Accurate pore pressure calculation in undrained analysis

D. V. Griffiths and C. O. Li
Department of Engineering, University of Manchester, Manchester M13 9PL, UK
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ABSTRACT

A simple procedure for accurate calculation of pore pressures in undrained elasto-plastic materials is described. An 8-node element is used, with 'reduced' integration during stress redistribution, and 'full' integration to form the global stiffness matrix for the modified Newton–Raphson procedure. An analysis of passive earth pressure in an undrained soil is used to demonstrate the algorithm and computed results are compared with a closed-form solution.

INTRODUCTION

For two-phase saturated soil systems, a method of including a nearly incompressible pore fluid is used to introduce a large bulk modulus into the effective elastic stress–strain matrix as suggested by Naylor[1,2], i.e. in 2-D (assuming symmetrical matrices):

$$\mathbf{D}=\frac{E'}{(1+\nu')(1-2\nu')}\times\begin{bmatrix} 1-\nu'+K_e & \nu'+K_e & 0 & \nu'+K_e \\ & 1-\nu'+K_e & 0 & \nu'+K_e \\ & & \frac{1-2\nu'}{2} & 0 \\ \text{symmetrical} & & & 1-\nu'+K_e \end{bmatrix} \quad (1)$$

where $\mathbf{D}$ is the 'total' stress strain matrix, E', ν' are the effective elastic properties, and K_e is the apparent fluid bulk modulus.

Addition of the bulk modulus K_e into the effective $\mathbf{D}'$ matrix is equivalent to giving the composite soil/fluid mixture the following 'total' elastic properties:

$$E=E'\left[\frac{3\beta_{ps}+1+\nu'}{2\beta_{ps}(1+\nu')+1+\nu'}\right] \quad (2)$$

$$\nu=\frac{\beta_{ps}+\nu'}{2\beta_{ps}+1} \quad (3)$$

where

$$\beta_{ps}=(1+\nu')(1-2\nu')\frac{K_e}{E'} \quad (4)$$

From (3), for large values of β_{ps}, $\nu\rightarrow\frac{1}{2}$ and the material approaches elastic incompressibility.

In an earlier study[3], it was suggested that the value:

$$\beta_{ps}\simeq 20 \quad (5)$$

was sufficient to give virtually incompressible behaviour without any numerical difficulties. Using this value of β_{ps} together with a typical effective Poisson's ratio ν' of 0.3, a total Poisson's ratio ν of 0.4951 is given from (3).

In the analysis of elasto-plastic soils, further volumetric constraints are imposed by the plastic flow rule. For example, if a dilation angle of zero is prescribed, the material will yield with zero volume change.

Accurate computation of pore pressures in virtually incompressible materials such as undrained soils, is a problem that has received much attention in the finite element literature. Various computational devices have been proposed for overcoming numerical difficulties associated with incompressibility (see e.g. Hughes[4]).

In a typical modified Newton–Raphson iterative scheme, two main integrations are required (see e.g. Smith and Griffiths[5]),

(i) *Formation of stiffness matrices*

$$\iint \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}x\,\mathrm{d}y \quad (6)$$

where $\mathbf{B}$ is the element strain/displacement relationship.

(ii) *Stress redistribution*

$$\iint \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}'^{p}\,\mathrm{d}x\,\mathrm{d}y \quad (7)$$

where $\boldsymbol{\sigma}'^{p}$ are the 'plastic' stresses.

Previous calculations[3] in which both integrations (6) and (7) were performed using 'reduced' integration, led to stable stress calculation, but gave pore pressures which tended to oscillate about the correct solution. Naylor[1] suggested an averaging process to overcome this difficulty.

Pore pressure Δu is calculated from the relation:

$$\Delta u=K_e\Delta\varepsilon^{v} \quad (8)$$

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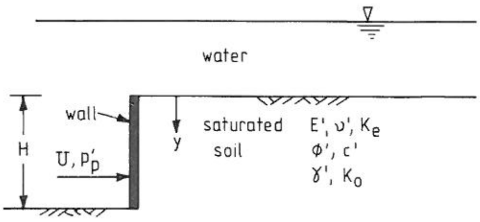

Figure 1 Undrained passive earth pressure problem

where $\Delta\varepsilon^v$ is the volumetric strain of the composite material computed at the reduced integration Gauss points. In general for a yielding soil, volume changes will result from both elastic and plastic strains. For a non-dilative soil with zero plastic volume change, the volume changes can only originate from the elastic stage of the stress–strain behaviour. As K_e (or β_{ps}) gets bigger, $\Delta\varepsilon^v$ gets smaller, but the product tends to a constant within the limitations of the computer accuracy.

In this paper, a form of selective integration with 8-node elements is proposed whereby the stiffness integration of (6) is performed using 'full' (3 × 3) integration, whereas the plastic stress redistribution integration of (7) remains with 'reduced' (2 × 2) integration. It will be shown that a considerable improvement in the accuracy of the computed pore pressure is obtained.

The assessment of accuracy in pore pressure calculations is helped considerably if a closed form solution is available for comparison. In the next section a solution is presented for a passive earth pressure problem involving undrained frictional soil. This is followed by finite element solutions to the same problem, where the performance of the full/reduced integration scheme is demonstrated.

PASSIVE EARTH PRESSURE

Consider the case of a smooth vertical wall of height H translated horizontally into a bed of undrained soil as shown in *Figure 1*. The soil is assumed to be saturated with properties defined by the following parameters:

- E', ν' effective elastic properties,
- K_e apparent fluid bulk modulus,
- ϕ', c' effective shear strength parameters,
- γ' submerged unit weight,
- K_0 'at rest' earth pressure coefficient.

We wish to find the limiting effective passive force P'_p and water force U on the wall at failure.

It is assumed that the soil is cohesionless, and that the stress paths followed by soil elements adjacent to the wall take place at constant mean effective stress. As shown in *Figure 2*, if $K_0 < 1$, the stress path initially falls to the p'-axis as the element passes through an isotropic stress state and subsequently reaches the failure line, defined:

$$q = -\frac{(K_p - 1)}{(K_p + 1)} p' \qquad (9)$$

where

$$q = (\sigma'_v - \sigma'_h)/2 \qquad (10)$$

$$p' = (\sigma'_v + \sigma'_h)/2 \qquad (11)$$

and

$$K_p = \tan^2(45 + \phi'/2) \qquad (12)$$

Figure 2 has been made dimensionless by dividing all stresses by the initial mean effective stress p'_0.

Consider a typical element of soil, adjacent to the wall which has fully consolidated under its own self weight. The vertical and horizontal stresses acting on the element are given by:

$$\sigma'_{v0} = \gamma' y \quad \text{and} \quad \sigma'_{h0} = K_0 \gamma' y \qquad (13)$$

where y is the depth of the element below the top of the wall. As the wall is translated into the soil σ_h increases, but σ_v remains constant, hence at failure,

$$(K_0\gamma' y + \Delta\sigma_{hf} - \Delta u_f) = K_p(\gamma' y - \Delta u_f) \qquad (14)$$

where $\Delta\sigma_{hf}$ = change in total horizontal stress at failure, Δu_f = excess pore pressure at failure.

From elastic theory in plane strain, $\Delta\sigma_{hf}$ and Δu_f are related through the expression[3]:

$$\Delta u_f = \frac{\beta_{ps}}{2\beta_{ps} + 1} \Delta\sigma_{hf} \qquad (15)$$

where β_{ps} is the dimensionless parameter defined in (4).

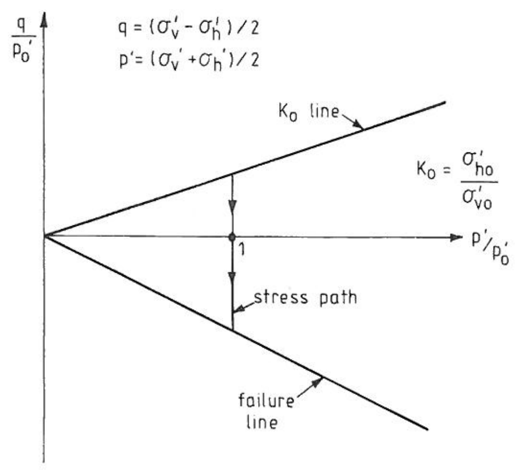

Figure 2 Effective stress path for typical soil element

Combining (14) and (15) and rearranging gives the pore pressure at failure:

$$\Delta u_f = \frac{\gamma' y(K_p - K_0)\beta_{ps}}{(K_p+1)\beta_{ps}+1} \tag{16}$$

The effective stress at failure is given from:

$$\sigma'_{hf} = \sigma_{h0} + \Delta\sigma_{hf} - \Delta u_f \tag{17}$$

hence

$$\sigma'_{hf} = \frac{\gamma' y K_p[(K_0+1)\beta_{ps}+1]}{(K_p+1)\beta_{ps}+1} \tag{18}$$

Both (16) and (18) give a triangular distribution of effective stress and pore pressure behind the wall, varying from zero at the surface to a maximum at the base.

It follows that the net forces on the wall are given by:

$$U = \tfrac{1}{2}\gamma' H^2\left[\frac{(K_p-K_0)\beta_{ps}}{(K_p+1)\beta_{ps}+1}\right] \tag{19}$$

and

$$P'_p = \tfrac{1}{2}\gamma' H^2 K_p\left[\frac{(K_0+1)\beta_{ps}+1}{(K_p+1)\beta_{ps}+1}\right] \tag{20}$$

Rankine's drained solution can be retrieved from (19) and (20) as a special case, by letting $\beta_{ps}=0$.

These equations form a basis for comparison with finite element calculation as will be shown in the next section.

COMPUTED RESULTS

In an earlier publication[3], it was shown that good agreement with (20) could be obtained using 8-node elements with reduced integration throughout.

It was found that even with quite large values of β_{ps}, the finite element computations gave stable effective stresses, but unstable pore pressures when sampled at the reduced integration Gauss-points.

Unstable pore pressures imply unstable volumetric strains, from (8). In a conventional displacement finite element approach, therefore, the problem must originate in the global stiffness matrix. 'Reduced' integration is commonly used in finite element analysis of non-linear materials in order to overcome the troublesome volumetric constraint imposed by incompressible plastic flow. The 'relaxation' brought about by the 'reduced' integration, introduces small errors into the volumetric strain energy approximation. These errors, although small, are magnified by the 'large' fluid bulk modulus when the product given by (8) is formed. Selective reduced integration techniques such as those described by Hughes[4] would not help matters, because the volumetric components of the stiffness matrix remain under-integrated.

A compromise is therefore suggested, whereby the global stiffness matrix is formed using exact integration (i.e. 3 × 3), but 'reduced' integration is retained for the plastic stress redistribution. As a full modified Newton–Raphson iterative algorithm is used, this scheme represents very little additional computation because the global stiffness matrix is formed once only.

The mesh shown in *Figure 3* was used to demonstrate the effects of the two integration schemes. Eight node quadrilateral elements were used throughout, and the program mentioned previously[5] was modified slightly to give the option of using full integration in the stiffness matrix formation. Only two lines of the program needed to be changed.

The following soil properties were assigned to the mesh:

$\phi' = 30°$
$c' = 0$
$\gamma' = 10\ \text{kN/m}^3$
$K_0 = 0.5$ and 1.0
$E' = 10^5\ \text{kN/m}^2$ $\quad \beta_{ps} = 52$
$\nu' = 0.3$ $\quad E = 1.15\times 10^5\ \text{kN/m}^2$
$K_e = 100\ E'$ $\quad \nu = 0.498$

A condition of no plastic volume change was enforced by letting the dilation angle equal zero. The 2 m high wall was simulated by prescribing equal horizontal displacement increments to the 9 nodes at the wall/soil interface.

After each displacement increment and convergence of the algorithm, the horizontal effective stresses and excess pore pressures were computed at each of the 8 Gauss points closest to the wall (*Figure 3*). The computed pore pressures and

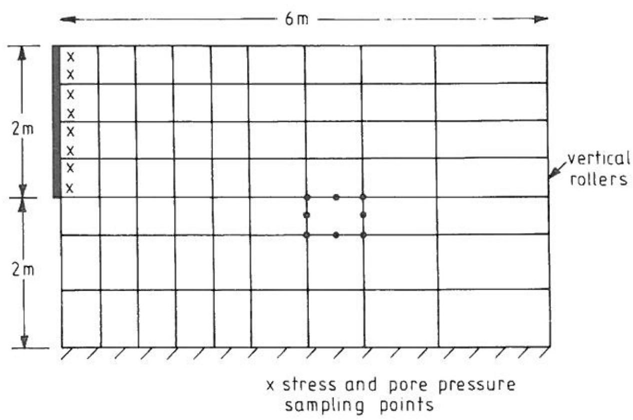

Figure 3 Mesh used for finite element analysis

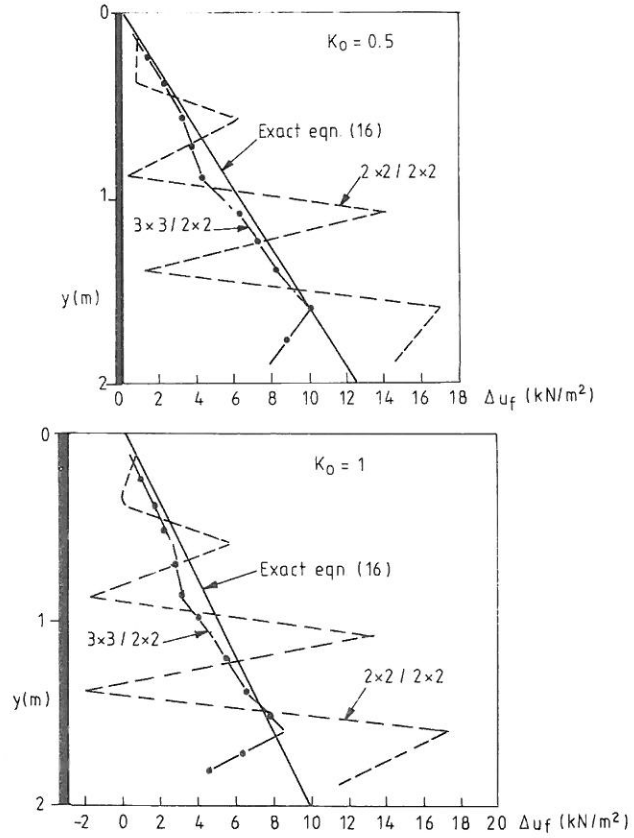

Figure 4 Comparison of computed and analytical pore pressures

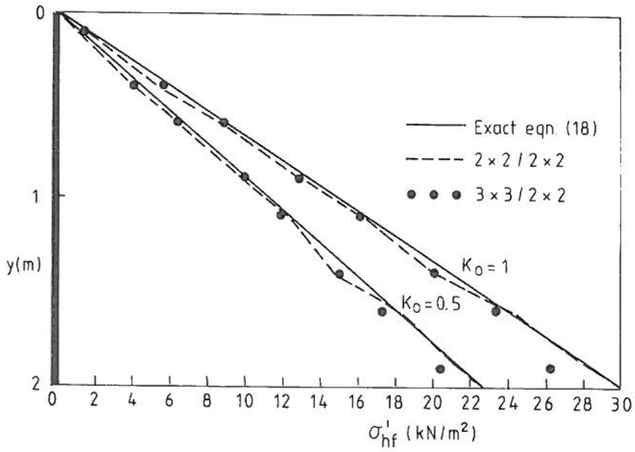

Figure 5 Comparison of computed and analytical effective stress

effective stresses at failure for two different K_0 values are summarized in *Figures 4* and *5* respectively together with the closed form solutions from (16) and (18). A striking improvement in the computed pore pressure response was obtained by using 'full' integration in the global stiffness matrix formulation. The tendency for the computed pore pressure to fall towards the base of the wall was due to the singularity introduced by the displacement discontinuity at the bottom of the fourth row of elements.

As expected, the effective stresses in *Figure 5* were not influenced by the order of integration, with both cases agreeing closely with the analytical solution.

Similar behaviour regarding pore pressure smoothing has also been observed by Li and Griffiths[6] for rapid draw-down problems.

CONCLUSION

In the context of undrained analysis, whereby a large fluid bulk modulus is added into the soil effective stress/strain matrix, a method of accurate pore pressure calculation has been demonstrated. When using 8-noded elements, a considerable improvement in the computed pore pressures was observed when the global stiffness matrix was formed using 'full' (3 × 3) integration as opposed to 'reduced' (2 × 2) integration. This modification made little difference to the runtimes, because the exactly integrated global matrix was formed once only for the modified Newton–Raphson procedure. The modification made little difference to the effective stresses however, which could be computed quite satisfactorily using 'reduced' integration throughout. Although it is recommended that the global stiffness matrix should be formed using 'exact' integration, it is still suggested that 'reduced' integration be used during plastic stress redistribution.

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Equation (1) in the paper is incorrect and should be as follows:

$$\mathbf{D} = \frac{E'}{(1+\nu')(1-2\nu')} \begin{bmatrix} (1-\nu') & \nu' & 0 & \nu' \\ \nu' & (1-\nu') & 0 & \nu' \\ 0 & 0 & (1-2\nu')/2 & 0 \\ \nu' & \nu' & 0 & (1-\nu') \end{bmatrix} + \begin{bmatrix} K_e & K_e & 0 & K_e \\ K_e & K_e & 0 & K_e \\ 0 & 0 & 0 & 0 \\ K_e & K_e & 0 & K_e \end{bmatrix}$$