}2$, a pronounced worst case occurs at a spatial correlation length of about $\Theta_{wc} = 0{\cdot}2$. Owing to the sharp increase in p_f for $0{\cdot}01 \leq \Theta \leq 0{\cdot}1$, four more points with $\Theta = 0{\cdot}02$, 0·04, 0·06 and 0·08 were calculated, as shown in Fig. 3. It can be seen from Fig. 3 that the new values of p_f are all smaller than 0·8 and have not affected the location of the maximum p_f. As $\overline{\text{FS}}$ is increased, the Θ_{wc} also increases, but the worst-case effect becomes less pronounced, and is barely noticeable for

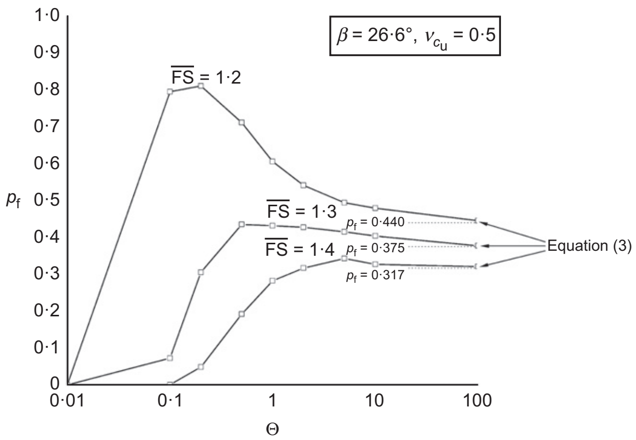

Fig. 2. Probability of failure plotted against dimensionless spatial correlation length with different mean factors of safety

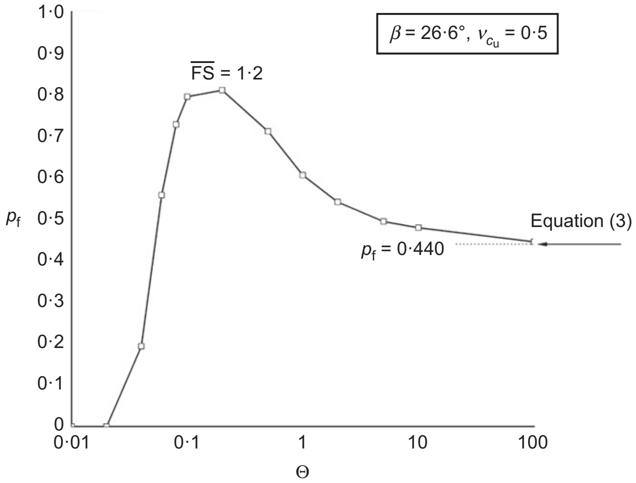

Fig. 3. Probability of failure plotted against dimensionless spatial correlation length with $\overline{\text{FS}} = 1{\cdot}2$

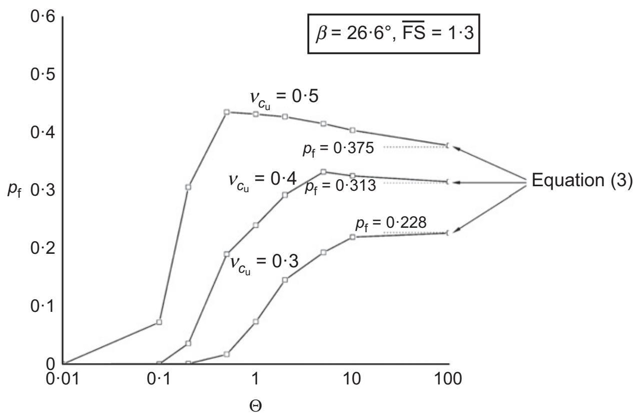

Fig. 4. Probability of failure plotted against dimensionless spatial correlation length with different coefficients of variation

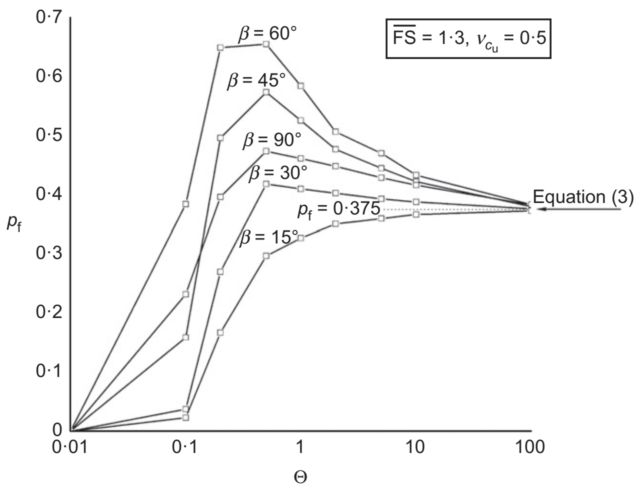

Fig. 5. Probability of failure plotted against dimensionless spatial correlation length with different slope angles

$\overline{FS} = 1{\cdot}4$. Fig. 2 demonstrates that the SRV approach may be unconservative when $\overline{FS}$ is relatively low ($\overline{FS} < 1{\cdot}4$) for undrained slopes with $v_{c_u} = 0{\cdot}5$.

Consideration of limiting values of Θ is instructive for understanding the range of probabilities and for validation. When $\Theta \to 0$, owing to local averaging (e.g. Griffiths & Fenton, 2004), the slope becomes essentially 'deterministic' with a homogeneous strength fixed at the median of c_u. As the undrained shear strength is assumed to be lognormally distributed, the median of c_u is equal to $\mu_{c_u}/(1 + v_{c_u}^2)^{1/2}$. For the undrained slopes shown in Fig. 2, the median corresponds to $FS > 1$, so $p_f \to 0$. As might be expected, when $\Theta \to \infty$, the RFEM solutions converge on the analytical solutions from equation (3) as horizontal dotted lines.

The reason for the worst-case phenomenon is that at extreme values of Θ the results of p_f are fixed, as explained above; however, intermediate spatial correlation lengths facilitate the formation of additional failure mechanisms, leading to more failure simulations in the Monte-Carlo process, and hence a higher p_f.

Figure 4 shows the p_f plotted against Θ for the $\beta = 26{\cdot}6°$ slope with different coefficients of variation when $\overline{FS} = 1{\cdot}3$. This value of $\overline{FS}$ is chosen as a likely minimum acceptable value in practice for less important or temporary slopes (e.g. USACE, 2003). The $v_{c_u} = 0{\cdot}5$ result is the same as the middle plot in Fig. 2. As v_{c_u} is decreased, Θ_{wc} increases, but the worst-case effect becomes less pronounced, and is barely noticeable for $v_{c_u} = 0{\cdot}3$. The figure demonstrates that the SRV approach may be unconservative when v_{c_u} is relatively high ($v_{c_u} > 0{\cdot}3$) for undrained slopes with $\overline{FS} = 1{\cdot}3$.

Influence of slope angle

To investigate the influence of slope angle on the worst-case phenomenon, the slope angle was varied in the range $15° \leq \beta \leq 90°$. Fig. 5 shows the p_f plotted against Θ for several slope angles with the mean factor of safety and the coefficient of variation set to $\overline{FS} = 1{\cdot}3$ and $v_{c_u} = 0{\cdot}5$, respectively. It can be seen from Fig. 5 that a worst-case spatial correlation length was observed for all cases (except for $\beta = 15°$); however, the $\beta = 60°$ result gives the most pronounced worst case, which has implications for reliability-based design. For large spatial correlation lengths ($\Theta \to \infty$), probabilities of failure in all cases approach the value given by equation (3) of $p_f = 0{\cdot}375$. This apparent interdependence between the slope angle and Θ_{wc} is an area of continued investigation.

CONCLUDING REMARKS

This technical note has investigated the worst-case spatial correlation length for undrained slopes by RFEM. The worst-case spatial correlation length is the value that leads to the highest probability of failure. It was shown that the worst-case phenomenon is most pronounced when the mean factor of safety is relatively low (e.g. $\overline{FS} < 1{\cdot}4$) and the coefficient of variation of undrained strength is relatively high (e.g. $v_{c_u} > 0{\cdot}3$). The worst-case spatial correlation length increases with increasing mean factor of safety and decreasing coefficient of variation. The existence of a worst-case correlation length implies that simplified probabilistic analysis of slopes using, for example, the SRV approach, may lead to unconservative design. An unexpected finding in the current work is that there is also a slope angle effect relating to the worst-case phenomenon. For the slope considered in this note, the worst-case phenomenon was more pronounced for a slope of 60° than either 45° or 90°. This interesting result is a subject of ongoing investigation.

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NOTATION

c_u	undrained shear strength
H	slope height
p_f	probability of failure
v_{c_u}	coefficient of variation for c_u
β	slope angle
γ	unit weight
Θ	dimensionless spatial correlation length
Θ_{wc}	worst-case spatial correlation length
θ	spatial correlation length
μ_{c_u}	mean of c_u
ρ	correlation coefficient
σ_{c_u}	standard deviation of c_u
τ	absolute distance between two points
$\Phi(.)$	standard normal cumulative distribution function
ϕ_u	total stress friction angle (= 0)

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