INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS, VOL. 14, 587–594 (1990)

TECHNICAL NOTE ON PRACTICAL APPLICATIONS

STRESS STRAIN CURVE GENERATION FROM SIMPLE TRIAXIAL PARAMETERS

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SUMMARY

A simple hyperbolic function is often used to illustrate typical stress–strain behaviour of geomaterials during monotonic loading. This approach has the disadvantage that the failure condition is approached asymptotically, whereas in reality failure must occur at a finite value of strain. A modified hyperbolic function with one adjustable parameter is proposed in this paper. The function is shown to be capable of spanning all likely soil stress–strain data up to and including peak strength.

INTRODUCTION

Recent constitutive soil models, especially those involving multisurface plasticity[1–3] require stress–strain curves, typically obtained from triaxial tests, in order to generate the parameters associated with each yield surface. If such models are to be implemented in finite element codes for the solution of complex boundary value problems,[4] a great many stress–strain curves are required corresponding to each element or even Gauss point within the mesh. When dealing with frictional soils, where confinement has a big influence on stiffness and strength, it is impractical for detailed laboratory data to be available for all locations. In such cases, it becomes necessary to generate stress–strain curves automatically using some mathematical function relating stress to strain. The minimum amount of information required to generate such a curve comprises (i) the initial gradient, and (ii) the stress and strain levels at failure. Clearly the shear stress to cause failure in a 'Mohr–Coulomb' material and the initial gradient will both depend on the confining pressure (see Appendix).

The best known and most widely used function is a hyperbola,[5] which has been applied to some quite complex soil deformation problems, most notably by Duncan and Chang.[6] The simple hyperbola has the disadvantage that it is not able to model failure very realistically. This is because, if the function is forced to pass through a particular point corresponding to the nominal failure stress and strain, the asymptote may be considerably greater than the true failure stress level. This has prompted the authors to consider alternative functions for describing typical shear stress/shear-strain curves for soil from first loading through to failure. The effect of shear stresses on volumetric strains is not considered in this paper.

0363–9061/90/080587–08$05.00

Received 16 January 1989
Revised 27 October 1989

CURVE GENERATION: GENERAL CONSIDERATIONS

A curve is to be generated relating q and $\bar{\varepsilon}$, where these are the principal stress and strain differences defined as

$$q=\sigma_1-\sigma_3 \tag{1}$$

$$\bar{\varepsilon}=\varepsilon_1-\varepsilon_3 \tag{2}$$

For monotonic loading with no softening the function $q(\bar{\varepsilon})$ must satisfy the following conditions:

(i) $q(0)=q_0$

(ii) $\dfrac{\mathrm{d}q}{\mathrm{d}\bar{\varepsilon}}(0)=2G_0$

(iii) $q(\bar{\varepsilon}_{\mathrm{ult}})=q_{\mathrm{ult}}$

(iv) $\dfrac{\mathrm{d}q}{\mathrm{d}\bar{\varepsilon}}(\bar{\varepsilon}_{\mathrm{ult}})=0$

(v) $\dfrac{\mathrm{d}q}{\mathrm{d}\bar{\varepsilon}}>0$ and $\dfrac{\mathrm{d}^2q}{\mathrm{d}\bar{\varepsilon}^2}<0$ for $0\leq\bar{\varepsilon}<\bar{\varepsilon}_{\mathrm{ult}}$

where q_0 is the initial value of q, allowing for the possibility of anisotropic initial stress conditions, G_0 is the initial shear modulus, $\bar{\varepsilon}_{\mathrm{ult}}$ is the strain value when failure is reached and q_{ult} is the stress value at failure (see Appendix). Conditions (i) to (iv) give two points through which the curve must pass together with the corresponding gradients. Condition (v) merely states that the curve must be smooth with no points of inflexion.

REVIEW OF THE SIMPLE HYPERBOLA

The basic curve has the equation

$$q=\frac{\bar{\varepsilon}}{\dfrac{1}{2G_0}+\dfrac{\bar{\varepsilon}}{(q_{\mathrm{ult}}-q_0)}}+q_0 \tag{3}$$

which immediately satisfies conditions (i), (ii) and (v), but as the function is asymptotic to q_{ult}, conditions (iii) and (iv) are not satisfied.

If the hyperbola is forced to pass through a 'failure' point with co-ordinates $(\bar{\varepsilon}_{\mathrm{ult}}, q_{\mathrm{ult}})$, thus satisfying condition (iii), the modified curve will tend to a new asymptote q_a given by

$$q_a=(q_{\mathrm{ult}}-q_0)\frac{\bar{\varepsilon}_{\mathrm{ult}}}{\bar{\varepsilon}_{\mathrm{ult}}-\dfrac{q_{\mathrm{ult}}-q_0}{2G_0}}+q_0 \tag{4}$$

where $q_a>q_{\mathrm{ult}}$.

Such a curve will always be capable of overestimating the failure stress q_{ult} as shown in Figure 1. A horizontal cut-off line could be introduced at q_{ult}, but this would have the undesirable effect of introducing a discontinuity.

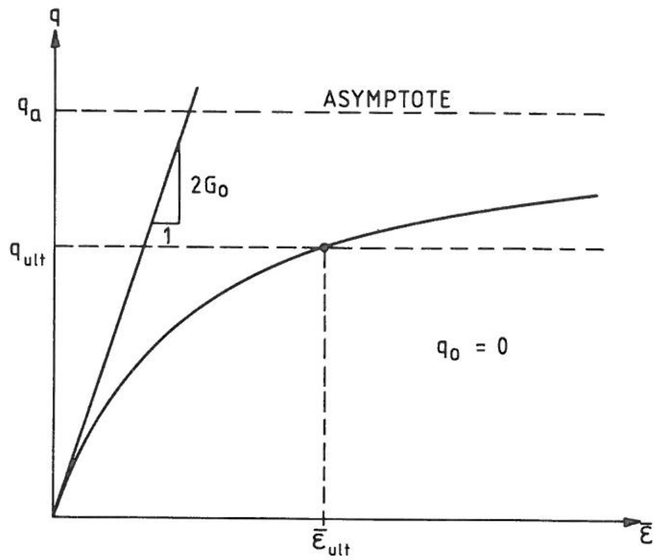

Figure 1. Hyperbolic stress–strain curve, equation (3)

MODIFIED HYPERBOLIC FUNCTION

A modified hyperbolic function is proposed with the equation

$$q = q_1 \frac{\bar{\varepsilon}}{a+\bar{\varepsilon}} - q_1 \frac{a}{(a+\bar{\varepsilon}_{ult})^2} \frac{1}{\bar{\varepsilon}_{ult}^{\alpha}} \frac{\bar{\varepsilon}^{1+\alpha}}{1+\alpha} + q_0 \tag{5}$$

where

$$a = \frac{q_1}{2G_0} \tag{6}$$

and q_1 is a reference stress which depends on the curve-fitting parameter α.

Inspection of equation (5) and its derivatives indicates that conditions (i), (ii), (iv) and (v) are satisfied provided that α and q_1 are both positive. In order to satisfy condition (iii), we make the substitution $\bar{\varepsilon} = \bar{\varepsilon}_{ult}$ and solve for q_1 to give

$$q_1 = \frac{-b\left[(q_{ult}-q_0) - \dfrac{b}{2}\right] - b\left\{b\left[\dfrac{b}{4} - \dfrac{(q_{ult}-q_0)}{1+\alpha}\right]\right\}^{1/2}}{(q_{ult}-q_0) - \dfrac{b\alpha}{1+\alpha}} \tag{7}$$

where

$$b = 2G_0 \bar{\varepsilon}_{ult} \tag{8}$$

In order for q_1 to remain positive it is readily shown that α must be chosen according to the following criteria:

$$\text{If } 0 < \frac{(q_{ult}-q_0)}{2G_0 \bar{\varepsilon}_{ult}} \leq \frac{1}{4}, \qquad \text{then } \alpha > 0 \tag{9}$$

$$\text{If } \frac{1}{4} < \frac{(q_{ult}-q_0)}{2G_0 \bar{\varepsilon}_{ult}} < \frac{1}{2}, \qquad \text{then } \alpha \geq \frac{4(q_{ult}-q_0)}{2G_0 \bar{\varepsilon}_{ult}} - 1 \tag{10}$$

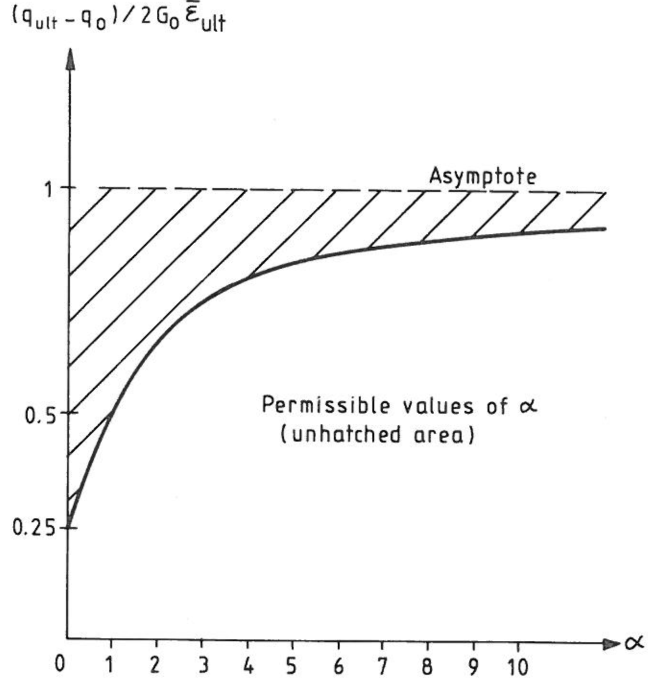

Figure 2. Range of permissible values of α for modified hyperbola

$$\text{If } \frac{1}{2} \le \frac{(q_{\text{ult}} - q_0)}{2G_0 \bar{\varepsilon}_{\text{ult}}} < 1, \qquad \text{then } \alpha > \frac{1}{1 - \dfrac{(q_{\text{ult}} - q_0)}{2G_0 \bar{\varepsilon}_{\text{ult}}}} - 1 \tag{11}$$

Conditions (9), (10) and (11) are summarized in Figure 2. In order to use this method, the following sequence could be followed:

(a) compute $(q_{\text{ult}} - q_0)/2G_0 \bar{\varepsilon}_{\text{ult}}$
(b) choose α according to condition (9), (10) or (11)
(c) compute q_1 from equation (7)
(d) compute the relationship between q and $\bar{\varepsilon}$ from equation (5).

The value of α chosen at step (b) gives a small amount of control over the shape of the curve between the initial tangent and the failure point where the curve flattens out. In general, small values of α concentrate the changes in gradient towards $\bar{\varepsilon}_{\text{ult}}$ whereas larger values of α concentrate the changes in gradient near the origin, as indicated in Figure 3. Further examination of α however, shows that this parameter has no significant effect on the characteristics of the stress–strain curve. It is therefore recommended that at stage (b), the parameter α is chosen to take the value

$$\alpha = 1{\cdot}1 \left\{ \frac{1}{1 - \dfrac{(q_{\text{ult}} - q_0)}{2G_0 \bar{\varepsilon}_{\text{ult}}}} - 1 \right\} \tag{12}$$

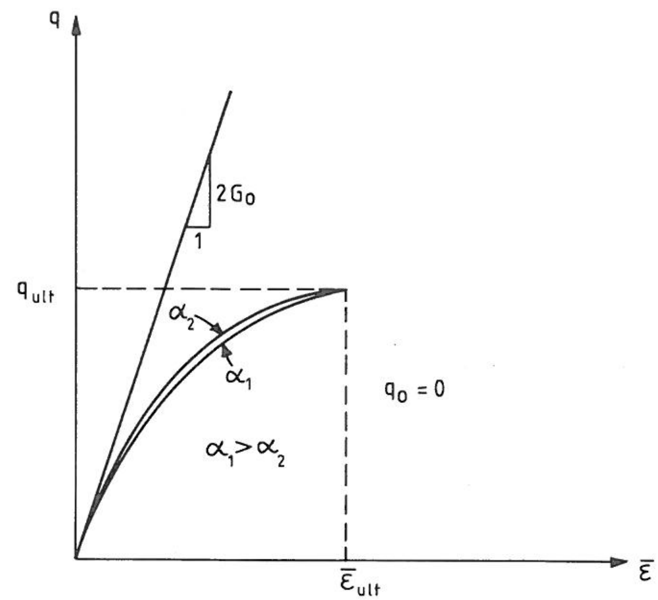

Figure 3. Modified hyperbolic functions, equation (5)

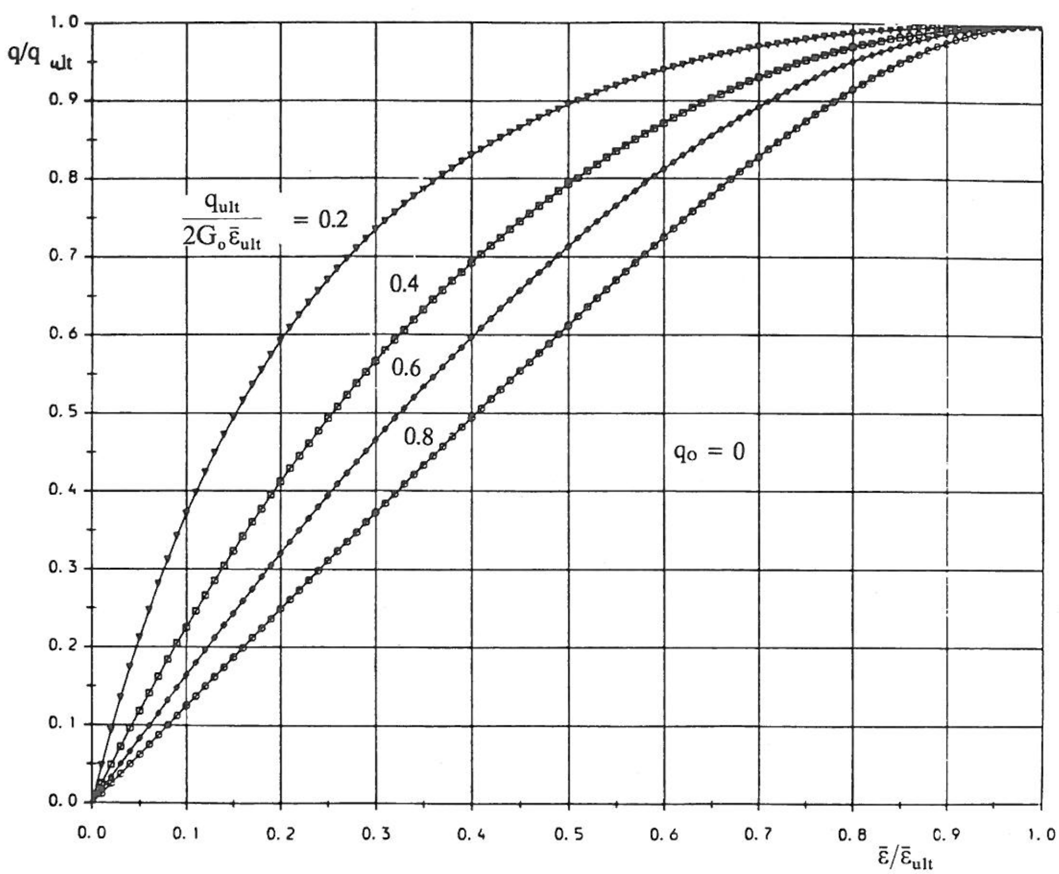

Figure 4. Modified hyperbola with a range of initial gradients

which satisfies all of conditions (9), (10) and (11). Using this value of α, and assuming $q_0 = 0$, a number of curves are plotted in Figure 4. The number shown against each of the curves is the initial stiffness ratio $q_{ult}/(2G_0\bar{\varepsilon}_{ult})$, which can be varied in the range 0 to 1. Clearly, the curves of Figure 4 span all likely soil test data up to and including peak strength.

CONCLUDING REMARKS

The popular hyperbola for representing shear stress–strain behaviour of soil has been reviewed, and shown to be restrictive owing to the lack of control over the asymptote. It appeared that under some circumstances the failure stress could be considerably overestimated. A modified hyperbolic function with one fixed parameter has been proposed, and is shown to be quite versatile, having all the necessary features of a soil stress–strain curve up to failure.

ACKNOWLEDGEMENTS

The work was supported in part by the U.S. National Science Foundation under grant ECE 85-12311, sub-contract NCEER 86-3034-A5 and by the Kajima Corporation, Tokyo.

APPENDIX

Definitions

p'	effective mean stress $(\sigma'_1 + \sigma'_2 + \sigma'_3)/3$
q	principal stress difference $(\sigma'_1 - \sigma'_3)$.
p'_0, q_0	initial values of p' and q
ϕ'_c	effective friction angle in triaxial compression
ϕ'_e	effective friction angle in triaxial extension
c'	effective cohesion
s	effective stress path (dp'/dq)
q^c_{ult}	q at failure in triaxial compression
q^e_{ult}	q at failure in triaxial extension
K_p	passive earth pressure coefficient $(\tan^2(45° + \phi'/2))$

Assuming that $\sigma'_1 > \sigma'_3$, and that the material has a shear strength given by the Mohr–Coulomb criterion,

$$\sigma'_1 = \sigma'_3 K_p + 2c'\sqrt{K_p} \tag{13}$$

we wish to find q_{ult} under triaxial conditions given

(i) initial stress p'_0, q_0
(ii) shear strength ϕ', c'
(iii) stress path s

Triaxial compression (q positive)

$$q = (\sigma'_1 - \sigma'_3) \tag{14}$$

$$p' = (\sigma'_1 + 2\sigma'_3)/3 \tag{15}$$

Hence

$$\sigma'_1 = (3p' + 2q)/3 \tag{16}$$

$$\sigma'_3 = (3p' - q)/3 \tag{17}$$

Substitution into (13) gives the equation of the failure line, i.e.

$$q = 3\,\frac{(K_p^c - 1)}{K_p^c + 2}\,p' + \frac{6c'\sqrt{K_p^c}}{K_p^c + 2} \tag{18}$$

Accounting for initial stress conditions and stress path, we obtain

$$q_{ult}^c = \frac{3(K_p^c - 1)(p_0' - q_0 s) + 6c'\sqrt{K_p^c}}{K_p^c + 2 - 3s(K_p^c - 1)} \tag{19}$$

Triaxial extension (*q negative*)

$$q = -(\sigma_1' - \sigma_3') \tag{20}$$

$$p' = (2\sigma_1' + \sigma_3')/3 \tag{21}$$

Hence

$$\sigma_1' = (3p' - q)/3 \tag{22}$$

$$\sigma_3' = (3p' + 2q)/3 \tag{23}$$

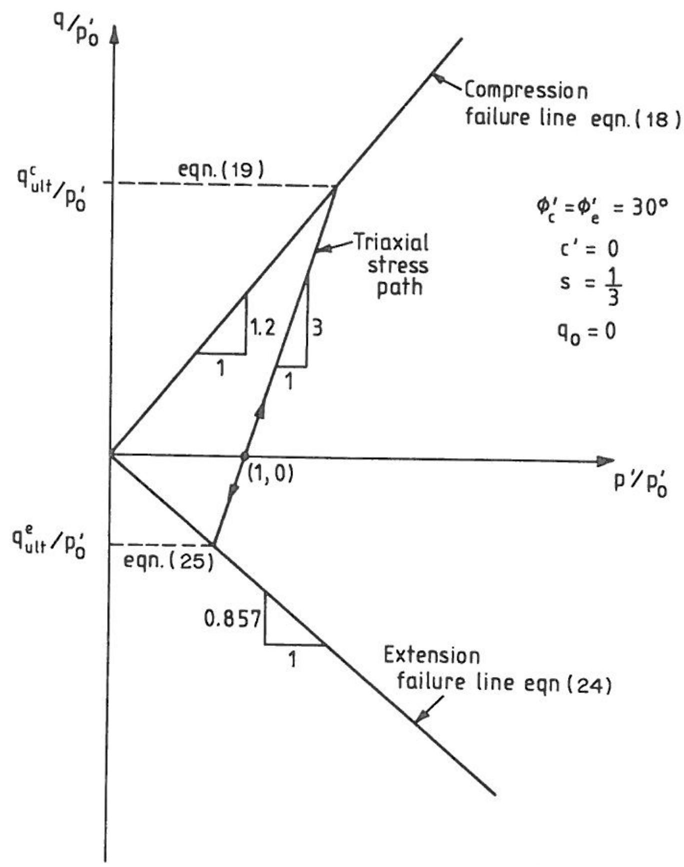

Figure 5. Failure lines and stress path in a triaxial plane

Substitution into (13) gives the equation of the failure line, i.e.

$$q=-3\frac{(K_p^e-1)}{2K_p^e+1}p'-\frac{6c'\sqrt{K_p^e}}{2K_p^e+1} \tag{24}$$

Accounting for initial stress conditions and stress path, we obtain

$$q_{\mathrm{ult}}^e=\frac{-3(K_p^e-1)(p'_0-q_0s)-6c'\sqrt{K_p^e}}{2K_p^e+1+3s(K_p^e-1)} \tag{25}$$

It should be noted from equations (18) and (24) that the compression and extension failure lines are not symmetrical about the p'-axis, and do not in general meet on the axis unless $c'=0$ or $\phi'_c=\phi'_e$. Figure 5 shows a typical plot in the triaxial plane for the case $\phi'_c=\phi'_e=30°$.

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