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Computation of bearing capacity factors using finite elements

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The Paper shows how finite elements, in conjunction with elasto-plastic theory, can give excellent collapse load predictions for footings resting on c–ϕ soils. The problem is approached by considering each of the three components of bearing capacity appearing in Terzaghi's classical equation. The results show that the contributions to bearing capacity of the N_c and N_q terms can be computed with confidence and economy. The N_γ term, which is theoretically less sound than the other two, is also computed to be in close agreement with published approximate solutions but takes more computing effort. This is shown to be particularly so when the soil friction angle is increased. The superposition assumption in Terzaghi's equation is assessed using finite elements. The conservative nature of the equation is confirmed, with the inherent error shown to occur solely because of non-linearities in the N_γ term. It is thought that the techniques used in obtaining the results given may be applied with confidence to a wide range of soil collapse problems for which no known solutions exist.

Le présent article décrit la manière dont des éléments finis, associés à la théorie élastoplastique, permettent d'obtenir d'excellentes prévisions des charges de rupture dans le cas de semelles reposant sur des sols c–ϕ. Pour résoudre le problème, on considère chacune des trois termes de la capacité portante telles qu'ils figurent dans l'équation classique de Terzaghi. Les résultats montrent que l'on peut calculer de manière sûre et simple la contribution des termes N_c et N_q à la capacité portante. Le terme N_γ, qui est théoriquement moins précis que les deux autres termes, semble, d'après les calculs, concorder étroitement avec les solutions approximatives publiées mais qui exigent des calculs plus complexes, notamment lorsque l'angle de frottement interne du sol s'accroit. L'hypothèse de superposition dans l'équation de Terzaghi est contrôlée à l'aide d'élements finis. Le caractère conservatif de l'équation se trouve confirmé et l'on constate que l'erreur inhérente ne se produit qu'en raison de la non-linéarité du terme N_γ. Les auteurs estiment que les techniques utilisées pour l'obtention des résultats indiqués peuvent être appliquées en toute confiance à un grand éventail de problèmes relatifs à la rupture du sol pour lesquels aucune solution connue n'existe.

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NOTATION

B	footing width
E	Young's modulus
K_0	at rest earth pressure coefficient
N_c, N_q, N_γ	bearing capacity factors
c	cohesion
q_s	surface surcharge
q_{ult}	ultimate bearing capacity
β	angle of ground adjacent to a footing
γ	unit weight
δ	soil/footing friction angle
ν	Poisson's ratio
ϕ	soil internal friction angle

INTRODUCTION

The bearing capacity of shallow foundations is generally calculated using Terzaghi's (1943) equation in which the total bearing resistance is approximated by superposition of three basic components

$$q_{ult} = cN_c + q_s N_q + \frac{\gamma B}{2} N_\gamma \qquad (1)$$

The three components are expressed in terms of the dimensionless bearing capacity factors N_c, N_q and N_γ which are available in tables or charts and assumed to be functions of the soil friction angle only. Terzaghi's values are frequently used in practice, but many other values exist, and in the case of N_γ these can vary greatly. Chen (1975) reported bearing capacity factors as obtained by different investigators, most of whom used limit analyses or slip line methods.

Finite elements have also been used (Davidson & Chen, 1976; Zienkiewicz, Humpheson & Lewis, 1975; Zienkiewicz, Norris, Winnicki, Naylor & Lewis, 1978), in conjunction with plasticity theory, to predict bearing capacity. These have tended to be single calculations on a soil with a given set of properties and have, generally speaking, agreed well with classical solutions using equation (1). However, the solutions did not indicate the relative ability of finite elements with respect to each of the three components in equation (1). Few finite element solutions are available for cohesionless soils with self-weight, and a recent attempt by Christian, Hagmann & Marr (1977) was unsuccessful.

The present work shows how finite elements can be used to compute each of the bearing capacity factors in turn as a function of the soil friction angle. This has been done by considering three fundamental types of problem

(*a*) weightless soil with cohesion and friction
(*b*) weightless, cohesionless soil with friction receiving a uniform surface surcharge
(*c*) cohesionless soil with friction and self-weight.

In each of these cases only one of the terms of equation (1) is applicable, enabling a direct evaluation of the particular factor once the ultimate bearing capacity is computed. The intention of such parametric studies is to give some general conclusions regarding the amenability of certain types of bearing capacity problem to satisfactory and economical solution by finite elements.

The influence of footing roughness has also been considered, as it is known that N_γ in particular is highly sensitive to this property.

The superposition principle assumed to hold by equation (1) has been examined using finite elements by comparing the computed results of bearing capacity obtained in a single calculation with the result obtained by simple addition of the three individual components.

SOIL CONSTITUTIVE RELATION

The solutions presented here are for plane strain conditions, with the soil behaving as an elastic–perfectly plastic material. The Mohr–Coulomb failure surface was used in conjunction with a non-associated (zero plastic volume change) flow rule. As the aim of the calculations was to predict the bearing pressure necessary to cause a general shear failure rather than the settlements before failure, the stress–strain behaviour assumed to act within the failure surface was considered relatively unimportant.

The elastic properties assigned to the soil within the failure surface were chosen arbitrarily to be $E = 2 \times 10^5$ kN/m^2 and $\nu = 0{\cdot}35$. The strength parameters ϕ and c that defined the failure surface were varied so that the performance of a range of material strengths could be observed.

NUMERICAL SOLUTION TECHNIQUE

Plasticity was introduced using the viscoplastic technique (Zienkiewicz & Cormeau, 1972, 1974) which has been shown (Humpheson, 1976; Griffiths, 1980, 1981) to be an efficient and versatile way of solving plasticity problems in geomechanics. The method, which falls into the initial strain family of solution techniques, iterates using equivalent elastic solutions until any stresses that originally violated yield have returned to the failure surface within quite strict tolerances. The convergence criterion was implemented by observing the change in the body forces from one iteration to the next. In the viscoplastic algorithm, these self-equilibrating body forces are incremented at each iteration by an amount directly related to the magnitude by which the stresses still violate yield. As the stresses return to the failure surface the increment of body forces diminishes. In the present work, convergence was said to have occurred when the change in body forces, non-dimensionalized with respect to the largest absolute value, nowhere exceeded 0·1%.

Equilibrium and continuity were also satisfied in the usual way using a displacement finite element formulation.

Eight-node quadrilateral, isoparametric elements were used throughout, with reduced (two-point) Gaussian quadrature in both the stiffness and the relaxation phases of the calculation.

BOUNDARY CONDITIONS

The basic mesh used in the analyses of bearing capacity is shown in Fig. 1. Bearing resistance was mobilized by applying prescribed vertical displacements at nodes to the right of the centre line assuming symmetry. By altering the number of nodes displaced, different footing widths could be simulated. For smooth footings ($\delta = 0$) the nodes were displaced vertically but allowed free movement horizontally. For the rough case ($\delta \geqslant \phi$) the added restraints shown in Fig. 2 were introduced. In all cases the bearing pressure mobilized by a given vertical displacement was obtained by averaging the vertical stress component occurring in the first row of integrating points below the displaced nodes.

The rationale for using displacement rather than load control was based on numerical convenience and, to some extent, physical reality. Load control allows the applied stresses to be adjusted finely by the application of nodal forces, but implies a perfectly flexible foundation. Stresses in excess of the bearing capacity are physically impossible, and failure of the numerical process to converge reflects this. Failure under load control is signalled by a dramatic increase in the number of iterations per load step, with correspondingly high computer times.

Under displacement control, which implies a perfectly rigid footing, equilibrium and yield can always be satisfied with relatively few iterations, even if failure has been reached. Failure under displacement control is indicated by a levelling out

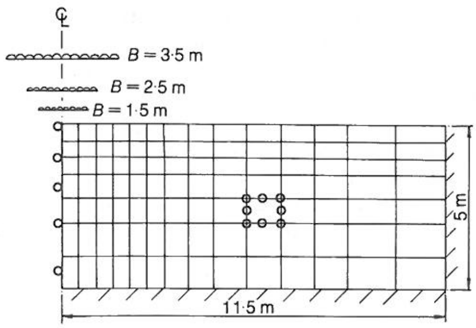

Fig. 1. Mesh used for analyses of bearing capacity

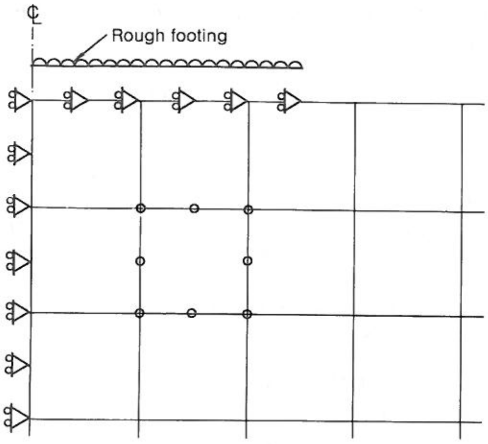

Fig. 2. Boundary conditions for rough footings

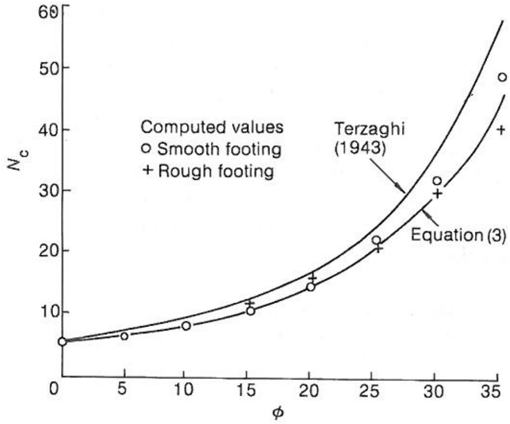

Fig. 3. Computed and theoretical N_c values

Table 1. N_c for rough and smooth footings (from equation (3))

ϕ	N_c
0	5·1
5	6·5
10	8·3
15	11·0
20	14·8
25	20·7
30	30·1
35	46·1

of the averaged stresses beneath the footing which, having reached the bearing capacity, remain at that value despite further displacement increments.

It would also be possible to place an actual footing on the soil within the finite element mesh. The footing would be assigned a stiffness and a strength well in excess of those of the soil below. Loads could then be applied to the soil via the footing instead of directly on to the soil as was done in the present work. Although physically this approach appears more realistic, in the Author's experience it can complicate matters numerically. The interaction between the footing and soil can cause unwanted effects such as tension, and the finite flexibility of the footing must mean that it can be neither truly flexible nor rigid. Also, footing roughness can be varied only by using a suitable interface element.

As far as fundamental bearing capacity calculations are concerned, the present approach, in which no actual footing is used, has justified itself on grounds of simplicity.

COMPUTATION OF N_c

In order to isolate the contribution of the N_c term to the total bearing capacity of a footing, it was necessary to eliminate the other two terms in equation (1) by assuming the soil to be weightless with no surcharge acting. The soil was then attributed strength parameters ϕ and c and the footing displaced incrementally until failure was reached.

Having computed the bearing capacity, the N_c value corresponding to the particular friction angle used in the analysis was obtained using

$$N_c = q_{ult}/c \tag{2}$$

The computed N_c values corresponding to a range of friction angles are shown in Fig. 3.

This class of problem has a closed form solution (Prandtl, 1921; Hill, 1950) given by

$$N_c = \cot\phi\left[\exp(\pi\tan\phi)\tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-1\right] \tag{3}$$

and summarized in Table 1.

The familiar Prandtl load ($N_c = 5{\cdot}14$) may be obtained from equation (3) which tends to this value in the limit as ϕ approaches zero.

Figure 3 shows that the theoretical results were well reproduced using finite elements. The computed values of N_c were also found to be completely insensitive to changes in cohesion or footing width. To illustrate this, Fig. 4 shows the stress distribution at failure under two footing widths for a friction angle of 20°. The stresses in each case were essentially constant apart from numerical flutter towards the edge of the wider footing. The agreement with theory was excellent for the case considered.

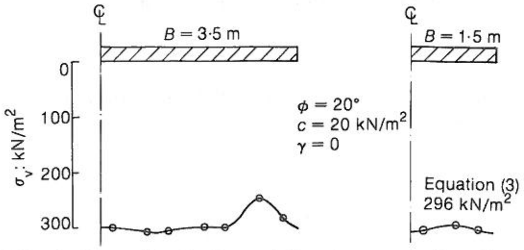

Fig. 4. Stress distribution at failure in computation of N_c

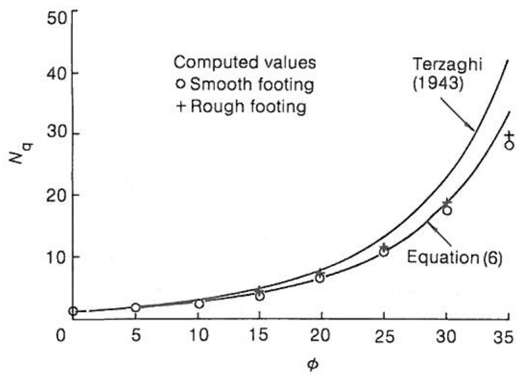

Fig. 5. Computed and theoretical N_q values

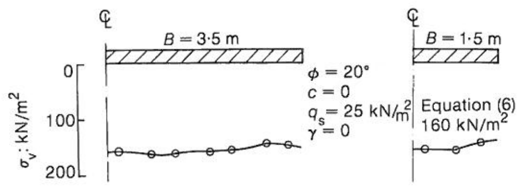

Fig. 6. Stress distribution at failure in computation of N_q

The analyses were repeated with the footing assumed to be perfectly rough; almost identical N_c values were obtained. This was not unexpected because equation (3) was originally derived using Prandtl's (1921) mechanism for rough footings and this includes a triangular wedge which sticks to the footing base. In view of this, Terzaghi's (1943) N_c values for rough footings, which are slightly higher than those in equation (3), would appear slightly unconservative.

In general, the N_c factor has been shown to be highly reproducible using finite elements. This was true not only for footings, but also for other boundary value problems in which failure is induced in a weightless $c-\phi$ material. Indeed, equation (3) is a special case of the more general equation

$$N_c = \cot\phi\left\{\exp\left[(\pi-2\beta)\tan\phi\right]\tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-1\right\} \tag{4}$$

where β is the acute angle (zero for surface footings) between the footing and the ground adjacent to it. When, for example, $0<\beta<\pi/2$, the solution is that for a footing at the crest of a slope. Very close agreement between finite element solutions and equation (4) has been achieved (Griffiths, 1980).

COMPUTATION OF N_q

To isolate the N_q term in equation (1), a weightless, cohesionless soil was considered the bearing capacity of which was derived solely from a uniform surface surcharge q_s. The initial stress state attributed to the soil was therefore q_s in the vertical direction and $K_0 q_s$ in each of the two horizontal directions. The at rest earth pressure coefficient K_0 was put equal to unity in the present work.

The soil was then attributed a friction angle ϕ and the footing displaced incrementally until failure was reached. The N_q value corresponding to the particular angle used in the analysis was obtained using

$$N_q = q_{ult}/q_s \tag{5}$$

The computed results relating N_q and ϕ are plotted in Fig. 5 and compared with Prandtl's (1921) solution of

$$N_q = \tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\exp(\pi\tan\phi) \tag{6}$$

which is also summarized in Table 2.

Excellent agreement was obtained between the finite element solutions and equation (6) and the N_q values computed were found to be completely insensitive to the surcharge or footing width. The limiting stress distributions for two footing widths with a friction angle of 20° are given in Fig. 6. The distributions were found to be essentially constant and even more stable than in the N_c analysis. The introduction of footing roughness made practically no difference to the computed N_q values.

As with N_c, it seemed that finite element techniques were confidently able to predict the component of bearing resistance occurring due to surcharge.

COMPUTATION OF N_γ

The bearing capacity factor N_γ, which gives the contribution to bearing capacity due to soil self-weight, has no closed form solution.

For weightless c–ϕ soils, or weightless cohesionless soils with a constant surcharge, the initial shear strength of the soil before loading is constant. In these cases simple upper bound slip line fields can be constructed which comprise straight lines in the regions of active and passive yielding. These lines are then connected by log spirals in the transitional zones of radial shear. The mechanisms of Prandtl (1921) and Hill (1950) are of this type, and are exact for the weightless soil problem leading to equations (3) and (6) for N_c and N_q.

The problem with N_γ is that a frictional cohesionless soil in which self-weight is included has an initial shear strength that increases with depth from zero at the ground surface. This has the effect of making curved the parts of the Hill and Prandtl mechanisms that were previously straight (Chen, 1975). These mechanisms which gave exact solutions to the weightless soil problem can now at best be only upper bounds.

Although footing roughness did not influence N_c and N_q, the indications are that it has a considerable effect on N_γ. For example, the Prandtl mechanism (rough) predicts values of N_γ that are approximately twice those according to the Hill mechanism (smooth). Slip line solutions by Hansen & Christiensen (1969) for rough and smooth footings present similar conclusions. Values of N_γ for smooth and rough footings are summarized in Tables 3 and 4 respectively.

Finite element predictions of N_γ were obtained by taking a cohesionless soil and giving it an initial vertical stress state that was the product of the soil's unit weight and the distance of the element below ground level. Horizontal stresses were related to this through the at rest coefficient K_0 which was put equal to unity. The footing was then displaced until the average stress under the displaced nodes levelled out. N_γ was then calculated using

$$N_\gamma = 2q_{\text{ult}}/\gamma B \qquad (7)$$

Computed values of N_γ for smooth and rough footings are shown in Figs 7 and 8 respectively. Reasonably good agreement was obtained with approximate solutions of other workers, but a small dependence of N_γ on footing width B was indicated by the calculations. This dependence, which was not observed in the N_c and N_q analyses, was to be expected neither from Terzaghi's equation nor from dimensional considerations. The linear relationship between q_{ult} and B implied by equation (7) suggests a triangular limiting stress

Table 2. N_q for rough and smooth footings (from equation (6))

ϕ	N_q
0	1·0
5	1·6
10	2·5
15	3·9
20	6·4
25	10·7
30	18·4
35	33·3

Table 3. N_γ for smooth footings

ϕ	N_γ	
	Hansen & Christiensen (1969)	Hill (1950) mechanism
0	0	0
5		0·1
10		0·5
15	0·7	1·2
20	1·6	2·7
25	3·5	5·9
30	7·5	12·7
35	18·0	28·6

Table 4. N_γ for rough footings

ϕ	N_γ		
	Hansen & Christiensen (1969)	Terzaghi (1943)	Prandtl (1921) mechanism
0		0	0
5		0·1	0·5
10		0·7	1·3
15	1·2	2·0	2·9
20	2·9	4·8	6·2
25	7·0	9·8	13·0
30	15·0	20·0	27·7
35	35·0	43·0	61·5

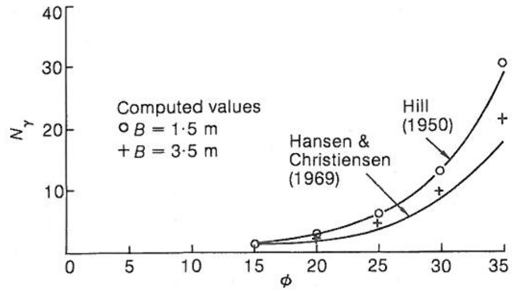

Fig. 7. Computed N_γ values for smooth footings

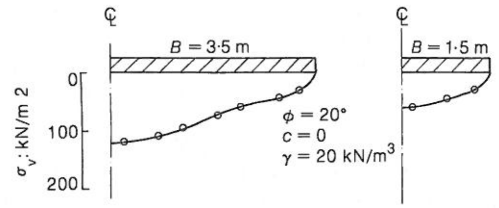

Fig. 9. Stress distribution at failure in computation of N_γ

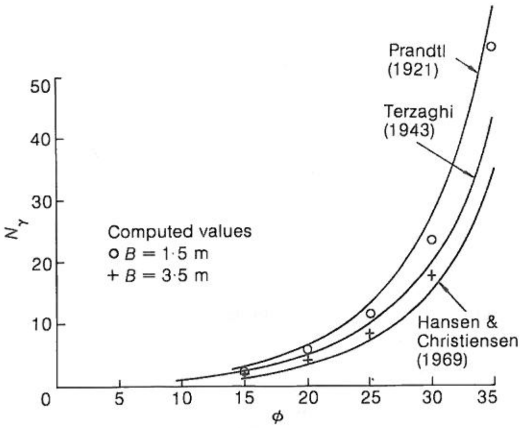

Fig. 8. Computed N_γ values for rough footings

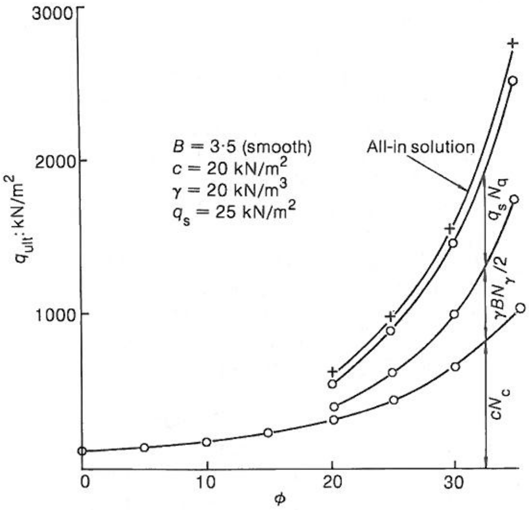

Fig. 10. Superposition of N_c, N_q and N_γ terms

distribution (Scott, 1963) which varies from zero at the footing edge to a maximum at the centre line.

The computed vertical stress distribution (Fig. 9) was found to increase in a non-linear manner from zero at the edge to a maximum at the centre line. Fig. 9 also indicates likely reasons for the dependence of q_{ult} on B. Not only do the averaged stresses increase non-linearly, but also the different coarseness of the mesh for the two cases considered mean that three values were averaged for $B = 1{\cdot}5$ m and seven for $B = 3{\cdot}5$ m.

The general shape of the stress distributions in Fig. 9 was to be expected because a material that relies solely on its self-weight for frictional strength can sustain no normal stress at the edge of a loaded surface area. It is therefore unrealistic to analyse the performance of a surface footing on cohesionless soil by applying a uniform stress. Load control methods may be used providing no stress is applied at the footing edge. For example, the footing problem was repeated by applying a linearly varying vertical stress which was a maximum at the centre line and zero at the footing edge. Results similar to those obtained using displacement control and given in Figs 7 and 8 were obtained.

A further uncertainty in the determination of N_γ lies in the choice of an appropriate friction angle. Ko & Davidson (1973) suggest that Terzaghi's (1943) values of N_γ for rough footings are conservative when using a friction angle measured under triaxial conditions, but unconservative when using a more appropriate friction angle measured in plane strain. The plane strain value would normally be a few degrees higher than its triaxial counterpart. All the computed results presented here are for plane strain conditions with the Mohr–Coulomb criterion being satisfied with respect to the stresses occurring in that plane. The friction angle referred to in the present work is therefore the plane strain value.

The choice of ϕ is also complicated by the fact that it is dependent on confining pressure and strain level. Increased confining pressure causes a reduction in ϕ and Fig. 9 implies that, at failure, ϕ is higher under the edges rather than the centre of

Table 5. Assessment of superposition; $\phi = 20^\circ$

c: kN/m^2	γ: kN/m^3	q_s: kN/m^2	q_{ult}: kN/m^2	q_{ult}: assuming superposition kN/m^2	Superposition error: %
20	0	0	292		
0	20	0	78		
0	0	25	158		
20	20	25	585	528	10
20	20	0	409	370	10
20	0	25	448	450	0
0	20	25	268	236	12

the footing. Also, dense sands exhibit post-peak strain softening and progressive failure (Rowe & Peaker, 1965; Lee & Seed, 1967), so all points at failure in the soil mass may have different operating friction angles depending on the strain level experienced. Attempts to incorporate these features in finite element calculations have been attempted (Griffiths, 1981), but not in the present work, where ϕ must be considered an average value for the soil mass.

In view of the theoretical problems associated with N_γ, it was not surprising that this class of problem posed the greatest problems computationally. Convergence was extremely slow using displacement control because of the shear concentration imposed at the footing edge. This was improved only slightly by attempting a load control approach in which the applied stress varied linearly.

As a general rule, the larger the friction angle, the slower the convergence. Footing roughness exacerbated the situation to such an extent that an angle of friction of 35° seemed the limit for which reasonable solutions could be obtained using the raw viscoplastic algorithm. Although convergence was also slower in the N_c and N_q analyses as ϕ was increased, it still took considerably longer to obtain an N_γ value for the same friction angle.

SUPERPOSITION OF N_c, N_q AND N_γ

Having studied the ability of the finite element method to compute each of the components of bearing capacity, the accuracy of the superposition approximation implicit in equation (1) was assessed. It is generally assumed that the superposition assumption is conservative. Sokolovsky (1956) found that due to non-linearity, the yield condition corresponding to the superposed stresses was equivalent to that of a material with an angle of friction smaller than the angle used to obtain the component stress states. Davis & Booker (1971) also considered superposition with respect to plasticity theory, and concluded that the error was always on the safe side and no greater than 30%.

The finite element approach was to compute the bearing capacity of a soil including surcharge, self-weight and cohesion, and to compare this with the sum of the individual parts obtained previously. The results are given in Fig. 10 for a smooth footing 3·5 m wide. Table 5 gives the computed results of different combinations of properties for a soil with a friction angle of 20°. The value that would have been obtained assuming superposition and the error introduced by such an assumption are also given.

The superposition assumption was always conservative and of the order of 10% for a friction angle of 20°. Errors introduced by superposition were due entirely to non-linearities in N_γ. The linear stress distributions in the N_c and N_q terms were found to superpose almost exactly.

CONCLUSIONS

The results presented show that the finite element method can be used to predict the bearing capacity of surface footings with confidence. By grouping the solutions into three fundamental parts corresponding to each of the components of bearing capacity, excellent agreement with closed form solutions can be obtained for N_c and N_q.

Many different approximate values of N_γ are available in the literature and this reflects the theoretical uncertainty associated with this parameter. Finite element solutions presented here gave good agreement with the lowest N_γ values available and confirmed the dependence of N_γ on footing roughness. The computed values of N_γ showed a slight dependence on footing width. This result, which would not be expected from dimensional considerations, was thought to be due to the non-linear stress distribution beneath the footing at failure and the greater number of stress sampling points available for the wider footing.

A brief assessment using finite elements of the superposition assumption indicated that the assumption was conservative due to non-linearities in the N_γ term. The N_c and N_q terms were found to superpose almost exactly.

Although all three classes of problem showed a marked increase in computer solution time as the friction angle was increased, this effect was considerably worse in the N_γ analyses where 35° represented the largest friction angle for which reasonable solutions could be obtained. This was partly because the very weak soil near the ground surface received a high shear concentration beneath the edge of the rigid footing, which required considerable stress redistribution.

Uncertainties in the value of N_γ are put in perspective if it is considered that for $\phi < 20°$ its value is so low that its contribution to the total bearing capacity could easily be swamped by a moderate presence of cohesion or surcharge. For larger ϕ values, N_γ increases very rapidly and, especially in view of the high safety factors associated with bearing capacity predictions, settlement is more likely to govern the design in all but the narrowest footings.

The good comparison obtained between finite element and closed form or well-known approximate solutions is particularly encouraging in view of the highly idealized soil properties and the simple iteration procedures used. It enables other types of problem involving collapse predictions to be tackled with confidence. Such problems would be those with irregular boundaries or loading condition which exploit the full potential of the finite element method, and for which no known solutions exist.

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