COMMUNICATIONS IN APPLIED NUMERICAL METHODS, Vol. 5, 53–60 (1989)

ADVANTAGES OF CONSISTENT OVER LUMPED METHODS FOR ANALYSIS OF BEAMS ON ELASTIC FOUNDATIONS

D. V. GRIFFITHS

Engineering Department, Simon Laboratories, University of Manchester, Manchester, M13 9PL, UK

SUMMARY

A simple method for analysing laterally loaded piles is to consider the problem as a beam on an elastic foundation. In this representation, the finite element method is used to solve the governing differential equation, in preference to the more popular finite difference approaches. The pile is modelled using beam elements, and two different methods are compared for introducing the soil stiffness. The 'consistent' approach, where the governing differential equation is discretized using a Galerkin formulation, is shown to be considerably more accurate than a 'lumped' approach. The improvement is particularly noticeable when the finite element discretization becomes coarser. Some examples are presented to illustrate the differences, and comparisons are made with closed-form solution.

INTRODUCTION

Provided axial forces are much smaller than those values that would cause buckling, the behaviour of laterally loaded piles is analogous to the classical problem of a beam on an elastic foundation. Particularly relevant to geotechnical engineering is the case where the soil stiffness increases with depth. The simplest assumption to make is that there is a linear relationship between depth and stiffness, although other power laws could be used.

Closed-form solutions[1,2] containing hyperbolic functions give displacements and moments within the pile provided the foundation stiffness variation is represented by simple functions. Even for a linear variation in foundation stiffness, these solutions become quite cumbersome, and numerical solution techniques are often preferred. For nonlinear foundation behaviour, however, where the springs reduce in stiffness as a function of displacement or reach an ultimate value, numerical methods represent the only means of tackling the problem.

The most popular numerical approach, and the one that has received the most attention, is the finite difference method (see e.g. Reference 3). Here the governing differential equation is written in finite difference form and applied at a number of discrete points along the length of the pile. This leads to a set of linear simultaneous equations in the unknown transverse deflection at each grid point.

An alternative approach to solving the problem numerically is to use the finite element method (see e.g. reference 4). In this case beam elements are used to discretize the pile, allowing a translation and a rotation at each node. The shape functions of these simple elements are first-order Hermite polynomials, and allow a cubic variation of transverse deflection between nodes. Incorporation of the foundation stiffness can be achieved either by lumping at the nodes using

0748–8025/89/010053–08$05.00

Received 24 June 1988

'p–y' springs or by a consistent approach in which the governing differential equation is fully discretized using a Galerkin formulation. These two approaches are now briefly reviewed.

Lumped approach

In this method, the global stiffness matrix is built up in the usual way from the beam element stiffness matrices. Once this is done, the foundation stiffness is added to the appropriate diagonal terms corresponding to translations only. This takes account of the effect of the foundation on transverse defections, but ignores the effects of rotations. A typical distribution of spring stiffnesses for a pile in a soil of uniform properties is shown in Figure 1(a), and for a soil of linearly increasing modulus in Figure 1(b). These values were obtained by considering the 'fixed-end' reactions that would act on the beams for the given loading distribution.

Consistent approach

This method enables the foundation and beam stiffnesses to be merged at the element level. Thus for a uniform foundation stiffness, the governing equation is

$$EI\,\frac{d^4y}{dx^4} + ky = w \tag{1}$$

where the symbols are as defined in Figure 2. As shown, a typical element has four degrees of freedom with a translation (y) and rotation ($\theta = dy/dx$) at each node. Discretization of equation (1) leads to an element stiffness matrix in which all terms contain contributions from the beam and the foundation:

$$\mathbf{K}\,\boldsymbol{\delta} = \mathbf{P} \tag{2}$$

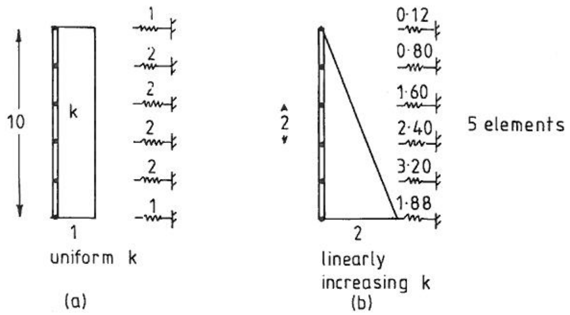

Figure 1. Equivalent spring stiffnesses

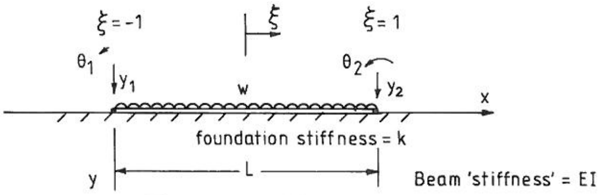

Figure 2. Definition of problem

where

$$\boldsymbol{\delta} = [\,y_1 \quad \theta_1 \quad y_2 \quad \theta_2\,]^{\mathrm{T}}$$

and

$$\mathbf{P} = [\,P_1 \quad M_1 \quad P_2 \quad M_2\,]^{\mathrm{T}}$$

P_1 and M_1 being the shear force and moment acting at node 1, and so on.

The stiffness matrix **K** can be written as

$$\mathbf{K} = \mathbf{KM} + \mathbf{MM} \tag{3}$$

where

$$\mathbf{KM} = EI \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ & 4/L & -6/L^2 & 2/L \\ & & 12/L^3 & -6/L^2 \\ & \text{symmetrical} & & 4/L \end{bmatrix} \tag{4}$$

and

$$\mathbf{MM} = \frac{kL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ & \text{symmetrical} & & 4L^2 \end{bmatrix} \tag{5}$$

It may be noted that **KM** represents the beam stiffness matrix, and **MM** closely resembles the beam mass matrix due to the term ky in equation (1). These matrices are fully populated, so there is full coupling between translations and rotations that occur between beam and foundation.

If the foundation stiffness k, or the pile stiffness EI, varies across each element, one approach would be to assume a step-function approximation in which properties are constant within each element, but differ from one element to the next. Alternatively, the element stiffness matrices **KM** and **MM** may be formed numerically,[4] in which case the actual variation of the properties can be more accurately represented with relatively few elements.

The most efficient way of performing this integration is to use Gaussian quadrature, by replacing the beam co-ordinate x in the range $[0, L]$ by the local co-ordinate ξ in the range $[-1, 1]$, as shown in Figure 2, using the transformation

$$x = \tfrac{1}{2} L(\xi + 1) \tag{6}$$

Assuming that beam displacements are given by

$$y = N_1 y_1 + N_2\theta_1 + N_3 y_2 + N_4\theta_2 \tag{7}$$

where

$$\begin{aligned} N_1 &= \tfrac{1}{4}(\xi^3 - 3\xi + 2), \qquad & N_2 &= \tfrac{1}{8} L(\xi^3 - \xi^2 - \xi + 1) \\ N_3 &= \tfrac{1}{4}(-\xi^3 + 3\xi + 2), \qquad & N_4 &= \tfrac{1}{8} L(\xi^3 + \xi^2 - \xi - 1) \end{aligned} \tag{8}$$

then typical terms of the element stiffness matrices are given by

$$\left.\begin{aligned} \mathrm{KM}_{KL} &= \frac{8}{L^3}\int_{-1}^{1} EI(\xi)\,\frac{\mathrm{d}^2 N_K}{\mathrm{d}\xi^2}\,\frac{\mathrm{d}^2 N_L}{\mathrm{d}\xi^2}\,\mathrm{d}\xi \\ \mathrm{MM}_{KL} &= \frac{L}{2}\int_{-1}^{1} k(\xi) N_K N_L \,\mathrm{d}\xi \end{aligned}\right\} K, L = 1, 2, 3, 4 \tag{9}$$

If the function $k(\xi)$ is linear, then typical terms in **MM** require the integration of seventh-order polynomials. Such polynomials are integrated exactly using four Gaussian integration points per element.

COMPARISON OF LUMPED AND CONSISTENT METHODS

A pile of length 10 m was used for the comparisons, and three different element discretizations were considered, as shown in Figure 3. Corresponding to each of these, three different relative stiffnesses were assigned to the pile/soil system according to Hetenyi's definition:[1]

$$\begin{array}{ll} \text{Short beams} & \beta L \leqslant 0{\cdot}6 \\ \text{Intermediate beams} & 0{\cdot}6 < \beta L \leqslant 5 \\ \text{Long beams} & 5 < \beta L \end{array} \tag{10}$$

where

$$\beta = \sqrt[4]{\left(\frac{k}{4EI}\right)} \tag{11}$$

Two distributions of foundation stiffness k were considered: in the first case $k = 1$ was maintained along the full length of the pile; in the second a linear distribution of k was taken, varying from zero at the ground surface to $k = 2$ at a depth of 10 m, as shown in Figure 4. In the second case the value of k at the mid-depth of 5 m ($k = 1$) was taken for defining the pile 'length' according to equation (11). In all cases, EI was taken to be constant, as shown in Table I.

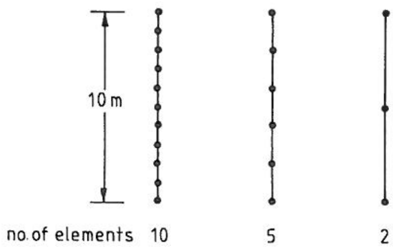

Figure 3. Mesh gradations

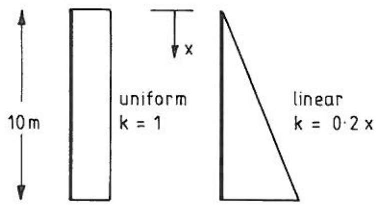

Figure 4. Foundation stiffness

Table I. Values of EI and effective length (assuming $k = 1$)

βL	EI
0·6	19290
2·5	64
5·0	4

The two loading cases, shown in Figure 5, namely a lateral force and a moment of one unit, were applied respectively at the tip.

Figures 6–8 show results of lateral deflection δ_h against pile depth x. In all cases using the consistent approach, it was found that the computed displacements at all pile depths were virtually indistinguishable from the closed-form solution.[1] This was also true in the case of the lumped solution with 10 elements. The results that have been plotted are the lumped solutions

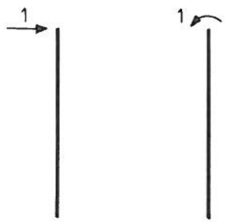

Figure 5. Loading cases

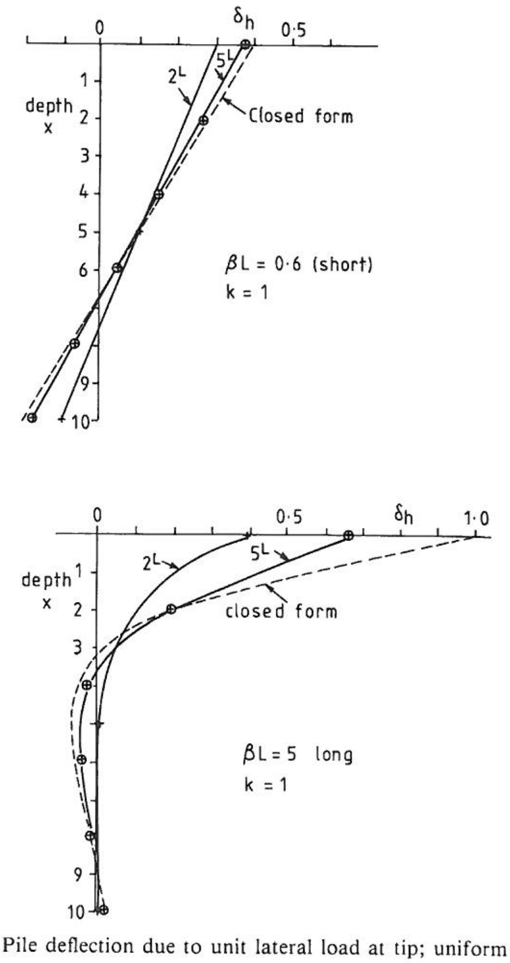

Figure 6. Pile deflection due to unit lateral load at tip; uniform stiffness

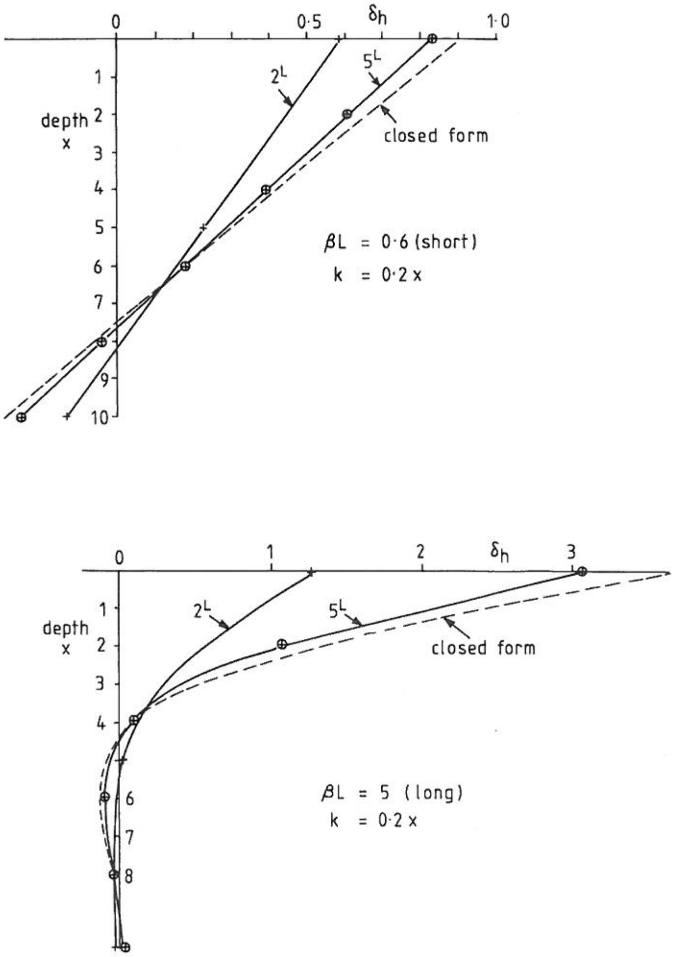

Figure 7. Pile deflection due to unit lateral load at tip; stiffness increasing with depth

using two and five elements (2^L and 5^L respectively) and the closed-form solution for 'short' and 'long' piles.

Figures 6 shows results for the pile subjected to unit lateral load in a foundation of uniform stiffness. The 'short' pile exhibited a simple rigid rotation and translation which were quite well reproduced by the numerical solutions. The two-element case, however, underestimated the tip deflection by 25 per cent. The agreement was even worse when a flexible pile was considered under the same loading conditions. In this case even the five-element solution underestimated the tip deflection by nearly 25 per cent. Figure 7 shows equivalent plots for unit lateral loading with a foundation stiffness increasing linearly with depth. A similar picture emerges of a rapid deterioration of agreement as the number of elements is reduced in the lumped case.

Finally, in Figure 8, the lateral deflection of the pile due to a unit moment applied at the tip is shown. In this case the two-element solution underestimated the tip deflection by over 75 per cent. A summary of the error in the computed tip deflection for the more interesting case of linearly increasing stiffness with depth is given in Table II.

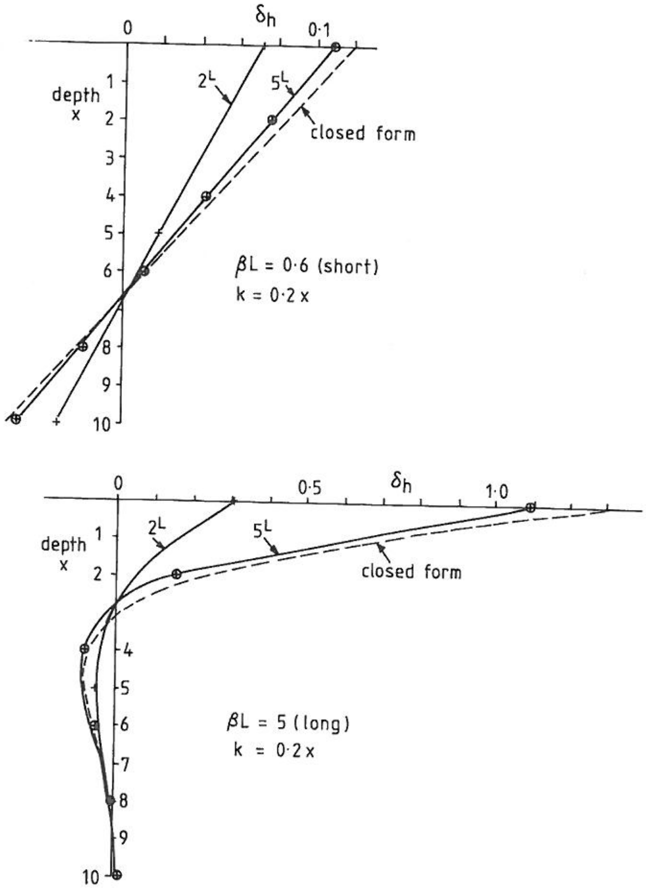

Figure 8. Pile deflection due to unit moment at tip

Table II. Percentage error in computed tip deflection ($k = 0{\cdot}2x$)

βL	5^L	5^C	2^L	2^C	5^L	5^C	2^L	2^C
	Unit load				Unit moment			
0·6	9	0	35	0	10	0	40	0
2·5	7	0	35	0	7	0	38	0
5·0	17	0	66	5	19	0	77	2

CONCLUDING REMARKS

A comparison between lumped and consistent methods of modelling soil stiffness in a laterally loaded pile analysis has been performed. The pile/soil system was treated as a beam on an elastic foundation with either uniform or linearly varying modulus. For both force and moment loading on the pile tip, the consistent approach gave consistently better results than the lumped approach when compared with closed-form solutions. The constrast became particularly significant as the number of finite elements was reduced. It appeared that, even though the closed-form solutions contain hyperbolic functions, the cubic polynomial interpolations used in the finite element discretization were quite adequate for the cases considered.

REFERENCES

1. M. Hetenyi, *Beams on Elastic Foundation*, University of Michigan Press, Ann Arbor, Mi, 1946.
2. S. P. Timoshenko, *Strength of Materials*, Part II, 3, Van Nostrand, New York, 1956.
3. L. C. Reese and C. S. Desai, 'Laterally loaded piles', in *Numerical Methods in Geotechnical Engineering*, (C. S. Desai and J. T. Christian, Eds.), McGraw-Hill New York; 1977, pp. 297–325.
4. I. M. Smith and D. V. Griffiths, *Programming the Finite Element Method*, 2nd Ed., Wiley, Chichester, 1988, pp. 23–24.